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Alternating Sums of the Reciprocals of Binomial Coefficients

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Abstract

We investigate summations of the form $\sum_{0\leq k\leq n}(-1)^k k^m \binom{n}{k}^{-1}$. We give closed formulae in terms of the Akiyama-Tanigawa matrix. Recurrence formulae, ordinary generating functions and some other results are also given.

1 Introduction and notation

Binomial coefficients play an important role in many areas of mathematics, such as combinatorics, number theory and special functions. The inverse binomial coefficients have an integral representation in terms of beta function

$$\binom{n}{k}^{-1} = (n+1) \int_0^1 x^k (1-x)^{n-k} \, dx. \tag{1}$$

and one can obtain (see [18])

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}.,$$

There are many papers dealing with sums involving inverses of binomial coefficients, see for instance [3, 11, 12, 14, 15, 17, 18, 21, 22]. For nonnegative integers n, m and p we consider the sums

$$T_n^{(m,p)} := \sum_{k=0}^{n} (-1)^k \, k^m \binom{p+n}{p+k}^{-1}. \tag{2}$$

These sums have been studied by many authors. Trif [20], using (1) proved for $m = 0$ that

$$T_n^{(0,p)} = \left((-1)^n + \binom{p+n+1}{p}^{-1} \right) \frac{p+n+1}{p+n+2}. \tag{3}$$

Sury, Wang and Zhao [19], studied (2) for $m = 1$ and $m = 2$, they obtain

$$T_n^{(1,p)} = \frac{p+n+1}{p+n+3} \left(\frac{(-1)^n (n+1)(p+n+3)}{p+n+2} - \binom{p+n+2}{p+1}^{-1} - (-1)^n \right), \tag{4}$$

and

$$\begin{aligned} T_n^{(2,p)} = (p+n+1) \Bigg(& \frac{(-1)^n (n+1)^2}{p+n+2} - \frac{(-1)^n (2n+3)}{p+n+3} \\ & + \frac{2}{p+n+4} \left(\binom{p+n+3}{p+2}^{-1} + (-1)^n \right) - \frac{1}{p+n+3} \binom{p+n+2}{p+1}^{-1} \Bigg). \end{aligned} \tag{5}$$

Our aim is to give a closed form and recurrence relation for the sums (2). In order to investigate the summation of the form $S_n^{(m)} := T_n^{(m,0)}$ and $T_n^{(m,p)}$, we shall use the following tools [1, 6]:

- The Stirling numbers of the first kind $s(n,k)$ (see A008275 in [13]), are defined by the generating function

  $$x(x-1) \cdots (x-n+1) = \sum_{k \geq 0} s(n,k) x^k,$$

  and satisfy the recurrence relation

  $$s(n+1,k) = s(n,k-1) - n s(n,k), \quad (1 \leq k \leq n),$$

  with $s(n,n) = 1$, $s(n,0) = 0$ for $n \geq 1$ and $s(n,k) = 0$ for $k < 0$ or $k > n$.

- The Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ (see A008277 in [13]), are defined by the generating function
$$\prod_{j=1}^{k} \frac{x}{1-jx} = \sum_{n\geq k} \left\{ {n \atop k} \right\} x^n,$$
and satisfy the recurrence relation
$$\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k \left\{ {n \atop k} \right\},$$
with $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1$.

They also verify the following important identity
$$x^n = \sum_{k=0}^{n} (-1)^{n+k} \left\{ {n \atop k} \right\} x(x+1)\cdots(x+k-1). \tag{6}$$

- The Eulerian numbers $\left\langle {n \atop k} \right\rangle$ (see A008292 in [13]) are defined by
$$\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k} (-1)^i (k-i)^n \binom{n+1}{i}, \quad (1 \leq k \leq n),$$
and satisfy the recursive identity
$$\left\langle {n \atop k} \right\rangle = k \left\langle {n-1 \atop k} \right\rangle + (n-k+1) \left\langle {n-1 \atop k-1} \right\rangle,$$
with $\left\langle {1 \atop 1} \right\rangle = 1$.

- The Worpitzky numbers $W_{n,k}$ (see A028246 in [13]), are defined by
$$W_{n,k} = \sum_{i=0}^{k} (-1)^{i+k} (i+1)^n \binom{k}{i}.$$
They can also be expressed through the Stirling numbers of the second kind as follows
$$W_{n,k} = k! \left\{ {n+1 \atop k+1} \right\}. \tag{7}$$
The Worpitzky numbers satisfy the recursive relation
$$W_{n,k} = (k+1) W_{n-1,k} + k W_{n-1,k-1} \quad (n \geq 1, k \geq 1). \tag{8}$$
Some simple properties are given
$$\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle x^k = \sum_{k=0}^{n} (x-1)^{n-k} k W_{n-1,k-1}, \tag{9}$$

$$\sum_{k=0}^{n}\binom{n}{k}\left\{{k \atop t}\right\}=\left\{{n+1 \atop t+1}\right\}, \tag{10}$$

and

$$\sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k+1}{t}=W_{n,n-t}. \tag{11}$$

- The Bernoulli numbers B_n are defined by the exponential generating function

$$\frac{x}{1-e^{-x}}=\sum_{n\geq 0}B_n\frac{x^n}{n!}.$$

The recursive relation is

$$\begin{aligned}B_0&=1,\\ B_n&=1-\sum_{k=0}^{n-1}\binom{n}{k}\frac{B_k}{n-k+1},\ \ (n\geq 1)\,.\end{aligned}$$

Thus we have $B_1=\frac{1}{2}, B_2=\frac{1}{6}, B_3=0$, and so on, they can also be expressed through the Worpitzky numbers

$$B_n=\sum_{k=0}^{n}(-1)^k\frac{W_{n,k}}{k+1}.$$

- The Akiyama-Tanigawa matrix $(A_{n,k})_{n,k\geq 0}$ associated with initial sequence $A_{0,k}=\frac{1}{k+1}$ is defined by (see [2, 5, 8, 10])

$$A_{n,k}=(k+1)\left(A_{n-1,k}-A_{n-1,k+1}\right),$$

or equivalently by [7]

$$\begin{aligned}A_{n,k}&=\frac{1}{k!}\sum_{i=0}^{k}(-1)^k s(k+1,i+1)B_{n+i},\\ &=\sum_{i=1}^{n}(-1)^{i-1}\binom{k+i+1}{k+1}^{-1}W_{n,i}. \end{aligned}\tag{12}$$

The Akiyama-Tanigawa matrix $A_{n,k}$ is then

$$A_{n,k} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \cdots \\ \frac{1}{6} & \frac{1}{6} & \frac{3}{20} & \frac{2}{15} & \frac{5}{42} & \cdots \\ 0 & \frac{1}{30} & \frac{1}{20} & \frac{2}{35} & \frac{5}{84} & \cdots \\ -\frac{1}{30} & -\frac{1}{30} & -\frac{3}{140} & -\frac{1}{105} & 0 & \cdots \\ 0 & -\frac{1}{42} & -\frac{1}{28} & -\frac{4}{105} & -\frac{1}{28} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

2 Explicit formula for $\mathbf{S}_n^{(m)}$

For any nonnegative integer m, we consider the sums

$$S_n^{(m)} := \sum_{k=0}^{n} (-1)^k k^m \binom{n}{k}^{-1}. \tag{13}$$

Note that $S_n^{(0)} = (1+(-1)^n)\frac{n+1}{n+2}$ (identity (14) of [19] or identity (3) here for $p=0$). The following expression for $S_n^{(m)}$ holds and, in terms of computational complexity, it is better than the sum defining $S_n^{(m)}$ for $n \geq m$ as the expression in the theorem involves $O(m)$ operations.

Theorem 1. *For any nonnegative integers n and m, we have*

$$S_n^{(m)} = (n+1)\sum_{j=0}^{m} \frac{(-1)^{m+j}}{n+j+2}\left(1+(-1)^n\binom{n+j+1}{j}\right) W_{m,j}. \tag{14}$$

Proof. We can write $S_n^{(m)}$ as follows

$$\begin{aligned} S_n^{(m)} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k}^{-1} \left((k+1)-1\right)^m \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k}^{-1} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (k+1)^i, \end{aligned}$$

and with (6), we obtain

$$
\begin{aligned}
S_n^{(m)} &= \sum_{k=0}^{n}(-1)^k \binom{n}{k}^{-1} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} \sum_{j=0}^{i} (-1)^{i+j} {i \brace j} (k+1)\cdots(k+j) \\
&= \sum_{k=0}^{n}\sum_{i=0}^{m}\sum_{j=0}^{i} \frac{(-1)^k}{n!} (-1)^{m+j} \binom{m}{i} {i \brace j} k!\,(k+1)\cdots(k+j)\,(n-k)! \\
&= \sum_{i=0}^{m}\sum_{j=0}^{i} (-1)^{m+j} \binom{m}{i} {i \brace j} (n+1)\cdots(n+j) \sum_{k=0}^{n}(-1)^k \binom{n+j}{k+j}^{-1}.
\end{aligned}
$$

Now, from (3) and after some rearrangement, we get

$$
S_n^{(m)} = (n+1)\sum_{j=0}^{m} \frac{(-1)^{m+j}}{n+j+2} \left(1 + (-1)^n \binom{n+j+1}{j}\right) j! \sum_{i=0}^{m} \binom{m}{i} {i \brace j}.
$$

From (7) and (10), the result holds.

$\square$

3 Recurrence relation for $S_n^{(m)}$

Theorem 2. *For any nonnegative integers m and n, we have*

$$
S_{n+1}^{(m)} = S_n^{(m)} - \frac{1}{n+1} S_n^{(m+1)} - (-1)^n (n+1)^m . \tag{15}
$$

Proof. The proof is based on the Sprugnoli [16] observation $\binom{n+1}{k}^{-1} = \binom{n}{k}^{-1} - \frac{k}{n+1}\binom{n}{k}^{-1}$. $\square$

The recurrence relation for $S_n^{(m)}$ is given in the following

Theorem 3. *For any nonnegative integers m and n, we have*

$$
S_{n+1}^{(m)} = S_n^{(m)} - (-1)^n (n+1)^m + \sum_{j=0}^{m+1} \frac{(-1)^{m+j}}{n+j+2} \left(1 + (-1)^n \binom{n+j+1}{j}\right) W_{m+1,j}, \tag{16}
$$

with the initial condition $S_0^{(m)} = \delta_{0m}$, where δ_{ij} is the Kronecker symbol.

Proof. This follows immediately from (14) and (15). $\square$

Setting $m = 1$ in (16), we have the following

Corollary 4. *If n is nonnegative integer, then*

$$
S_{n+1}^{(1)} = S_n^{(1)} + \frac{n - (-1)^n \left(2n^4 + 17n^3 + 49n^2 + 57n + 24\right)}{(n+2)(n+3)(n+4)}.
$$

Our next goal is to calculate the ordinary generating functions of $S_n^{(m)}$.

4 Ordinary generating functions of $S_n^{(m)}$

In 2002, Mansour [9], generalized the idea of Sury [18] and gave an approach based on calculus to obtain the generating function for some combinatorial identities.

Theorem 5 (Mansour [9]). *Let $r, n \geq k$ be any nonnegative integer numbers, and let $f_r(n,k)$ be given by*

$$f_r(n,k) = \frac{(n+r)!}{n!} \int_{u_1}^{u_2} p^k(t)\, q^{n-k}(t)\, dt, \tag{17}$$

where $p(t)$ and $q(t)$ are two functions defined on $[u_1, u_2]$. Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be any two sequences, and let $A(x), B(x)$ be the corresponding ordinary generating functions. Then

$$\sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} f_r(n,k)\, a_k b_{n-k} \right] x^n = \frac{d^r}{dx^r} \left[x^r \int_{u_1}^{u_2} A(xp(t))\, B(xq(t))\, dt \right]. \tag{18}$$

We apply Theorem 5, for $a_n = (-1)^n n^m$ $(m \geq 1)$ and $b_n = 1$, we have

$$\begin{aligned} A(x) &= \frac{1}{(1+x)^{m+1}} \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-x)^{k+1} \\ &= \sum_{k=0}^{m} \frac{(-1)^{m+k}}{(1+x)^{k+1}} W_{m,k}, \\ B(x) &= \sum_{n\geq 0} x^n = \frac{1}{1-x}. \end{aligned}$$

For $r = 1$, formula (18) gives:

$$\sum_{n\geq 0} S_n^{(m)} x^n = \frac{d}{dx} \left[x \int_0^1 \frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-xt)^{k+1}}{(1+xt)^{m+1}(1-x+xt)} dt \right]. \tag{19}$$

Making the substitution $xt = y$ in the right-hand side of (19), we obtain

$$\sum_{n\geq 0} S_n^{(m)} x^n = \frac{d}{dx} \left[\int_0^x \frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-y)^{k+1}}{(1+y)^{m+1}(1-x+y)} dy \right],$$

Since the degree of the denominator is at least one higher than that of the numerator, this fraction decomposes into partial fractions of the form

$$\frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-y)^{k+1}}{(1+y)^{m+1}(1-x+y)} = \frac{\alpha^{(m)}(x)}{1-x+y} + \sum_{s=0}^{m} \frac{\alpha_s^{(m)}(x)}{(1+y)^{m-s+1}}, \tag{20}$$

We note in passing that (20) is equivalent to

$$\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle (-y)^{k+1} = (1+y)^{m+1}\,\alpha^{(m)}(x) + (1-x+y)\sum_{s=0}^{m}(1+y)^{s}\,\alpha_s^{(m)}(x) \tag{21}$$
$$= \sum_{k=0}^{m}(-1)^{m+k+1}\,y\,(1+y)^{m-k}\,W_{m-1,k-1}.$$

For $y=-1$ and using the fact that $W_{p,p}=p!$ for $p\geq 0$, we immediately obtain the well-known identity

$$\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle = m!.$$

Next, if we set $y=0$ in (21) then we obtain a relation between $\alpha^{(m)}(x)$ and $\alpha_s^{(m)}(x)$

$$\sum_{s=0}^{m}\alpha_s^{(m)}(x) = \frac{\alpha^{(m)}(x)}{x-1}. \tag{22}$$

Proposition 6. *For $m\geq 1$, we have*

$$\alpha_s^{(m)}(x) = \sum_{i=0}^{s}\frac{(-1)^{i+s+1}}{x^{i+1}}\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle\binom{k+1}{s-i} \tag{23}$$
$$= \sum_{j=m-s}^{m}\frac{(-1)^{m+j+1}}{x^{s-m+1+j}}W_{m,j},$$

and

$$\alpha^{(m)}(x) = \frac{1}{x^{m+1}}\sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle (1-x)^{k+1} \tag{24}$$
$$= \sum_{j=0}^{m}\frac{(-1)^{m+j}}{x^{j+1}}W_{m,j},$$
$$= -\alpha_m^{(m)}(x).$$

Proof. We verify that (23) and (24) satisfy (21). Denote the right-hand side of (21) by $R^{(m)}(y)$, after some rearrangement, we get

$$R^{(m)}(y) = \sum_{k=0}^{m}\left\langle {m \atop k}\right\rangle\left[\frac{(1+y)^{m+1}}{x^{m+1}}(1-x)^{k+1}\right.$$
$$\left. + (1-x+y)\sum_{s=0}^{m}(1+y)^{s}\sum_{j=0}^{s}\frac{(-1)^{j+1}}{x^{s-j+1}}\binom{k+1}{j}\right],$$

using binomial formula and for $k \leq m$, we obtain

$$R^{(m)}(y) = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left[\sum_{s=m+1}^{m+1} \frac{(1+y)^s}{x^s} \sum_{j=0}^{s} \binom{k+1}{j} (-1)^j x^j \right.$$
$$\left. - \frac{(1-x+y)}{x} \sum_{s=0}^{m} \frac{(1+y)^s}{x^s} \sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} \right]$$

$$= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left[\sum_{s=m+1}^{m+1} \frac{(1+y)^s}{x^s} \sum_{j=0}^{s} \binom{k+1}{j} (-1)^j x^j \right.$$
$$+ \sum_{s=0}^{m} \frac{(1+y)^s}{x^s} \sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j}$$
$$\left. - \sum_{s=0}^{m} \frac{(1+y)^{s+1}}{x^{s+1}} \sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} \right]$$

$$= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left[\sum_{s=0}^{m+1} \frac{(1+y)^s}{x^s} \sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} \right.$$
$$\left. - \sum_{s=0}^{m} \frac{(1+y)^{s+1}}{x^{s+1}} \sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} \right]$$

$$= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left[\sum_{s=0}^{m+1} \frac{(1+y)^s}{x^s} \left(\sum_{j=0}^{s} (-1)^j x^j \binom{k+1}{j} \right. \right.$$
$$\left. \left. - \sum_{j=0}^{s-1} (-1)^j x^j \binom{k+1}{j} \right) \right].$$

Finally,

$$R^{(m)}(y) = \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \left[\sum_{s=0}^{k+1} (1+y)^s \left((-1)^s \binom{k+1}{s} \right) \right]$$
$$= \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-y)^{k+1}.$$

According to (7) and (11), we have

$$\begin{aligned}
\alpha_s^{(m)}(x) &= \sum_{i=0}^{s} \frac{(-1)^{i+s+1}}{x^{i+1}} \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle \binom{k+1}{s-i} \\
&= \sum_{i=0}^{s} \frac{(-1)^{i+s+1}(m-s+i)!}{x^{i+1}} \left\{ {m+1 \atop m-s+i+1} \right\} \\
&= \sum_{i=0}^{s} \frac{(-1)^{i+s+1}}{(x)^{i+1}} W_{m,m-s+i} \\
&= \sum_{j=m-s}^{m} \frac{(-1)^{m+j+1}}{x^{s-m+1+j}} W_{m,j}.
\end{aligned}$$

It follows from (9) that

$$\begin{aligned}
\alpha^{(m)}(x) &= \frac{1}{x^{m+1}} \sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (1-x)^{k+1} \\
&= \frac{1-x}{x^{m+1}} \sum_{k=1}^{m} (-1)^{m+k} x^{m-k} k W_{m-1,k-1} \\
&= (1-x) \sum_{k=0}^{m} (-1)^{m+k} \frac{k}{x^{k+1}} W_{m-1,k-1} \\
&= \sum_{k=0}^{m} (-1)^{m+k} \frac{k}{x^{k+1}} W_{m-1,k-1} - \sum_{k=0}^{m} (-1)^{m+k} \frac{k}{x^{k}} W_{m-1,k-1} \\
&= \sum_{k=0}^{m} (-1)^{m+k} \frac{k}{x^{k+1}} W_{m-1,k-1} + \sum_{k=0}^{m-1} (-1)^{m+k} \frac{k+1}{x^{k+1}} W_{m-1,k}.
\end{aligned}$$

Using (8), we get $\alpha^{(m)}(x)$ as desired. This completes the proof. $\square$

Now, integrating the right-hand side of (20) over y, we obtain

$$\int_0^x \frac{\sum_{k=0}^{m} \left\langle {m \atop k} \right\rangle (-y)^{k+1}}{(1+y)^{m+1}(1-x+y)} dy = \alpha_m^{(m)}(x) \ln(1-x^2) + \sum_{s=0}^{m-1} \frac{\alpha_s^{(m)}(x)}{m-s} \left[1-(1+x)^{s-m}\right]. \tag{25}$$

By differentiating (25) we get the ordinary generating function of $S_n^{(m)}$

$$\begin{aligned}
\sum_{n\geq 0} S_n^{(m)} x^n = \ln(1-x^2) \frac{d}{dx} \alpha_m^{(m)}(x) + \sum_{s=0}^{m-1} \frac{\frac{d}{dx}\alpha_s^{(m)}(x)}{m-s} \left(\left[1-(1+x)^{s-m}\right]\right) \\
+ \sum_{s=0}^{m-1} \left((1+x)^{s-m-1} \alpha_s^{(m)}(x)\right) - \frac{2x}{1-x^2} \alpha_m^{(m)}(x),
\end{aligned} \tag{26}$$

with

$$\frac{d}{dx}\alpha_s^{(m)}(x) = \sum_{j=m-s}^{m} \frac{(s-m+1+j)(-1)^{m+j}}{x^{s-m+2+j}} W_{m,j}.$$

With Proposition 6, we can now rewrite (26) as follows

Theorem 7. *For any real numbers x and for all nonnegative integer m, we have*

$$\sum_{n\geq 0} S_n^{(m)} x^n = \left(\sum_{j=0}^{m} \frac{(1+j)(-1)^{m+j}}{x^{2+j}} W_{m,j}\right) \ln(1-x^2) + \sum_{0\leq j\leq s\leq m-1} \frac{(-1)^j}{x^{s-j+2}} W_{m,m-j} \left(\frac{s-j+1}{m-s}\left(1-(1+x)^{s-m}\right) - x(1+x)^{s-m-1}\right) + \frac{2}{1-x^2}\sum_{j=0}^{m} \frac{(-1)^{m+j}}{x^j} W_{m,j}. \quad (27)$$

In particular for $m=0$ and $m=1$, we get

$$\sum_{n\geq 0} S_n^{(0)} x^n = \frac{2}{1-x^2} + \frac{\ln(1-x^2)}{x^2},$$

and

$$\sum_{n\geq 0} S_n^{(1)} x^n = \frac{2+3x}{x(1+x)^2} + \frac{2-x}{x^3}\ln\left(1-x^2\right).$$

5 The asymptotic expansion

In the previous sections, $S_n^{(m)}$ becomes more complex when, m grows, so it is important to have asymptotic expansion of $S_n^{(m)}$.

Theorem 8. *For $m>0$, we have*

$$S_{2n}^{(m)} \sim (2n)^m \text{ and } S_{2n+1}^{(m)} \sim -(2n+1)^m.$$

Proof. Write

$$k^m = c_0 + c_1(k+1) + c_2(k+1)(k+2)\cdots + c_m(k+1)\cdots(k+m),$$

where c_i's depend on m ($c_m = 1$). we immediately have

$$S_n^{(m)} = c_0\sum_{k=0}^{n}(-1)^k\binom{n}{k}^{-1} + c_1\sum_{k=0}^{n}(-1)^k(k+1)\binom{n}{k}^{-1} + \cdots + \sum_{k=0}^{n}(-1)^k(k+1)\cdots(k+m)\binom{n}{k}^{-1}.$$

After some rearrangement and using the fact that $(k+1)\binom{n}{k}^{-1} = (n+1)\binom{n+1}{k+1}^{-1}$, we have

$$S_n^{(m)} = c_0 T_n^{(0,0)} + c_1(n+1)T_n^{(0,1)} + c_2(n+1)(n+2)T_n^{(0,2)} + \cdots + (n+1)\cdots(n+m)T_n^{(0,m)}.$$

Since $T_{2n}^{(0,p)} \to 1$ and $T_{2n+1}^{(0,p)} \to -1$, the result holds. □

6 A connection to the Akiyama-Tanigawa matrix

In this section we consider $T_n^{(m,p)}$. The following lemma will be useful in the proof of the main theorem of this section.

Lemma 9. *For $m \geq 1$, we have*

$$\sum_{k=0}^{n} k^m z^k = \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} - z^{n+1} \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} \sum_{k=0}^{s} (-1)^{s+k} W_{s,k} (1-z)^{-k-1}.$$

Proof. Recall that, for $m \geq 1$

$$\sum_{k=0}^{\infty} k^m z^k = \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} = \sum_{k=0}^{m} (-1)^{m+k} W_{m,k} (1-z)^{-k-1},$$

we have

$$\begin{aligned}
\sum_{k=0}^{n} k^m z^k &= \sum_{k=0}^{\infty} k^m z^k - \sum_{k=n+1}^{\infty} k^m z^k \\
&= \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} - z^{n+1} \sum_{i=0}^{\infty} (i+n+1)^m z^i \\
&= \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} - z^{n+1} \sum_{i=0}^{\infty} \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} i^s z^i \\
&= \sum_{k=0}^{m} W_{m,k} \left(\frac{z}{1-z}\right)^{k+1} - z^{n+1} \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} \sum_{i=0}^{\infty} i^s z^i,
\end{aligned}$$

as desired. □

For an alternative proof see Boyadzhiev [4]. The main result of this section is to prove the following theorem which expresses explicitly the alternating sums of the reciprocals of binomial coefficients, $T_n^{(m,p)}$, in terms of the Akiyama-Tanigawa matrix $A_{n,k}$.

Theorem 10. *For nonnegative integers n, m and p, we have*

$$T_n^{(m,p)} = \binom{n+p}{p}^{-1} \delta_{0m} + (n+p+1) \sum_{s=0}^{m} (-1)^{n+s} \binom{m}{s} (n+1)^{m-s} A_{s,n+p+1} - \frac{n+p+1}{n+1} \sum_{s=0}^{m} (-1)^s \binom{n+s+p+2}{p+s+1}^{-1} W_{m,s}, \quad (28)$$

where $A_{i,j}$ is the Akiyama-Tanigawa matrix.

Proof. By the Beta function we can write

$$T_n^{(m,p)} = \sum_{k=0}^{n} (-1)^k k^m (p+n+1) \int_0^1 x^{p+k} (1-x)^{n-k} \, dx$$
$$= (p+n+1) \int_0^1 x^p (1-x)^n \sum_{k=0}^{n} k^m \left(\frac{-x}{1-x} \right)^k dx.$$

Using the lemma, we get

$$T_n^{(m,p)} = (p+n+1) \int_0^1 x^p (1-x)^n \left(\sum_{k=0}^{m} W_{m,k} (-x)^{k+1} \right.$$
$$\left. - \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} \sum_{k=0}^{s} (-1)^{s+k} W_{s,k} (-x)^{n+1} (1-x)^{k-n} \right) dx$$
$$= \frac{(p+n+1)}{n+1} \sum_{k=0}^{m} (-1)^{k+1} \binom{n+k+p+2}{p+k+1}^{-1} W_{m,k}$$
$$- (p+n+1) \sum_{s=0}^{m} \sum_{k=0}^{s} \binom{m}{s} (n+1)^{m-s} \frac{(-1)^{n+s+k+1}}{k+1} \binom{n+k+2+p}{k+1+p}^{-1} W_{s,k}.$$

Finally, from (12) we obtain

$$T_n^{(m,p)} = \frac{(p+n+1)}{n+1} \sum_{k=0}^{m} (-1)^{k+1} \binom{n+k+p+2}{p+k+1}^{-1} W_{m,k}$$
$$+ (n+p+1) \sum_{s=0}^{m} \binom{m}{s} (n+1)^{m-s} \frac{(-1)^{n+s}}{(n+p+1)!} \sum_{k=0}^{n+p+1} (-1)^k s(n+p+2, k+1) B_{s+k}.$$

As desired, this completes the proof. □

Setting $p = 0$ in (28) we can rewrite (14) as follows

Corollary 11.

$$S_n^{(m)} = \delta_{0m} - A_{m+1,n} + \sum_{s=0}^{m} (-1)^{n+s} \binom{m}{s} (n+1)^{m-s+1} A_{s,n+1}.$$

7 A recurrence relation For $T_n^{(m,p)}$

Theorem 12. *For any nonnegative integers m, n and p*

$$T_{n+1}^{(m,p)} = \frac{n+1}{n+p+1} T_n^{(m,p)} - \frac{1}{n+p+1} T_n^{(m+1,p)} - (-1)^n (n+1)^m. \tag{29}$$

Proof. Using the identity

$$\binom{n+p+1}{k+p}^{-1} = \frac{n+1}{n+p+1}\binom{n+p}{k+p}^{-1} - \frac{k}{n+p+1}\binom{n+p}{k+p}^{-1},$$

we get the relation (29). □

Now, from (28) and (29), we have the recurrence relation for $T_n^{(m,p)}$

Theorem 13. *For nonnegative integers n, m and p, we have*

$$T_{n+1}^{(m,p)} = \frac{n+1}{n+p+1}T_n^{(m,p)} - (-1)^n (n+1)^{m+1} \sum_{s=0}^{m+1} \frac{(-1)^s}{(n+1)^s}\binom{m+1}{s} A_{s,n+p+1} + \frac{1}{n+1}\sum_{s=0}^{m+1} (-1)^s \binom{n+s+p+2}{p+s+1}^{-1} W_{m+1,s} - (-1)^n (n+1)^m,$$

with the initial condition $T_0^{(m,p)} = \binom{n+p}{p}^{-1}\delta_{0m}$.

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References

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, 1964.

[2] S. Akiyama and Y. Tanigawa, Multiple zeta values at non-positive integers. *Ramanujan J.* **5** (4) (2001), 3327–351.

[3] H. Belbachir, M. Rahmani, and B. Sury, Sums involving moments of reciprocals of binomial coefficients. *J. Integer Sequences* **14** (2011), Article 11.6.6.

[4] K. H. Boyadzhiev, Power sum identities with generalized Stirling numbers. *Fib. Quart.* **46/47** 4 (2008/09), 326–330.

[5] K. W. Chen, Algorithms for Bernoulli numbers and Euler numbers, *J. Integer. Seq* **4** (2001), Article 01.1.6.

[6] R. Graham, D. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley Publishing Company, 1994.

[7] Y. Inaba, Hyper-sums of powers of integers and the Akiyama-Tanigawa matrix. *J. Integer Sequences* **8** (2005), Article 05.2.7.

[8] M. Kaneko, The Akiyama-Tanigawa algorithm for Bernoulli numbers, *J. Integer. Seq* **3** (2000), Article 00.2.9.

[9] T. Mansour, Combinatorial identities and inverse binomial coefficients. *Adv. in Appl. Math.* **28** (2002), 196–202.

[10] D. Merlini, R. Sprugnoli, and M. C. Verri, The Akiyama-Tanigawa transformation, *Integers* **5** (1) (2005), Paper A5.

[11] J. Pla, The sum of inverses of binomial coefficients revisited. *Fib. Quart.* **35** (1997), 342–345.

[12] M. A. Rockett, Sums of the inverses of binomial coefficients. *Fib. Quart.* **19** (1981), 433–437.

[13] N. J. A. Sloane, *The On–Line Encyclopedia of Integer Sequences*. Published electronically at `http://www.research.att.com/~njas/sequences`, 2011.

[14] A. Sofo, General properties involving reciprocals of binomial coefficients. *J. Integer Sequences* **9** (2004), Article 06.4.5.

[15] R. Sprugnoli, Sums of the reciprocals of central binomial coefficients. *Integers* **6** (2006), Paper A27.

[16] R. Sprugnoli, Moments of Reciprocals of Binomial Coefficients. *J. Integer Sequences* **14** (2011), Article 11.7.8.

[17] T. B. Staver, Om summasjon av potenser av binomialkoeffisientene. *Norsk Mat. Tidssker* **29** (1947), 97–103.

[18] B. Sury, Sum of the reciprocals of the binomial coefficients. *European J. Combin.* **14** (1993), 351–353.

[19] B. Sury, Tianming Wang, and Feng-Zhen Zhao. Some identities involving reciprocals of binomial coefficients. *J. Integer Sequences* **7** (2004), Article 04.2.8.

[20] T. Trif, Combinatorial sums and series involving inverses of binomial coefficients. *Fib. Quart.* **38** (2000), 79–84.

[21] J.-H. Yang and F.-Z. Zhao. Sums involving the inverses of binomial coefficients. *J. Integer Sequences* **9** (2006), Article 06.4.2.

[22] F.-Z. Zhao and T. Wang. Some results for sums of the inverses of binomial coefficients. *Integers* **5** (2005), Paper A22.

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