JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 3, Number 3, July 1990

EULER'S "EXEMPLUM MEMORABILE INDUCTIONIS FALLACIS" AND q-TRINOMIAL COEFFICIENTS

GEORGE E. ANDREWS

1. Introduction

In [5], R. J. Baxter and the author introduced q-analogs of the coefficients in the Laurent polynomial

$$(1 + x + x^{-1})^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^j . \tag{1.1}$$

The coefficients in (1.1) are called trinomial coefficients (although the same name is used for the coefficients arising in $(x + y + z)^n$). Now trinomial coefficients have a rather sparse literature. There are occasional references to them in combinatorics books (e.g., [11]; indeed Euler found them worthy of a 20 page account [12]). However they have seen little development perhaps for two reasons: (1) their elementary properties mimic closely those of binomial coefficients, and (2) there is no nice simple formula for them as there is for binomial coefficients: $\binom{n}{m} = n!/(m!(n-m)!)$. Indeed the simplest formulas for $\binom{n}{m}_2$ (easily derived from (1.1)) are

$$\binom{n}{m}_2 = \sum_{j\geq 0} \frac{n!}{j!(j+m)!(n-2j-m)!}, \tag{1.2}$$

and

$$\binom{n}{m}_2 = \sum_{j\geq 0} (-1)^j \binom{n}{j} \binom{2n-2j}{n-m-j}, \tag{1.3}$$

neither of which is especially attractive.

It came as a real surprise that q-analogs of these numbers were to play a crucial role in the solution of a model in statistical mechanics [5]. For example

Received by the editors October 26, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11P68; Secondary 05A10, 05A15, 05A30.
The author was partially supported by National Science Foundation Grant DMS-820269.

[5, §4], we required a representation of the polynomials defined by

$$Y_m(a, b; q) = q^{m(3-b)} \sum_{j=4-b}^{3} Y_{m-1}(a, j; q), \tag{1.4}$$

$$Y_0(a, b; q) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

(where a and b are among $\{1, 2, 3\}$), so that, for example,

$$Y_\infty(3, 3; q) = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 3 \pmod 7}}^{\infty} (1 - q^n)^{-1} \tag{1.6}$$

would be an immediate corollary. The representation we found was [5, p. 319, equation (4.7), $j = k = 1$]

$$\begin{aligned} Y_m(3; q) = &\sum_{\mu=-\infty}^{\infty} q^{14\mu^2+\mu} \binom{m; 7\mu; q}{7\mu}_2 \\ &- \sum_{\mu=-\infty}^{\infty} q^{14\mu^2+13\mu+3} \binom{m; 7\mu + 3; q}{7\mu + 3}_2, \end{aligned} \tag{1.7}$$

where

$$\binom{m; B; q}{A}_2 = \sum_{j \geq 0} \frac{q^{j(j+B)}(q)_n}{(q)_j (q)_{j+A} (q)_{n-2j-A}}, \tag{1.8}$$

and

$$(A)_j = (A; q)_j = (1 - A)(1 - Aq) \cdots (1 - Aq^{j-1}). \tag{1.9}$$

As we have described in [6], consideration of these polynomials was literally thrust upon us. We should emphasize the similarity between the above and Schur's polynomial proof of the Rogers-Ramanujan identities [13, §4]. The polynomials Schur examined were given by $G_1(q) = G_2(q) = 1$ and $G_n(q) = G_{n-1}(q) + q^{n-2} G_{n-2}(q)$ for $n > 2$. Schur showed that

$$G_{n+1}(q) = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda+1)/2} \begin{bmatrix} n \\ \left\lfloor \frac{n - 5\lambda}{2} \right\rfloor \end{bmatrix}_q, \tag{1.10}$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$ and

(1.11)

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{cases} \dfrac{(1 - q^A)(1 - q^{A-1}) \cdots (1 - q^{A-B+1})}{(1 - q^B)(1 - q^{B-1}) \cdots (1 - q)}, & 0 \leq B \leq A, \\ 0 \quad \text{otherwise.} \end{cases}$$

Since generalizations of Schur's identity (1.10) have been utilized extensively in physics [3, 8] and additive number theory [1, 4, 7, 9, 10], one may naturally ask whether the q-trinomial coefficients, appearing in a similar context such as (1.7), may have diverse applications rivalling those of the q-binomial coefficients.

It would seem that to answer this question one would need to identify the q-trinomial coefficients as the generating functions of certain partition-like objects. While there are certain rather artificial interpretations (see §7), none has yielded any combinatorial explanation of (1.7) as the combinatorics of the q-binomial coefficients explains (1.10) [1]. This is surely the key aspect of these polynomials, and unfortunately it will not be resolved in this paper. However we shall consider related questions which we hope will provide further evidence for the power of q-trinomial coefficients and consequently for the importance of understanding them fully.

Obviously one way to proceed is to attempt to investigate any unresolved mysteries related to the trinomial coefficients themselves. As Richard Guy and Donald Knuth pointed out, Euler [12, pp. 54–55] presented such a mystery in two truly surprising pages devoted to Exemplum Memorabile Inductionis Fallacis. Euler first computed $\binom{m}{0}_2$ for $0 \le m \le 9$:

$$1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, \ldots .$$

He then triples each entry in a row shifted one to the right:

$$\begin{array}{l} 1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, \ldots \\ \quad 3, 3, 9, 21, 57, 153, 423, 1179, 3321, \ldots , \end{array}$$

and starting with the first two-entry column, he subtracted the first row from the second:

$$2, 0, 2, 2, 6, 12, 30, 72, 182, \ldots ,$$

each of which may be factored into two consecutive integers:

$$1 \cdot 2, 0 \cdot 1, 1 \cdot 2, 1 \cdot 2, 2 \cdot 3, 3 \cdot 4, 5 \cdot 6, 8 \cdot 9, 13 \cdot 14, \ldots .$$

The first factors make up the Fibonacci sequence F_n defined by $F_{-1} = 1$, $F_0 = 0$, $F_n = F_{n-1} + F_{n-2}$ for $n > 0$.

Surprisingly, however, this marvelous rule

$$3\binom{m+1}{0}_2 - \binom{m+2}{0}_2 = F_m(F_m + 1), \qquad -1 \le m \le 7, \tag{1.12}$$

is *false* for $m > 7$. In order to understand (1.12) we define

$$E_m(a, b) = \sum_{\lambda=-\infty}^{\infty} \left(\binom{m}{10\lambda + a}_2 - \binom{m}{10\lambda + b}_2 \right). \tag{1.13}$$

As part of Theorem 2.1, we show that

$$2E_{m+1}(0, 1) = F_m(F_m + 1), \tag{1.14}$$

from which (1.12) follows by inspection (see Corollary 2.2). However our real concern lies with the q-analogs of the various formulae appearing in Theorem 2.1. What develops in this instance is not another proof of the Rogers-Ramanujan identities, but rather new identities for the even and odd parts of the Rogers-Ramanujan functions. To be more explicit, we recall the first Rogers-Ramanujan function:

$$(1.15)\qquad \begin{aligned} G(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} \\ &= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}. \end{aligned}$$

Then for the even part of $G(q)$ we have (see (6.22))

$$(1.16)\qquad \frac{1}{2}(G(q)+G(-q)) = \frac{\sum_{\lambda=-\infty}^{\infty}(q^{60\lambda^2-4\lambda} - q^{60\lambda^2+44\lambda+8})}{\prod_{n=1}^{\infty}(1-q^{2n})},$$

and for the odd part we have (see (6.22))

$$(1.17)\qquad \frac{1}{2}(G(q)-G(-q)) = \frac{\sum_{\lambda=-\infty}^{\infty}(q^{60\lambda^2+16\lambda+1} - q^{60\lambda^2+64\lambda+17})}{\prod_{n=1}^{\infty}(1-q^{2n})}.$$

These identities, which can be proved by other means, are limiting cases of the following Schur polynomial identities (see (6.16) and (6.17)):
(1.18)

$$\begin{aligned} &\frac{1}{2}(G_{2m+1}(q^{1/2}) + G_{2m+1}(-q^{1/2})) \\ &\quad = \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2-2\lambda}\binom{m;\ 10\lambda;\ q}{10\lambda}_2 - \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+22\lambda+4}\binom{m;\ 10\lambda+4;\ q}{10\lambda+4}_2, \end{aligned}$$

(1.19)

$$\begin{aligned} &\frac{q^{-1/2}}{2}(G_{2m+1}(q^{1/2}) - G_{2m+1}(-q^{1/2})) \\ &\quad = \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+8\lambda}\binom{m;\ 10\lambda+1;\ q}{10\lambda+1}_2 - \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+32\lambda+8}\binom{m;\ 10\lambda+5;\ q}{10\lambda+5}_2. \end{aligned}$$

By investigating Euler's Exemplum Memorabile Inductionis Fallacis we are led to results which raise more questions than they answer. In particular, the legacy of Schur's polynomials [1, 4, 8, 9, 10] suggests that q-trinomial coefficients can provide combinatorial explanations of (1.7), (1.18), and (1.19).

Just as Schur [13, equation (30)] proved the simpler

$$(1.20)\qquad \sum_{\mu=-\infty}^{\infty} (-1)^{\mu} q^{\mu(3\mu+1)/2} \begin{bmatrix} n \\ \lfloor \frac{n-3\mu}{2} \rfloor \end{bmatrix} = 1$$

(a finite version of Euler's Pentagonal Number Theorem), we shall also (in §§3 and 5) look at the much simpler case of (1.13) in which the 10 is replaced by 6.

In §2 we shall examine $E_m(a, b)$ and Euler's remarkable example of a misleading induction. Section 3 briefly treats the sequences of (1.13) when 10 is replaced by 6. In §4 we provide the necessary background and new lemmas for the q-trinomial coefficients. In §5 we provide q-analogs of the results in §3. In §6 we treat q-analogs of the $E_m(a, b)$; this yields, among other things, (1.18) and (1.19). Our work raises new questions and these are briefly described in the conclusion.

I wish to thank Ray and Christine Ayoub for translating Euler's Exemplum Memorabile Inductionis Fallacis for me.

2. The $E_m(a, b)$ and Euler's misleading induction

Actually the only hard part of understanding (1.12) is writing down (1.13). Once that has been done the following theorem is quite easy.

Theorem 2.1.

$$2E_m(1, 2) = 2E_{m-1}(0, 3) = 2E_{m+1}(0, 1) = F_m(F_m + 1), \tag{2.1}$$

$$E_m(1, 4) = E_{m+1}(2, 3) = F_{m+1}F_m, \tag{2.2}$$

$$2E_m(3, 4) = 2E_{m-1}(2, 5) = 2E_{m+1}(4, 5) = F_m(F_m - 1), \tag{2.3}$$

$$2E_m(1, 3) = F_{2m} + F_m, \tag{2.4}$$

$$2E_m(2, 4) = F_{2m} - F_m, \tag{2.5}$$

$$2E_m(1, 5) = F_{2m+1} - F_{m-1}, \tag{2.6}$$

$$2E_m(0, 4) = F_{2m+1} + F_{m-1}, \tag{2.7}$$

$$2E_m(0, 2) = F_{2m-1} + F_{m+1}, \tag{2.8}$$

$$2E_m(3, 5) = F_{2m-1} - F_{m+1}, \tag{2.9}$$

$$E_m(0, 5) = F_{2m-1} + F_mF_{m-1}. \tag{2.10}$$

Remark. This result gives us $E_m(a, b)$ for every pair of integers a, b because clearly

$$E_m(a, b) = -E_m(b, a), \qquad E_m(10r + a, 10s + b) = E_m(a, b),$$

and

$$E_m(10 - a, b) = E_m(a, b) = E_m(a, 10 - b).$$

Proof. We recall the Euler-Binet formula for the Fibonacci numbers [12, p. 54]

$$F_n = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}}, \tag{2.11}$$

where $\phi = (1 + \sqrt{5})/2$ and $\bar{\phi} = (1 - \sqrt{5})/2$ are roots of the equation

$$x^2 - x - 1 = 0. \tag{2.12}$$

Thus each extreme right-hand entry of (2.1)–(2.10) is of the form

$$A(\phi^2)^m + B(\bar{\phi}^2)^m + C\phi^m + D\bar{\phi}^m + E(-1)^m. \tag{2.13}$$

Therefore each of these sequences satisfies a fifth-order recurrence whose auxiliary equation [15, p. 153] is

$$(x^2 - x - 1)(x^2 - 3x + 1)(x + 1) = x^5 - 3x^4 - x^3 + 5x^2 + x - 1, \tag{2.14}$$

since ϕ^2 and $\bar{\phi}^2$ are the roots of $x^2 - 3x + 1 = 0$.

On the other hand, if $\xi = e^{2\pi i/10}$, then

(2.15)

$$\begin{aligned}
E_m(a,b) &= \frac{1}{10}\sum_{\mu=-\infty}^{\infty}\sum_{j=0}^{9}\binom{m}{\mu}_2(\xi^{j(\mu-a)}-\xi^{j(\mu-b)})\\
&= \frac{1}{10}\sum_{j=0}^{9}(1+\xi^j+\xi^{-j})^m(\xi^{-aj}-\xi^{-bj})\\
&= \frac{1}{5}\sum_{j=1}^{4}(1+2\cos(\pi j/5))^m(\cos(\pi aj/5)-\cos(\pi bj/5))\\
&\qquad\qquad+\frac{1}{10}(-1)^m((-1)^a-(-1)^b)\\
&= \frac{1}{5}\sum_{j=1}^{2}\{(1+2\cos(\pi j/5))^m-(1-2\cos(\pi j/5))^m\}\\
&\qquad\qquad\times(\cos(\pi aj/5)-\cos(\pi bj/5))+\frac{1}{10}(-1)^m((-1)^a-(-1)^b)\\
&= \alpha(\phi^2)^m+\beta(\overline{\phi}^2)^m+\gamma\phi^m+\delta\overline{\phi}^m+\varepsilon(-1)^m,
\end{aligned}$$

since $\phi^2 = 1+2\cos(\pi/5)$, $\overline{\phi}^2 = 1-2\cos(2\pi/5)$, $\phi = 1+2\cos(2\pi/5)$, and $\overline{\phi} = 1-2\cos(\pi/5)$.

Hence by (2.13) and (2.15) we see that every entry in (2.1)–(2.10) satisfies the fifth-order recurrence whose auxiliary polynomial is (2.14). Therefore our theorem follows by mathematical induction provided all the assertions are true for $m \le 6$, and this is easily verified by inspection. □

Corollary 2.2. *For* $m \le 7$, *Equation* (1.14) *holds.*

Proof. From (1.1) we see that

$$\binom{m}{j}_2 = \binom{m-1}{j-1}_2+\binom{m-1}{j}_2+\binom{m-1}{j+1}_2. \tag{2.16}$$

Hence

(2.17)

$$\begin{aligned}
3\binom{m+1}{0}_2-\binom{m+2}{0}_2 &= 3\binom{m+1}{0}_2-\binom{m+1}{-1}_2-\binom{m+1}{0}_2-\binom{m+1}{1}_2\\
&= 2\binom{m+1}{0}_2-2\binom{m+1}{1}_2\\
&= 2E_{m+1}(0,1)\quad(\text{for } m\le 7)\\
&= F_m(F_m+1)\quad(\text{by (2.1)}).\ \square
\end{aligned}$$

Obviously similar corollaries can be produced from all the other assertions in Theorem 2.1; however none seems quite this dramatic.

Also, the reader may wonder why we have treated differences of two trinomial coefficients in our sums. The following formula (proved exactly like Theorem 2.1) makes clear how much simpler the differencing results turn out:

$$\sum_{\lambda=-\infty}^{\infty} \binom{m}{10\lambda + a}_2 = \begin{cases} \frac{1}{2}F_m^2 + \frac{1}{2}F_{m-1}^2 + \frac{1}{2}F_{m-1}F_m + \frac{2}{5}F_{m-1} + \frac{1}{5}F_m + \frac{3^m}{10} & \text{if } a = 0, \\ \frac{1}{2}F_{m-1}F_m + \frac{1}{2}F_m^2 - \frac{1}{10}F_{m-1} + \frac{1}{5}F_m + \frac{3^m}{10} & \text{if } a = 1, \\ \frac{1}{2}F_{m-1}F_m - \frac{1}{10}F_{m-1} - \frac{3}{10}F_m + \frac{3^m}{10} & \text{if } a = 2, \\ -\frac{1}{2}F_{m-1}F_m - \frac{1}{10}F_{m-1} - \frac{3}{10}F_m + \frac{3^m}{10} & \text{if } a = 3, \\ -\frac{1}{2}F_{m-1}F_m - \frac{1}{2}F_m^2 - \frac{1}{10}F_{m-1} + \frac{1}{5}F_m + \frac{3^m}{10} & \text{if } a = 4, \\ -\frac{1}{2}F_{m-1}^2 - \frac{1}{2}F_{m-1}F_m - \frac{1}{2}F_m^2 + \frac{2}{5}F_{m-1} + \frac{1}{5}F_m + \frac{3^m}{10} & \text{if } a = 5. \end{cases} \tag{2.18}$$

Additionally, we have been unable to find any reasonable q-analogs of any part of (2.18).

3. Recurrences related to 6

Obviously the sequences defined by (1.13) are special cases of

$$\mathscr{E}_m(k; a, b) = \sum_{\lambda=-\infty}^{\infty} \left(\binom{m}{k\lambda + a}_2 - \binom{m}{k\lambda + b}_2 \right). \tag{3.1}$$

The case $k = 7$ is considered extensively in [5]. Elsewhere we shall treat $k = 4$ and 5, which have somewhat different q-analogs. For later work in this paper, we shall find it useful to treat the case $k = 6$.

Theorem 3.1.

$$\begin{aligned} \mathscr{E}_{m+1}(6; 0, 1) = \mathscr{E}_{m+1}(6; 0, 1) = \mathscr{E}_{m+1}(6; 2, 3) = \mathscr{E}_m(6; 1, 2) \\ = \mathscr{E}_{m-1}(6; 0, 3) = (2^m - (-1)^m)/3, \end{aligned} \tag{3.2}$$

$$\mathscr{E}_m(6; 0, 2) = \mathscr{E}_m(6; , 1, 3) = 2^{m-1}. \tag{3.3}$$

Proof. Now each entry satisfies a recurrence whose auxiliary polynomial is $x^2 - x - 2 = 0$. The rest is as before, only easier. □

4. q-trinomial coefficients

In [5], R. J. Baxter and the author studied six q-analogs of $\binom{m}{j}_2$. To our surprise, there are intricate facts and relationships among the q-analogs which we found quite refractory. We shall require a selection of these results including three q-analogs. First, the definitions [5, Equations (2.7), (2.8), (2.9), (2.13), and (2.14)]:

$$\binom{m; B; q}{A}_2 = \sum_{j \geq 0} \frac{q^{j(j+B)}(q)_m}{(q)_j (q)_{j+A} (q)_{m-2j-A}}, \tag{4.1}$$

$$T_0(m, A, q) = \sum_{j=0}^{m} (-1)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m - 2j \\ m - A - j \end{bmatrix}, \tag{4.2}$$

and

$$T_1(m, A, q) = \sum_{j=0}^{m} (-q)^j \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \begin{bmatrix} 2m-2j \\ m-A-j \end{bmatrix}, \tag{4.3}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix}_q = \begin{bmatrix} A \\ B \end{bmatrix} = \frac{(q)_A}{(q)_B (q)_{A-B}} \tag{4.4}$$

and

$$(A; q)_n = (A)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}). \tag{4.5}$$

These are indeed q-analogs of the trinomial coefficients since [5, p. 299]

$$\binom{m; B; 1}{A}_2 = T_0(m, A, 1) = T_1(m, A, 1) = \binom{m}{A}_2. \tag{4.6}$$

There are easily established symmetry relations [5, p. 299, Equation (2.15)]

$$\binom{m; B; q}{-A}_2 = q^{A(A+B)} \binom{m; B+2A; q}{A}_2, \tag{4.7}$$

while, immediately from (4.4), we see that

$$T_i(m, A, q) = T_i(m, -A, q) \qquad (i = 0 \text{ or } 1). \tag{4.8}$$

We shall require only two of the several Pascal triangle recurrences [5, Equations (2.17) and (2.19)]:

$$\begin{aligned} T_1(m, A, q) = {} & T_1(m-1, A, q) + q^{m+A} T_0(m-1, A+1, q) \\ & + q^{m-A} T_0(m-1, A-1, q), \end{aligned} \tag{4.9}$$

$$\begin{aligned} T_0(m, A, q) = {} & T_0(m-1, A-1, q) + q^{m+A} T_1(m-1, A, q) \\ & + q^{2m+2A} T_0(m-1, A+1, q). \end{aligned} \tag{4.10}$$

Next we have an identity which reduces to a tautology when $q = 1$ [5, p. 301, Equation (2.20)]:

$$\begin{aligned} & T_1(m, A, q) - q^{m-A} T_0(m, A, q) - T_1(m, A+1, q) \\ & \quad + q^{m+A+1} T_0(m, A+1, q) = 0, \end{aligned} \tag{4.11}$$

and two important identities [5, p. 305, Equations (2.33_i) and (2.34_i)]:

$$\binom{m; A; q^2}{A}_2 = q^{m^2 - A^2} T_0(m, A, q^{-1}) \tag{4.12}$$

and

$$\binom{m; A-1; q^2}{A}_2 = q^{m(m-1) - A(A-1)} T_1(m, A, q^{-1}). \tag{4.13}$$

Also, we need to consider a sum of the $T_0(m, A, q)$, so we define

$$U(m, A, q) = T_0(m, A, q) + T_0(m, A+1, q). \tag{4.14}$$

Limiting values of our q-analogs are also needed [5, pp. 309–310, Equations (2.48), (2.53), (2.54)]:

$$\lim_{m\to\infty}\binom{m\,;\,A\,;\,q}{A}_2=\frac{1}{(q)_\infty}, \tag{4.15}$$

$$\lim_{m\to\infty}U(m\,,\,A\,,\,q)=\frac{(-q\,;\,q^2)_\infty}{(q^2\,;\,q^2)_\infty}. \tag{4.16}$$

In addition, two new lemmas are needed.

Lemma 4.1. *For* $m\geq 1$,

(4.17)

$$\begin{aligned}U(m\,,\,A\,,\,q)&=(1+q^{2m-1})U(m-1\,,\,A\,,\,q)\\&\quad+q^{m-A}T_1(m-1\,,\,A-1\,,\,q)+q^{m+A+1}T_1(m-1\,,\,A+2\,,\,q).\end{aligned}$$

Proof.

$$\begin{aligned}&U(m\,,\,A\,,\,q)-(1+q^{2m-1})U(m-1\,,\,A\,,\,q)\\&\quad=T_0(m\,,\,A\,,\,q)+T_0(m\,,\,A+1\,,\,q)\\&\qquad-(1+q^{2m-1})(T_0(m-1\,,\,A\,,\,q)+T_0(m-1\,,\,A+1\,,\,q))\\&\quad=T_0(m\,,\,A\,,\,q)-T_0(m-1\,,\,A+1\,,\,q)\\&\qquad+T_0(m\,,\,A+1\,,\,q)-T_0(m-1\,,\,A\,,\,q)\\&\qquad-q^{2m-1}T_0(m-1\,,\,A+1\,,\,q)-q^{2m-1}T_0(m-1\,,\,A\,,\,q)\\&\quad=(q^{m-A}T_1(m-1\,,\,A\,,\,q)+q^{2m-2A}T_0(m-1\,,\,A-1\,,\,q))\\&\qquad+(q^{m+A+1}T_1(m-1\,,\,A+1\,,\,q)+q^{2m+2A+2}T_0(m-1\,,\,A+2\,,\,q))\\&\qquad-q^{2m-1}T_0(m-1\,,\,A+1\,,\,q)-q^{2m-1}T_0(m-1\,,\,A\,,\,q)\end{aligned}$$

(by two applications of equation (4.10)—with A replaced by $-A$ in the first and by $A+1$ in the second)

$$\begin{aligned}&=q^{m-A}(T_1(m-1\,,\,A\,,\,q)+q^{m-A}T_0(m-1\,,\,A-1\,,\,q)\\&\qquad\qquad-q^{m+A-1}T_0(m-1\,,\,A\,,\,q))\\&\quad+q^{m+A+1}(T_1(m-1\,,\,A+1\,,\,q)+q^{m+A+1}T_0(m-1A+2\,,\,q)\\&\qquad\qquad-q^{m-A-2}T_0(m-1\,,\,A+1\,,\,q))\\&=q^{m-A}T_1(m-1\,,\,A-1\,,\,q)+q^{m+A+1}T_1(m-1\,,\,A+2\,,\,q)\end{aligned}$$

(by two applications of (4.11)—with A replaced by $A-1$ in the first and by $A+1$ in the second, and both with m replaced by $m-1$). □

Lemma 4.2. *For* $m\geq 2$,

(4.18)

$$\begin{aligned}U(m\,,\,A\,,\,q)&=(1+q+q^{2m-1})U(m-1\,,\,A\,,\,q)-qU(m-2\,,\,A\,,\,q)\\&\quad+q^{2m-2A}T_0(m-1\,,\,A-2\,,\,q)+q^{2m+2A+2}T_0(m-2\,,\,A+3\,,\,q).\end{aligned}$$

Proof. By Lemma 4.1,

$$\begin{aligned}
&U(m, A, q) - (1+q+q^{2m-1})U(m-1, A, q) + qU(m-2, A, q)\\
&\quad = q^{m-A}T_1(m-1, A-1, q) + q^{m+A+1}T_1(m-1, A+2, q)\\
&\qquad - qU(m-1, A, q) + qU(m-2, A, q)\\
&\quad = q^{m-A}(T_1(m-2, A-1, q) + q^{m+A+2}T_0(m-2, A, q)\\
&\qquad\qquad + q^{m-A}T_0(m-2, A-2, q))\\
&\qquad + q^{m+A+1}(T_1(m-2, A+2, q) + q^{m+A+1}T_0(m-2, A+3, q)\\
&\qquad\qquad + q^{m-A-3}T_0(m-2, A+1, q))\\
&\qquad - q(T_0(m-2, A+1, q) + q^{m-1-A}T_1(m-2, A, q)\\
&\qquad\qquad + q^{2m-2A-2}T_0(m-2, A-1, q))\\
&\qquad - q(T_0(m-2, A, q) + q^{m+A}T_1(m-2, A+1, q)\\
&\qquad\qquad + q^{2m+2A}T_0(m-2, A+2, q))\\
&\qquad + qT_0(m-2, A, q) + qT_0(m-2, A+1, q)
\end{aligned}$$

(by two applications of (4.10) and two of (4.9))

$$\begin{aligned}
&\quad = q^{2m-2A}T_0(m-2, A-2, q) + q^{2m+2A+2}T_0(m-2, A+3, q)\\
&\qquad + q^{m-A}(T_1(m-2, A-1, q) - T_1(m-2, A, q)\\
&\qquad\qquad - q^{m-A-1}T_0(m-2, A-1, q) + q^{m+A-2}T_0(m-2, A, q))\\
&\qquad + q^{m+A+1}(T_1(m-2, A+2, q) - T_1(m-2, A+1, q)\\
&\qquad\qquad + q^{m-A-3}T_0(m-2, A+1, q) - q^{m+A}T_0(m-2, A+2, q))\\
&\qquad\qquad\qquad \text{(by cancelling and regrouping)}\\
&\quad = q^{2m-2A}T_0(m-2, A-2, q) + q^{2m+2A+2}T_0(m-2, A+3, q)
\end{aligned}$$

(by two applications of (4.11)—with A replaced by $A-1$ in the first and by $-A-2$ in the second, and both with m replaced by $m-2$). □

5. The q-analog of §3

In this section we shall prove an identity, Theorem 5.1, which is, in fact, the q-analog of (3.4). From this we shall also deduce identities (83) and (86) in L. J. Slater's compendium [14] of Rogers-Ramanujan type identities.

Theorem 5.1.

$$\begin{aligned}
\prod_{j=1}^{m}(1+q^{2j-1}) = &\sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+2\lambda}(T_0(m, 6\lambda, q) + T_0(m, 6\lambda+1, q))\\
&- \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+10\lambda+2}(T_0(m, 6\lambda+2, q) + T_0(m, 6\lambda+3, q)).
\end{aligned}$$

Proof. Let us denote the left-hand side of our theorem by L_m and the right-hand side of R_m. Clearly L_m is uniquely defined by the conditions

$$L_m = \begin{cases} 1 & \text{if } m = 0, \\ (1+q^{2m-1})L_{m-1} & \text{if } m > 0. \end{cases} \tag{5.1}$$

On the other hand, $R_0 = 1$ and

$$\begin{aligned}
R_m &= \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+2\lambda} U(m, 6\lambda, q) - \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+10\lambda+2} U(m, 6\lambda+2, q) \\
&= (1+q^{2m-1})R_{m-1} + \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+2\lambda+m-6\lambda} T_1(m-1, 6\lambda-1, q) \\
&\quad + \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+2\lambda+m+6\lambda+1} T_1(m-1, 6\lambda+2, q) \\
&\quad - \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+10\lambda+2+m-6\lambda-2} T_1(m-1, 6\lambda+1, q) \\
&\quad - \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+10\lambda+2+m+6\lambda+3} T_1(m-1, 6\lambda+4, q) \\
&= (1+q^{2m-1})R_{m-1}
\end{aligned}$$

since the first sum (with λ replaced by $-\lambda$) cancels the third, and the fourth sum (with λ replaced by $-\lambda-1$) cancels the second. □

Corollary 5.2.

$$\begin{aligned}
\sum_{j\geq 0} q^{2j^2} \begin{bmatrix} m \\ 2j \end{bmatrix} &= \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2-\lambda} \binom{m;\, 6\lambda;\, q}{6\lambda}_2 \\
&\quad - \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+7\lambda+1} \binom{m;\, 6\lambda+2;\, q}{6\lambda+2}_2,
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
\sum_{j\geq 0} q^{2j^2+2j} \begin{bmatrix} m \\ 2j+1 \end{bmatrix} &= \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+5\lambda} \binom{m;\, 6\lambda+1;\, q}{6\lambda+1}_2 \\
&\quad - \sum_{\lambda=-\infty}^{\infty} q^{12\lambda^2+13\lambda+3} \binom{m;\, 6\lambda+3;\, q}{6\lambda+3}_2.
\end{aligned} \tag{5.3}$$

Proof. Equation (5.2) is the even portion of Theorem 5.1 once it has had q replaced by q^{-1} and the result multiplied by q^{m^2}; equation (5.3) is the odd portion. □

Finally we note that if $m \to \infty$ in (5.2), then we obtain (using (4.15) and the Jacobi triple product [2, p. 21])

$$\sum_{j=0}^{\infty} \frac{q^{2j^2}}{(q)_{2j}} = \prod_{n=1}^{\infty}(1+q^{24n-11})(1+q^{24n-13})(1-q^{24n}) - q\sum_{n=1}^{\infty}(1+q^{24n-5})(1+q^{24n-19})(1-q^{24n}),$$

which is Slater's equation (83) [14, p. 160], and from (5.3) in the same way we obtain

$$\sum_{j=0}^{\infty} \frac{q^{2j^2+2j}}{(q)_{2j+1}} = \prod_{n=1}^{\infty}(1+q^{24n-7})(1+q^{24n-17})(1-q^{24n}) - q^2\prod_{n=1}^{\infty}(1+q^{24n-1})(1+q^{24n-23})(1-q^{24n}),$$

which is Slater's equation (86) [14, p. 161].

6. Results related to the Rogers-Ramanujan identities

In order to provide q-analogs for some of the results in §2, we require the following polynomials:

$$G_n(q) = \begin{cases} 0 & \text{if } n=0, \\ 1 & \text{if } n=1, \\ G_{n-1}(q)+q^{n-2}G_{n-2} & \text{if } n \geq 2, \end{cases} \tag{6.1}$$

and

$$H_n(q) = \begin{cases} 0 & \text{if } n=0, \\ 1 & \text{if } n=1, \\ H_{n-1}(q)+q^{n-1}H_{n-2}(q) & \text{if } n \geq 2. \end{cases} \tag{6.2}$$

Obviously

$$G_n(1) = H_n(1) = F_n. \tag{6.3}$$

Furthermore, it is a simple exercise in mathematical induction to prove that

$$G_{2n}(-1) = H_{2n}(-1) = F_n \tag{6.4}$$

and

$$G_{2n-3}(-1) = H_{2n+3}(-1) = F_n. \tag{6.5}$$

In [2, p. 50, Example 10], it is shown that

$$G_n(q) = \sum_{j\geq 0} q^{j^2} \begin{bmatrix} n-1-j \\ j \end{bmatrix} \tag{6.6}$$

and

$$H_n(q) = \sum_{j\geq 0} q^{j^2+j} \begin{bmatrix} n-1-j \\ j \end{bmatrix}. \tag{6.7}$$

Reversing the order of summation in (6.6) and (6.7) and replacing q by q^{-1} we obtain related reciprocal polynomials:

$$R_m(q) \equiv q^{m^2} G_{2m+1}(q^{-1}) = \sum_{j=0}^{m} q^{j^2} \begin{bmatrix} m+j \\ 2j \end{bmatrix} \tag{6.8}$$

and

$$S_m(q) = q^{m^2-1} H_{2m}(q^{-1}) = \sum_{j=0}^{m} q^{j^2+2j} \begin{bmatrix} m+j \\ 2j+1 \end{bmatrix}. \tag{6.9}$$

Theorem 6.1.

$$\begin{aligned} R_m(q) = &\sum_{\lambda=-\infty}^{\infty} q^{40\lambda^2+4\lambda} U(m, 10\lambda, q) \\ &- \sum_{\lambda=-\infty}^{\infty} q^{40\lambda^2+36\lambda+8} U(m, 10\lambda+4, q), \end{aligned} \tag{6.10}$$

$$\begin{aligned} S_m(q) = &\sum_{\lambda=-\infty}^{\infty} q^{40\lambda^2+12\lambda} U(m, 10\lambda+1, q) \\ &- \sum_{\lambda=-\infty} q^{40\lambda^2+28\lambda+4} U(m, 10\lambda+3, q). \end{aligned} \tag{6.11}$$

Proof. From (6.8) and (6.1) we can derive a defining second-order recurrence for $R_m(q)$:

$$R_m(q) = \begin{cases} 1 & \text{if } m = 0, \\ 1+q & \text{if } m = 1, \\ (1+q+q^{2m-1})R_{m-1}(q) - qR_{m-2}(q) & \text{if } m \geq 2. \end{cases} \tag{6.12}$$

Hence to prove (6.10) we only need show that the right-hand side satisfies the same defining recurrence. Inspection shows the validity of (6.10) for $m = 0$ and 1. So if $\rho_m(q)$ denotes the right-hand side of (6.10), then by Lemma 4.2 we obtain

$$\begin{aligned} &\rho_m(q) - (1+q+q^{2m-1})\rho_{m-1}(q) + q\rho_{m-2}(q) \\ &= \sum_{\lambda=-\infty}^{\infty} a^{40\lambda^2+4\lambda}(q^{2m-20\lambda} T_0(m-2, 10\lambda-2, q) \\ &\qquad + q^{2m+20\lambda+2} T_0(m-2, 10\lambda+3, q)) \\ &\quad - \sum_{\lambda=-\infty}^{\infty} q^{40\lambda^2+36\lambda+8}(q^{2m-20\lambda-8} T_0(m-2, 10\lambda+2, q) \\ &\qquad + q^{2m+20\lambda+10} T_0(m-2, 10\lambda+7, q)) \\ &= 0, \end{aligned} \tag{6.13}$$

since the first portion of the first sum (with λ replaced by $-\lambda$) cancels the first portion of the second sum, and the second portion of the second sum (with λ replaced by $-\lambda-1$) cancels the second portion of the first sum.

Thus $\rho_m(q)$ satisfies (6.12) and consequently (6.10) is valid.

In the same way, it is easily verified that $S_m(q)$ satisfies

$$S_m(q) = \begin{cases} 0 & \text{if } m=0, \\ 1 & \text{if } m=1, \\ (1+q+q^{22m-1})S_{m-1}(q) - qS_{m-2}(q) & \text{if } m \geq 2. \end{cases} \tag{6.14}$$

As before, we let $\sigma_m(q)$ denote the right-hand side of (6.11), and we need only show that $\sigma_m(q)$ satisfies (6.14). Now by Lemma 4.2 we obtain

$$\begin{aligned} &\sigma_m(q) - (1+q+q^{2m-1})\sigma_{m-1}(q) + q\sigma_{m-2}(q) \\ &= \sum_{\lambda=-\infty}^{\infty} q^{40\lambda^2+12\lambda}(q^{2m-20\lambda-2}T_0(m-2,\, 10\lambda-1,\, q) \\ &\qquad + q^{2m+20\lambda+4}T_0(m-2,\, 10\lambda+4,\, q)) \\ &\quad - \sum_{\lambda=-\infty}^{\infty} q^{40\lambda^2+28\lambda+4}(q^{2m-20\lambda-6}T_0(m-2,\, 10\lambda+1,\, q) \\ &\qquad + q^{2m+20\lambda+8}T_0(m-2,\, 10\lambda+6,\, q)) \\ &= 0, \end{aligned} \tag{6.15}$$

since the first portion of the first sum (with λ replaced by $-\lambda$) cancels the first portion of the second sum, and the second sum (with λ replaced by $-\lambda-1$) cancels the second portion of the first sum. □

Corollary 6.2.

(6.16)

$$\begin{aligned} &\frac{1}{2}(G_{2m+1}(q^{1/2} + G_{2m+1}(-q^{1/2})) \\ &\quad = \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2-2\lambda}\begin{pmatrix} m;\, 10\lambda;\, q \\ 10\lambda \end{pmatrix}_2 - \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+22\lambda+4}\begin{pmatrix} m;\, 10\lambda+4;\, q \\ 10\lambda+4 \end{pmatrix}_2, \end{aligned}$$

(6.17)

$$\begin{aligned} &\frac{q^{-1/2}}{2}(G_{2m+1}(q^{1/2}) - G_{2m+1}(-q^{1/2})) \\ &\quad = \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+8\lambda}\begin{pmatrix} m;\, 10\lambda+1;\, q \\ 10\lambda+1 \end{pmatrix}_2 - \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+32\lambda+8}\begin{pmatrix} m;\, 10\lambda+5;\, q \\ 10\lambda+5 \end{pmatrix}_2, \end{aligned}$$

(6.18)

$$\begin{aligned} &\frac{1}{2}(H_{2m}(q^{1/2}) + H_{2m}(-q^{1/2}) \\ &\quad = \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+4\lambda}\begin{pmatrix} m;\, 10\lambda+1;\, q \\ 10\lambda+1 \end{pmatrix}_2 - \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+16\lambda+2}\begin{pmatrix} m;\, 10\lambda+3;\, q \\ 10\lambda+3 \end{pmatrix}_2, \end{aligned}$$

$$(6.19)\quad \begin{aligned}\frac{q^{-1/2}}{2}(H_{2m}(q^{1/2}) - H_{2m}(-q^{1/2})) &= \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+14\lambda+1}\begin{pmatrix} m;\ 10\lambda+2;\ q \\ 10\lambda+2 \end{pmatrix}_2 \\ &- \sum_{\lambda=-\infty}^{\infty} q^{30\lambda^2+26\lambda+5}\begin{pmatrix} m;\ 10\lambda+4;\ q \\ 10\lambda+4 \end{pmatrix}_2 .\end{aligned}$$

Proof. These four identities arise by replacing q by q^{-1} in (6.10) and (6.11) and then extracting even functions and odd functions. □

Note that by (6.4) and (6.5) we see that Corollary 6.2 provides q-analogs of (2.7), (2.6), (2.4), and (2.5) respectively.

Finally we note that if $m \to \infty$ in (6.10), then we obtain (using (4.16) and Jacobi's triple product [2, p. 21])

$$(6.20)\quad \sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_{2j}} = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{20n-12})(1-q^{20n-8})(1-q^{20n})}{(1-q^{2n})},$$

which is Slater's equation (79) [14, p. 160]. If $m \to \infty$ in (6.11) we obtain (using (4.16) and Jacobi's triple product [2, p. 21])

$$(6.21)\quad \sum_{j=0}^{\infty} \frac{q^{j^2+2j}}{(q)_{2j+1}} = \prod_{n=1}^{\infty} \frac{(1+q^{2n-1})(1-q^{20n-4})(1-q^{20n-16})(1-q^{20n})}{(1-q^{2n})},$$

which is equivalent to Slater's equation (96) [14, p. 162].

To conclude this section we replace q by q^2 in both (6.16) and (6.17), and then multiply the second resulting equation by q and add it to the first; taking the limit as $m \to \infty$ we obtain

$$(6.22)\quad \begin{aligned}\sum_{j=0}^{\infty} \frac{q^{j^2}}{(q)_j} = \frac{1}{(q^2;q^2)_\infty}\Bigg\{ &\sum_{\lambda=-\infty}^{\infty} (q^{60\lambda^2-4\lambda} - q^{60\lambda^2+44\lambda+8}) \\ &+ q\sum_{\lambda=-\infty}^{\infty} (q^{60\lambda^2+16\lambda} - q^{60\lambda^2+64\lambda+16})\Bigg\}.\end{aligned}$$

This is the Rogers-Ramanujan series [2, p. 104]; however the right-hand side is split into even odd parts with each part a simple theta function quotient. This dissection is related clearly to Watson's representation of this series wherein even and odd parts can be readily discerned [16, p. 64, line 4].

Exactly the same treatment of (6.18) and (6.19) yields for the second Roger-Ramanujan series:

$$\sum_{j=0}^{\infty}\frac{q^{j^2+j}}{(q)_j}=\frac{1}{(q^2;q^2)_\infty}\left\{\sum_{\lambda=-\infty}^{\infty}(q^{60\lambda^2+8\lambda}-q^{60\lambda^2+32\lambda+4})\right.$$
$$\left.+q\sum_{\lambda=-\infty}^{\infty}(q^{60\lambda^2+28\lambda+2}-q^{60\lambda^2+52\lambda+10})\right\}. \tag{6.23}$$

7. Conclusion

Having unravelled Euler's "remarkable example of misleading induction", we are left with deeper mysteries. For example, what is a combinatorial explanation of Theorems 5.1 or 6.1, or Corollary 6.2? Surely before we can answer such a question we must find a full combinatorial explanation of the q-trinomial coefficients themselves. Standard partition-theoretic arguments [2, Chapter 2] show that

$$\binom{m;\,0;\,q}{0}=\sum_{j\geq 0}q^{j^2}\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}n-j\\ j\end{bmatrix} \tag{7.1}$$

is the generating function for all partitions in which the largest part is $\leq n$ and the number of parts plus the edge of the Durfee square is also $\leq n$. This interpretation can easily be generalized to $\binom{m;A;q}{A}_2$; however, it has yet to reveal anything significant about the various surprising identities involving q-trinomial coefficients. Indeed the combinatorics of the identities in §4 is still unknown.

References

1. G. E. Andrews, *Sieves in the theory of partitions*, Amer. J. Math. **94** (1972), 1214–1230.
2. ——, *The theory of partitions*, Encyclopedia of Mathematics and Its Applications (G.-C. Rota, ed.), vol. 2, Addison-Wesley, Reading, MA, 1976; reissued by Cambridge Univ. Press, London and New York, 1985.
3. ——, *The hard-hexagon model and Rogers-Ramanujan type identities*, Proc. Nat. Acad. Sci. U.S.A. **78** (1981), 5290–5292.
4. ——, *Uses and extensions of Frobenius' representation of partitions*, Enumeration and Design (D. M. Jackson and S. A. Vanstone, eds.), Academic Press, 1984, pp. 51–65.
5. G. E. Andrews and R. J. Baxter, *Lattice gas generalization of the hard hexagon model*. III. *q-trinomial coefficients*, J. Statist. Phys. **47** (1987), 297–330.
6. ——, *SCRATCHPAD explorations for elliptic theta functions* (Proc. 1986 Stanford Computers and Mathematics Conf.), Computers in Mathematics (D. Chudnovsky and R. Jenks, eds.), Marcel Dekker, New York (to appear).
7. G. E. Andrews, R. J. Baxter, D. M. Bressoud, W. H. Burge, P. J. Forrester, and G. Viennot, *Partitions with prescribed hook differences*, European J. Combin. **8** (1987), 341–350.
8. G. E. Andrews, R. J. Baxter, and P. J. Forrester, *Eight-vertex SOS model and generalized Rogers-Ramanujan-type identities*, J. Statist. Phys. **35** (1984), 193–266.
9. D. M. Bressoud, *Extension of the partition sieve*, J. Number Theory **12** (1980), 87–100.

10. W. H. Burge, *A correspondence between partitions related to generalizations of the Rogers-Ramanujan identities*, Discrete Math. **34** (1981), 9–15.

11. L. Comtet, *Advanced combinatorics*, Reidel, Dordrecht and Boston, 1974.

12. L. Euler, *Observations analyticae*, Novi Commentarii Academiae Scientarum Petropolitanae **11** (1765), 124–143; also in *Opera Omnia*, Series 1, vol. 15, Teubner, pp. 50–69.

13. I. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur theorie der Kettenbrüche*, S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl., 1917, pp. 302–321; reprinted in *Gesammelte Abhandlungen*, vol. 2, Springer, Berlin, 1973, pp. 117–136.

14. L. J. Slater, *Further identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. (2) **54** (1952), 147–167.

15. M. Spiegel, *Finite differences and difference equations*, McGraw-Hill, New York, 1971.

16. G. N. Watson, *Proof of certain identities in combinatory analysis*, J. Indian Math. Soc. **20** (1934), 57–69.

ABSTRACT. The trinomial coefficients are defined centrally by $\sum_{j=-m}^{\infty} \binom{m}{j}_2 x^j = (1+x+x^{-1})^m$. Euler observed that for $-1 \le m \le 7$, $3\binom{m+1}{0}_2 - \binom{m+2}{0}_2 = F_m(F_m+1)$, where F_m is the mth Fibonacci number. The assertion is false for $m > 7$. We prove general identities—one of which reduces to Euler's assertion for $m \le 7$. Our main object is to analyze q-analogs extending Euler's observation. Among other things we are led to finite versions of dissections of the Rogers-Ramanujan identities into even and odd parts.

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16802