International Journal of Mathematics and Mathematical Sciences
Volume 2012 (2012), Article ID 463659, 12 pages
http://dx.doi.org/10.1155/2012/463659

Research Article

Bernoulli Basis and the Product of Several Bernoulli Polynomials

Dae San Kim[1] and Taekyun Kim[2]

[1]Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
[2]Division of General Education, Kwangwoon University, Seoul 139-701, Republic of Korea

Received 25 June 2012; Accepted 9 August 2012

Academic Editor: Yilmaz Simsek

Abstract

We develop methods for computing the product of several Bernoulli and Euler polynomials by using Bernoulli basis for the vector space of polynomials of degree less than or equal to n.

1. Introduction

It is well known that, the nth Bernoulli and Euler numbers are defined by

$$\sum_{l=0}^{n}\binom{n}{l}B_l - B_n = \delta_{1,n}, \qquad \sum_{l=0}^{n}\binom{n}{l}E_l + E_n = 2\delta_{0,n}, \tag{1.1}$$

where $B_0 = E_0 = 1$ and $\delta_{k,n}$ is the Kronecker symbol (see [1–20]).

The Bernoulli and Euler polynomials are also defined by

$$B_n(x) = \sum_{l=0}^{n}\binom{n}{l}B_{n-l}x^l, \qquad E_n(x) = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}x^l. \tag{1.2}$$

Note that $\{B_0(x), B_1(1), \dots, B_n(x)\}$ forms a basis for the space $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg\, p(x) \le n\}$.

So, for a given $p(x) \in \mathbb{P}_n$, we can write

$$p(x) = \sum_{k=0}^{n} a_k B_k(x), \tag{1.3}$$

(see [8–12]) for uniquely determined $a_k \in \mathbb{Q}$.

Further,

$$a_k = \frac{1}{k!}\left\{p^{(k-1)}(1) - p^{(k-1)}(0)\right\}, \quad \text{where } p^{(k)}(x) = \frac{d^k p(x)}{dx^k},$$
$$a_0 = \int_0^1 p(t)\,dt, \quad \text{where } k = 1, 2, \dots, n. \tag{1.4}$$

Probably, $\{1, x, \dots, x^n\}$ is the most natural basis for the space $\mathbb{P}_n$. But $\{B_0(x), B_1(x), \dots, B_n(x)\}$ is also a good basis for the space $\mathbb{P}_n$, for our purpose of arithmetical and combinatorial applications.

What are common to $B_n(x)$, $E_n(x)$, x^n? A few proportion common to them are as follows:

(i) they are all monic polynomials of degree n with rational coefficients;
(ii) $(B_n(x))' = nB_{n-1}(x)$, $(E_n(x))' = nE_{n-1}(x)$, $(x^n)' = nx^{n-1}$;
(iii) $\int B_n(x)dx = B_{n+1}(x)/(n+1) + c$, $\int E_n(x)dx = E_{n+1}(x)/(n+1) + c$, $\int x^n dx = x^{n+1}/(n+1) + c$.

In [5, 6], Carlitz introduced the identities of the product of several Bernoulli polynomials:

$$B_m(x)\,B_n(x) = \sum_{r=0}^{\infty}\left\{\binom{m}{2r}n + \binom{n}{2r}m\right\}\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1} \times \frac{m!\,n!}{(m+n)!}B_{m+n} \quad (m+n \geq 2),$$

$$\begin{aligned}\int_0^1 B_m(x)\,B_n(x)\,B_p(x)\,B_q(x)\,dx &= (-1)^{m+n+p+q}\sum_{r,s=0}^{\infty}\left\{\binom{m}{2r}n + \binom{n}{2r}m\right\}\left\{\binom{p}{2s}q + \binom{q}{2s}p\right\} \\ &\times \frac{(m+n-2r-1)!\,(p+q-2s-1)!}{(m+n+p+q-2r-2s)!}B_rB_sB_{m+n+p+q-2r-2} \\ &+ (-1)^{m+p}\frac{m!\,n!}{(m+n)!}\frac{p!\,q!}{(p+q)!}B_{m+n}B_{p+q}.\end{aligned} \tag{1.5}$$

In this paper, we will use (1.4) to derive the identities of the product of several Bernoulli and Euler polynomials.

2. The Product of Several Bernoulli and Euler Polynomials

Let us consider the following polynomials of degree n:

$$p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)\,E_{j_1}(x)\cdots E_{j_s}(x), \tag{2.1}$$

where the sum runs over all nonnegative integers $i_1, \ldots, i_r,\ j_1, \ldots j_s$ satisfying $i_1 + \cdots + i_r + j_1 + \cdots + j_s = n,\ r + s = 1, r, s \geq 0$.

Thus, from (2.1), we have

$$\begin{aligned}p^{(k)}(x) =\ &(n+r+s-1)(n+r+s-2)\cdots(n+r+s-k) \\ &\times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}^{\infty} B_{i_1}(x)\cdots B_{i_r}(x)\,E_{j_1}(x)\cdots E_{j_s}(x).\end{aligned} \tag{2.2}$$

For $k = 1, 2, \ldots, n$, by (1.4), we get

$$\begin{aligned}a_k &= \frac{1}{k!}\left\{p^{(k-1)}(1) - p^{(k-1)}(0)\right\} \\ &= \frac{\binom{n+r+s}{k}}{n+r+s}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}\left\{B_{i_1}(1)\cdots B_{i_r}(1)\,E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} \\ &= \frac{\binom{n+r+s}{k}}{n+r+s}\left\{\sum_{\substack{0\leq a\leq r \\ 0\leq c\leq s \\ k+r-n-1\leq a\leq r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\right. \\ &\times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n+a+1-k-r}^{\infty} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \\ &\left. - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}.\end{aligned} \tag{2.3}$$

From (2.3), we note that

$$\begin{aligned}a_n &= \frac{\binom{n+r+s}{n}}{n+r+s}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1}\left\{B_{i_1}(1)\cdots B_{i_r}(1)\,E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} \\ &= \frac{\binom{n+r+s}{n}}{n+r+s}\left\{\left(-\frac{1}{2}+1\right)r - \left(-\frac{1}{2}\right)s - \left(-\frac{1}{2}\right)(r+s)\right\} \\ &= \frac{\binom{n+r+s}{n}}{n+r+s}(r+s) = \binom{n+r+s-1}{n},\end{aligned}$$

$$\begin{aligned} a_{n-1} &= \frac{1}{n+r+s}\binom{n+r+s}{n-1} \\ &\quad \times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=2} \left\{ B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s} \right\} \\ &= \frac{1}{n+r+s}\binom{n+r+s}{n-1}\left\{ \frac{1}{6}r + \frac{1}{2}\frac{1}{2}\binom{r+s}{2} - \frac{1}{6}r - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\binom{r+s}{2} \right\} = 0, \\ a_0 &= \int_0^1 p(t)\,dt \\ &= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}^{\infty} \sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s} \binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s} \\ &\quad \times \frac{B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}}{l_1+\cdots+l_r+p_1+\cdots+p_s+1}. \end{aligned} \tag{2.4}$$

Therefore, by (1.3), (2.1), (2.3), and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}$ with $n \geq 2$, we have*

$$\begin{aligned} &\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x) \\ &= \frac{1}{n+r+s}\sum_{k=1}^{n-2}\binom{n+r+s}{k} \\ &\quad \times \Bigg\{ \sum_{\substack{0\leq a\leq r \\ 0\leq c\leq s \\ k+r-n-1\leq a\leq r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a+1-k-r} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \\ &\qquad - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}^{\infty} B_{i_1}\cdots B_{i_r}E_{k_1}\cdots E_{j_s} \Bigg\} B_k(x) + \binom{n+r+s-1}{n} B_n(x) \\ &\quad + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} \sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s} \binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s} \\ &\quad \times \frac{B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}}{l_1+\cdots+l_r+p_1+\cdots p_s+1}. \end{aligned} \tag{2.5}$$

Let us take the polynomial $p(x)$ of degree n as follows:

$$p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t, \tag{2.6}$$

Then, from (2.6), we have

$$\begin{aligned} p^{(k)}(x) = &\ (n+r+s)(n+r+s-1)\cdots(n+r+s-k+1) \\ &\times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t, \end{aligned} \tag{2.7}$$

By (1.4) and (2.7), we get, for $k = 1, 2, \ldots, n$,

$$\begin{aligned} a_k &= \frac{1}{k!}\left\{ p^{(k-1)}(1) - p^{(k-1)}(0) \right\} \\ &= \frac{1}{n+r+s+1}\binom{n+r+s+1}{k} \\ &\quad \times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k+1} \left\{ B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t \right\} \end{aligned}$$

$$= \frac{\binom{\ \ }{k}}{n+r+s+1}\left\{\sum_{\substack{0\le a\le r\\0\le c\le s\\k+r-n-1\le a\le r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\right.$$
$$\times\sum_{t=0}^{n+a+1-k-r}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}$$
$$\left.-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}, \tag{2.8}$$

Now, we look at a_n and a_{n-1}.

$$a_n = \frac{\binom{n+r+s+1}{n}}{n+r+s+1}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=1}\left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)-B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\}$$
$$= \frac{\binom{n+r+s+1}{n}}{n+r+s+1}\left\{\frac{1}{2}(r+s)+1-\left(-\frac{1}{2}\right)(r+s)\right\}$$
$$= \frac{r+s+1}{n+r+s+1}\binom{n+r+s+1}{n}=\binom{n+r+s}{n},$$
$$a_{n-1} = \frac{\binom{n+r+s+1}{n-1}}{n+r+s+1}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=2}\left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)-B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\} \tag{2.9}$$
$$= \frac{\binom{n+r+s+1}{n-1}}{n+r+s+1}\left\{\frac{1}{6}r+1+\frac{1}{2}\frac{1}{2}\binom{r+s}{2}+\frac{1}{2}(r+s)-\frac{1}{6}r-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\binom{r+s}{2}\right\}$$
$$= \frac{1}{n+r+s+1}\binom{n+r+s+1}{n-1}\frac{r+s+2}{2}=\frac{1}{2}\binom{n+r+s}{n-1},$$

From (2.6), we note that

$$a_0 = \int_0^1 p(t)\,dt$$
$$= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}\sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s}\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s} \tag{2.10}$$
$$\times B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}\frac{1}{l_1+\cdots+l_r+p_1+\cdots p_s+t+1}.$$

Therefore, by (1.3), (2.6), (2.8), (2.9), and (2.10), we obtain the following theorem.

Theorem 2.2. *For $n\in\mathbb{N}$ with $n\ge 2$, one has*

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t$$
$$= \frac{1}{n+r+s+1}\sum_{k=1}^{n-2}\binom{n+r+s+1}{k}$$
$$\times\left\{\sum_{\substack{0\le a\le r\\0\le c\le s\\k+r-n-1\le a\le r}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\sum_{t=0}^{n+a+1-k-r}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\right.$$
$$\left.-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}B_k(x) \tag{2.11}$$
$$+\frac{1}{2}\binom{n+r+s}{n-1}B_{n-1}(x)+\binom{n+r+s}{n}B_n(x)$$

$$+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}\sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s}\left\{\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s}\right.$$
$$\times B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}\cdots E_{j_s-p_s}$$
$$\left.\times\frac{1}{l_1+\cdots+l_r+p_1+\cdots p_s+t+1}\right\}.$$

Consider the following polynomial of degree n:

$$p(x)=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}^{\infty}\frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x). \tag{2.12}$$

Then, from (2.12), one has

$$p^{(k)}(x)=(r+s)^k\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!}. \tag{2.13}$$

By (1.4) and (2.13), one gets, for $k=1,2,\ldots,n$,

$$\begin{aligned}
a_k&=\frac{1}{k!}\left\{p^{(k-1)}(1)-p^{(k-1)}(0)\right\}\\
&=\frac{(r+s)^{k-1}}{k!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k+1}\left\{\frac{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)-B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!}\right\}\\
&=\frac{(r+s)^{k-1}}{k!}\left\{\sum_{\substack{0\le a\le r\\0\le c\le s\\k+r-n-1\le a\le r}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a+1-k-r}\frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\,i_2!\cdots i_a!\,j_1!\cdots j_c!}\right.\\
&\qquad\left.-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}\frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}.
\end{aligned} \tag{2.14}$$

Now look at a_n and a_{n-1}:

$$\begin{aligned}
a_n&=\frac{(r+s)^{n-1}}{n!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1}\left\{\frac{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)-B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!}\right\}\\
&=\frac{(r+s)^{n-1}}{n!}\left\{\frac{1}{2}(r+s)-\left(-\frac{1}{2}\right)(r+s)\right\}=\frac{(r+s)^n}{n!},\\
a_{n-1}&=\frac{(r+s)^{n-2}}{(n-1)!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=2}\left\{\frac{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)-B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!}\right\}\\
&=\frac{(r+s)^{n-2}}{(n-1)!}\left\{\frac{1}{2}\frac{1}{6}r+\frac{1}{2}\frac{1}{2}\binom{r+s}{2}-\frac{1}{2}\frac{1}{6}r-\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\binom{r+s}{2}\right\}=0.
\end{aligned} \tag{2.15}$$

It is easy to show that

$$\begin{aligned}
a_0&=\int_0^1p(t)\,dt=\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}\frac{1}{i_1!\cdots i_r!\,j_1!\cdots j_s!}\\
&\times\sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s}\left\{\frac{B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}}{l_1+\cdots+l_r+p_1+\cdots p_s+1}\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s}\right\}.
\end{aligned} \tag{2.16}$$

Therefore, by (1.3), (2.14), and (2.15), one obtains the following theorem.

Theorem 2.3. *For $n\in\mathbb{N}$ with $n\ge 2$, one has*

$$\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}\frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)$$

$$=\sum^{n-2}\frac{(r+s)^{k-1}}{}\left\{\sum\binom{r}{}\binom{s}{}(-1)^c2^{s-c}\right.$$

$$\sum_{k=1} \quad k! \quad \left\{ \sum_{\substack{0\le a\le r\\ 0\le c\le s\\ k+r-n-1\le a\le r}} \binom{}{a}\binom{}{c} \right.$$

$$\times \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a+1-k-r} \frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\,i_2!\cdots i_a!\,j_1!\cdots j_c!}$$

$$- \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1} \frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!} \qquad (2.17)$$

$$\left.\times B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} B_k(x) + \frac{(r+s)^n}{n!}B_n(x)$$

$$+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}\sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s}\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s}$$

$$\times \frac{B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!\,\left(l_1+\cdots+l_r+p_1+\cdots p_s+1\right)}.$$

Take the polynomial $p(x)$ of degree n as follows:

$$p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} \frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!\,t!} B_{i_1}(x)\cdots B_{i_r}(x)\,E_{j_1}(x)\cdots E_{j_s}(x)\,x^t. \qquad (2.18)$$

Then, from (2.18), one gets

$$p^{(k)}(x) = (r+s+1)^k$$
$$\times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k} \frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!\,t!} B_{i_1}(x)\cdots B_{i_r}(x)\,E_{j_1}(x)\cdots E_{j_s}(x)\,x^t. \qquad (2.19)$$

By (1.4) and (2.19), one gets, for $k = 1,\ldots,n$,

$$a_k = \frac{1}{k!}\left\{p^{(k-1)}(1) - p^{(k-1)}(0)\right\}$$
$$= \frac{(r+s+1)^{k-1}}{k!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k+1}\frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!\,t!}$$
$$\times\left\{B_{i_1}(1)\cdots B_{i_r}(1)\,E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\}$$
$$= \frac{(r+s+1)^{k-1}}{k!}\left\{\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ k+r-n-1\le a\le r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\sum_{t=0}^{n+a+1-k-r}\frac{1}{(n+a+1-k-r-t)!}\right. \qquad (2.20)$$
$$\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t}\frac{1}{i_1!\,i_2!\cdots i_a!\,j_1!\cdots j_c!}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}$$
$$\left. - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}.$$

Now look at a_n and a_{n-1}:

$$a_n = \frac{(r+s+1)^{n-1}}{n!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=1}\frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!\,t!}$$
$$\times\left\{B_{i_1}(1)\cdots B_{i_r}(1)\,E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\}$$
$$= \frac{(r+s+1)^{n-1}}{n!}\left\{\frac{1}{2}(r+s)+1-\left(-\frac{1}{2}\right)(r+s)\right\}$$
$$= \frac{(r+s+1)^{n-1}}{n!}(r+s+1) = \frac{(r+s+1)^n}{n!},$$
$$a_{n-1} = \frac{(r+s+1)^{n-2}}{(n-1)!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=2}\frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!\,t!} \qquad (2.21)$$
$$\times\left\{B_{i_1}(1)\cdots B_{i_r}(1)\,E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\}$$

$$
\begin{aligned}
&= \frac{(r+s+1)^{n-2}}{(n-1)!}\left\{\frac{1}{2}\frac{1}{6}r+\frac{1}{2}+\frac{1}{2}\frac{1}{2}\binom{r+s}{2}+\frac{1}{2}(r+s)-\frac{1}{2}\frac{1}{6}r-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\binom{r+s}{2}\right\} \\
&= \frac{(r+s+1)^{n-2}}{(n-1)!}\frac{r+s+1}{2} \\
&= \frac{(r+s+1)^{n-1}}{2(n-1)!}.
\end{aligned}
$$

From (2.18), one can derive the following identity:

$$
\begin{aligned}
a_0 &= \int_0^1 p(t)\,dt \\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} \frac{1}{i_1!\cdots i_r!\,j_1!\cdots j_s!\,t!}\int_0^1 B_{i_1}(x)\cdots B_{i_r}(x)\,E_{j_1}(x)\cdots E_{j_s}(x)\,x^t\,dt \\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} \frac{1}{i_1!\cdots i_r!\,j_1!\cdots j_s!\,t!}\sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots \\
&\quad \times \sum_{p_s=0}^{j_s}\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s} B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}\frac{1}{l_1+\cdots+l_r+p_1+\cdots p_s+t+1}.
\end{aligned}
\tag{2.22}
$$

Therefore, by (1.3), (2.20), (2.21), and (2.22), one obtains the following theorem.

Theorem 2.4. *For* $n\in\mathbb{N}$ *with* $n\geq 2$, *one has*

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}\frac{1}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!\,t!}B_{i_1}(x)\cdots B_{i_r}(x)\,E_{j_1}(x)\cdots E_{j_s}(x)\,x^t \\
&= \sum_{k=1}^{n-2}\frac{(r+s+1)^{k-1}}{k!}\left\{\sum_{\substack{0\leq a\leq r\\ 0\leq c\leq s\\ k+r-n-1\leq a\leq r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\sum_{t=0}^{n+a+1-k-r}\frac{1}{(n+a+1-k-r-t)!}\right. \\
&\qquad \times \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t}\frac{1}{i_1!\,i_2!\cdots i_a!\,j_1!\cdots j_c!}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \\
&\qquad \left. -\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\,i_2!\cdots i_r!\,j_1!\cdots j_s!}\right\}B_k(x) \\
&\quad + \frac{(r+s+1)^{n-1}}{2(n-1)!}B_{n-1}(x) + (r+s+1)^n
\end{aligned}
$$