# WYTHOFF PAIRS

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## 1. INTRODUCTION

The author had been working on the safe combinations (Wythoff pairs) in Wythoff's game [11] when the researches of Silber [9, 10] came to his attention. As the two approaches differ somewhat, it is probably worthwhile to indicate briefly the author's alternative treatment, which may throw a little light on the general problem.

Both Silber and the author use the fundamental idea of the canonical Fibonacci representation of an integer. While much work has been done recently on Fibonacci representation theory and on Nim-related games, we will attempt to minimize our reference list.

Wythoff pairs have been analyzed in detail by Carlitz, Scoville and Hoggatt, e.g., in [3, 4], though without specific reference to Wythoff's game. For a better understanding of the principles used in our reasoning which follows, it is desirable to present a description of the nature and strategy of Wythoff's game.

## 2. WYTHOFF'S GAME

*Wythoff's game* was first investigated by W. A. Wythoff [11] in 1907. It is similar to Nim (see Bouton [2]) and may be described thus (Ball [1]):

Unspecified numbers of counters occur in each of two heaps. In each draw, a player may freely choose counters from either (i) one heap, or (ii) two heaps, provided that in this case he *must* take the same number from each.

For example, heaps of 1 and 2 can be reduced to 0 and 2, or 1 and 1, or 1 and 0. The player who takes the last counter wins the game.

As Coxeter [5] remarks: "An experienced player, playing against a novice, can nearly always win by remembering which pairs of numbers are "safe combinations": safe for him to leave on the table with the knowledge that, if he does not make any mistake later on, he is sure to win. (If both players know the safe combinations, the outcome depends on whether the initial heaps form a safe or unsafe combination.)"

The safe combinations for Wythoff's game are known to be the pairs:

(1)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | $n = 1$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $\cdots$ |
| (1) | (1,2), | (3,5), | (4,7), | (6,10), | (8,13), | (9,15), | (11,18), | (12,20), | $\cdots$ |

A safe pair may also be called a *Wythoff pair.*

There are several interesting things about the integers occurring in these safe combinations. They are:

(I) Members of the first pair of integers differ by 1, of the second pair by 2, of the third by 3, $\cdots$, of the $n^{th}$ pair by $n$.

(II) The $n^{th}$ pair is $(a_n, b_n) \equiv ([n\alpha], [n\alpha^2])$, where the symbol $[x]$ denotes the greatest integer which is less than, or equal to, $x$, and $\alpha = (1+\sqrt{5})/2 \doteq 1.618$ (so that $\alpha^2 = (3+\sqrt{5})/2 \doteq 2.618$). We recognize $\alpha$ as the "golden section" number which is a root of $x^2 - x - 1 = 0$ (i.e., $\alpha^2 = \alpha + 1$). Note that $b_n = a_n + n$, i.e., $[n\alpha^2] = [n\alpha] + n$.

(III) In the list of integers occurring in the ordered pairs for safe combinations, each integer appears exactly once (i.e., every interval between two consecutive positive integers contains just one multiple of either $\alpha$ or $\alpha^2$, as Ball [1] observes).

(IV) In every pair of a safe combination, the smaller number is the smallest integer not already used and the larger number is chosen so that the difference in the $n^{th}$ pair is $n$.

It might reasonably be asked: How does the "golden section" number $\alpha$ come into the solution of Wythoff's game? The answer is detailed in Coxeter [5] where the solution given by Hyslop (Glasgow) and Ostrowski (Göttingen) in 1927, in response to a problem proposed by Beatty (Toronto) a year earlier, is reproduced. (Coxeter notes that Wythoff himself obtained his solution "out of a hat" without any mathematical justification). Basically, the answer to our query, as given by Hyslop and Ostrowski quoted in [5], depends on the occurrence of the equations $(1/x) + (1/y) = 1$, $y = x + 1$ which, when $y$ is eliminated, yield our quadratic equation $x^2 - x - 1 = 0$.

That the Wythoff pairs are ultimately connected with Fibonacci numbers should not surprise us since, by Hoggatt [6], the $n^{th}$ Fibonacci number

$$F_n = \left[\frac{\alpha^n}{\sqrt{5}} + \frac{1}{2}\right]$$

for $n = 1, 2, 3, \cdots$, i.e., both Wythoff pairs and Fibonacci numbers involve $[x]$. (See (II) above.)

The first forty Wythoff pairs are listed in Silber [9]. From the rules of construction (I)–(IV) it is only a matter of patience for the interested reader to form as long a list of Wythoff pairs as is desired.

## 3. WYTHOFF PAIRS AS MEMBERS OF $\{H_m(p,q)\}$

Consider the generalized Fibonacci sequence $\{H_m(p,q)\}$ of integers (Horadam [7]):

(2)
$$\begin{matrix} H_0 & H_1 & H_2 & H_3 & H_4 & H_5 & H_6 & H_7 & \cdots \\ q & p & p+q & 2p+q & 3p+2q & 5p+3q & 8p+5q & 13p+8q & \cdots \end{matrix}$$

where

(3)
$$H_m = H_{m-1} + H_{m-2} \qquad (m \geq 2)$$

in which we omit $p,q$ when there is no possible ambiguity. The restriction $m \geq 2$ in (3) may be removed, if desired, to allow for negative subscripts.

The ordinary Fibonacci sequence $\{F_m\}$ with $F_0 = 0$, $F_1 = F_2 = 1$ occurs when $p = 1$, $q = 0$, i.e.,

(4)
$$F_m = H_m(1,0).$$

It is known [7] that

(5)
$$H_m = pF_m + qF_{m-1}.$$

Every positive integer $N$

(a) produces a $H_1$ $(= [N\alpha] = p)$ which is the first member of a Wythoff pair, i.e., sequences $\{H_m([N\alpha], N)\}$ yield all the Wythoff pairs; and

(b) is, by (IV), a member of a Wythoff pair and a member of some $H$-sequence (in fact, of infinitely many $H$-sequences of which the given $H$-sequence forms a part),

e.g., $52 = H_1(52, 32) = H_2(32, 20) = H_3(20, 12) = H_4(12, 8) = H_5(8, 4) = H_6(4, 4) = H_7(4, 0) = \cdots$

with infinite extension through negative values of $m$ if the restriction $m \geq 2$ in (3) is removed.

⌈Every positive integer $N$ is obviously also a member of infinitely many non-Wythoff pairs belonging to infinitely many different $H$-sequences, e.g.,

$$N = 2000 = 20F_{11} + 4F_{10} \; (= H_{11}(20,4)) = 20 \times 89 + 4 \times 55$$

is a member of all the $H$-sequences resulting from the solution, by Euclid's algorithm or by congruence methods, of the Diophantine equation $89x + 55y = 2000$. Some instances of this are

$$2000 = H_{11}(75, -85) = H_{11}(-35, 93)$$

yielding the non-Wythoff pairs (1235,2000) and (1237,2000) whereas (1236,2000) is a Wythoff pair $(\equiv (a_{764}, b_{764}))$.⌋

Now make the identification with the notation in [9]:

(6)
$$a_n = H_m, \quad b_n = H_{m+1}, \quad a_{b_n} = H_{m+2} \quad \text{for some } p,q.$$

For example, $n = 6$ yields the Wythoff pair $(a_6, b_6) \equiv (9, 15) \equiv (H_3, H_4)$ for $p = 3$, $q = 3$.

To save space, we will assume the results in [9] expressed in our notation:

*Theorem.* All pairs in an $H$-sequence after a Wythoff pair are Wythoff pairs, i.e., each Wythoff pair generates a sequence of Wythoff pairs.

*Theorem.* A Wythoff pair $(H_m, H_{m+1})$ is primitive if and only if (for $H_m = a_n$) $n = a_k$ for some positive integer $k$, and for some positive integers $p, q$.

A *primitive* Wythoff pair is a Wythoff pair which is not generated by any other Wythoff pairs. Thus, (1,2), (4.7), (6,10), (9,15), (88,143) are primitive Wythoff pairs.

## 4. ZECKENDORF'S CANONICAL REPRESENTATION

Zeckendorf's Theorem, quoted in Lekkerkerker [8], states: *(Zeckendorf's Theorem)* Every positive integer $N$ can be represented as the sum of distinct Fibonacci numbers, using no two consecutive numbers, and such a representation is unique.

Symbolically, this *canonical (Zeckendorf)* representation of $N$ is

(7) $$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r},$$

where

(8) $$k_1 > k_2 > \cdots > k_r \geqslant 2 \text{ ($r$ depending on $N$)}$$

and

(9) $$k_j - k_{j+1} \geqslant 2 \qquad (j = 1, 2, \cdots, r-1).$$

From [9], the criteria for a Wythoff pair $(H_m, H_{m+1})$ are that, in the canonical representation (7), with (8) and (9),

A.

(i) $$\begin{cases} H_m = F_{k_1} + F_{k_2} + \cdots + F_{k_r} \\ H_{m+1} = F_{k_1+1} + F_{k_2+1} + \cdots + F_{k_r+1} \end{cases}$$

(ii) $k_r$ is even.

For a primitive Wythoff pair, we have further that

B.

(i) $$k_r = 2$$

(ii) $$n = F_{k_1-z} + F_{k_2-z} + \cdots + F_{k_r-z},$$

where

(10) $$z = k_r - 1 \qquad (k_r - z = 1)$$

so the last term in $n$(B(ii)) is $F_1 = 1$.

*Examples.* (1) Non-Wythoff pair (62, 100)

$62 = F_{10} + F_5 + F_3$ so $k_r = 3$ which is not even, and A(ii) is therefore invalid (though A(i) holds).

(2) Non-Wythoff pair (62, 101)

$101 = F_{11} + F_6 + F_4 + F_1$ so A(i) is invalid (and so is A(ii)).

(3) Non-primitive Wythoff pair (1236, 2000) $\equiv (a_{764}, b_{764})$

$1236 = F_{16} + F_{13} + F_7 + F_4$ so B(i) is invalid (and so is B(ii)), though A(i), A(ii) are valid.

(4) Primitive Wythoff pair (108, 175) $\equiv (a_{67}, b_{67})$

$$108 = F_{11} + F_7 + F_5 + F_2, \qquad 175 = F_{12} + F_8 + F_6 + F_3$$

so A(i), A(ii), B(i) are valid and (175 – 108 =) $67 = F_{10} + F_6 + F_4 + F_1$ so B(ii) holds.

## 5. WYTHOFF PAIRS, ZECKENDORF'S REPRESENTATION AND $\{H_m(p,q)\}$

From (5) and (7) we have, for $k_1 \geqslant m > m - 1 \geqslant k_r$, (11)′ $$N = H_m(p,q) = pF_m + qF_{m-1} = F_{k_1} + F_{k_2} + \cdots + F_{k_r}\ .$$

A little thought reveals that

(12) $$N = H_m(p,q) = H_{m'}(p',q'),$$
where
(13) $$m' = z$$
(14) $$p' = H_{m-z+1}(p,q)$$
(15) $$q' = H_{m-z}(p,q)$$
in which $(p',\ p' + q')$ is a primitive Wythoff pair. That is, the sequence $\{H_{m'}(p',q')\}$ is generated by a primitive Wythoff pair.

The explanation of (12)–(15) is as follows. If, in (12), $p'$ is the first member of a primitive Wythoff pair, then by B(i) its canonical representation must end with $F_2$. Thus, by (11), $p'$ precedes $N$ by $k_r - 2 = z - 1$ places by (10), i.e., $p'$ is located in term position $m - (z - 1) = m - z + 1$ in the $H$-sequence. Hence, we have (14) and consequently (15). Clearly, $m' = (z - 1) + 1 = z$ giving (13).

It is now possible to determine, for any positive integer $N$, exactly which Wythoff pair generates the $H$-sequence in which that $N$ appears, as well as the location of $N$ in that sequence (as is done in [9]).

*Examples.*

(1) $$N = 52 = F_9 + F_7 + F_5$$
(so $z = 4$ by (10))
$$= 4F_7 + 0.\ F_6$$
by repeated use of $F_m = F_{m-1} + F_{m-2}$
$$= H_7(4,0)$$
by (11) (so $m = 7,\ p = 4,\ q = 0$)
$$= H_4(12,8)$$
by (12)–(15) since
$$m' = z = 4, \quad p' = H_4(4,0) = 12, \quad q' = H_3(4,0) = 8.$$

That is, $N = 52$ is the $4^{th}$ term in the sequence $\{H(12,8)\}$ generated by the primitive Wythoff pair $(12,20) \equiv (a_8, b_8)$:

| $H_0$ | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | $H_6$ | $H_7$ | $H_8$ | ··· |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 4 | 4 | 8 | 12 | 20 | 32 | (52) | 84 | ··· |

$\{H(12, 8)\}$ (from $H_4$ on)

$\{H(4, 0)\}$ (whole sequence)

(2) $$N = 1000 = F_{16} + F_7 = H_{11}(10, 2) = H_6(90, 56),$$

i.e., 1000 is the $6^{th}$ term in the sequence generated by the primitive Wythoff pair (90, 146) $\equiv (a_{56}, b_{56})$:

| $H_0$ | $H_1$ | $H_2$ | $H_3$ | $H_4$ | $H_5$ | $H_6$ | $H_7$ | $H_8$ | $H_9$ | $H_{10}$ | $H_{11}$ | ··· |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 10 | 12 | 22 | 34 | 56 | 90 | 146 | 236 | 382 | 618 | (1000) | ··· |

$\{H(90, 56)\}$ (from $H_6$ on)

$\{H(10, 2)\}$ (whole sequence)

Example (1) is given by Silber [9]. Comparing his zero-unit notation with our $H$-notation in relation to canonical representations, we see that our $z$ is a suggestive symbol as it is also the number of zeros at the right-hand end of the zero-unit notation for a canonical representation. Checking that (12, 20) and (90, 146) in the examples above are indeed primitive Wythoff pairs is straightforward.

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**[Continued from page 146.]**

and that $V_n = 2$ satisfies

$$V_{n+2} = 2V_{n+1} - V_n,$$

we can rewrite (1.2) as

$$2\sum_{k=1}^{n} U_k^3 = V_n\left(\sum_{k=1}^{n} U_k\right)^2.$$

This suggests the following result for integer sequences.

*Conjecture.* Let $U_k$, with $U_0 = 0$, $U_1 = 1$, and $V_k$, with $V_0 = 2$, $V_1 = P$, be two solutions of

$$W_{k+2} = PW_{k+1} + QW_k, \qquad k = 0, 1, \cdots,$$

where $P$ and $Q$ are integers with $P \geqslant 2$ and $P + Q \geqslant 1$. We then claim that

$$2\sum_{k=1}^{n} U_k^3 \leqslant V_n\left(\sum_{k=1}^{n} U_k\right)^2 \qquad (n = 1, 2, \cdots). \tag{3.1}$$

**Remarks.** For $P = 2$ and $Q = -1$, (3.1) gives (1.2). Using double induction, one can prove the conjecture for $P + Q \geqslant 3$, which leaves the two cases $P + Q = 2$ and $P + Q = 1$ open.

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