A PROPERTY OF WYTHOFF PAIRS

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The Wythoff pairs A_n and B_n are the ordered safe-pairs in the game. See for example [1].

$$A = \{A_n\} = \{[n\alpha]\} = \{1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, \ldots\}$$

$$B = \{B_n\} = \{[n\alpha^2]\} = \{2, 5, 7, 10, 13, 15, 18, 20, 23, \ldots\}$$

where $\alpha = (1 + \sqrt{5})/2$. $\alpha^2 = \alpha + 1$. The following properties will be assumed:

(i) The sets A and B are disjoint sets whose union is the set of positive integers.

(ii) $B_n = A_n + n$.

Lemma 1: $A_{A_n} + 1 = B_n$.

Proof: Consider the set of integers $1, 2, 3, \ldots, B_n$. Of these, n are B's, and the rest are $A_1, A_2, A_3, \ldots, A_j = B_n - 1$. Thus, $j + n = B_n$, but $A_n + n = B_n$, so that $A_{A_n} + 1 = B_n$.

If we consider the set of integers $1, 2, 3, \ldots, A_n$, there are n A's and $B_1, B_2, \ldots, B_j \le A_n - 1$; thus,

Lemma 2: There are $A_n - n$ B's less than A_n.

Theorem: $A_{A_n+1} - A_{A_n} = 2, \qquad A_{B_n+1} - A_{B_n} = 1;$

$B_{A_n+1} - B_{A_n} = 3, \qquad B_{B_n+1} - B_{B_n} = 2.$

Proof: It is easy to see that no two B's are adjacent. Consider $A_n + 1 = A_{n+1}$ or $A_n + 1 = B_j$, then

$$A_{n+1} - (n + 1) - (A_n - n) = 1 \quad \text{iff} \quad A_n + 1 = B_j.$$

Fix j, then since $A_n + 1$ is a strictly increasing sequence in n, there is at most one solution to $A_n + 1 = B_j$, and from $A_{A_n} + 1 = B_n$, we see $n = A_j$, so

$$A_{A_j+1} - A_{A_j} = 2 \quad \text{and} \quad A_{B_j+1} - A_{B_j} = 1.$$

From $A_n + n = B_n$, it easily follows that

$$B_{A_j+1} - B_{A_j} = 3 \quad \text{and} \quad B_{B_j+1} - B_{B_j} = 2.$$

We now show that $\{A_n\}$ and $\{B_n\}$ are self-generating sequences. We illustrate only with $B_n = [n\alpha^2] = \{2, 5, 7, 10, 13, \ldots\}$: $B_1 = 2$ and $B_2 - B_1 = 3$, so $B_2 = 5$; $B_3 - B_2 = 2$, so $B_3 = 7$; $B_4 - B_3 = 3$, so $B_4 = 10$; $B_5 - B_4 = 3$, so $B_5 = 13$. Now, knowing that

$$B_{n+1} - B_n \text{ is } 2 \text{ if } n \in B \text{ and } B_{n+1} - B_n = 3 \text{ if } n \notin B,$$

we can generate as many terms of the $\{B_n\}$ sequence as one would want only by knowing the earlier terms and which difference to add to these to obtain the next term.

REFERENCE

1. L. Carlitz, Richard Scoville, & V. E. Hoggatt, Jr., "Fibonacci Representations," *The Fibonacci Quarterly*, Vol. 10, No. 1 (January 1972), pp. 1-28.