J. Korean Math. Soc. **39** (2002), No. 5, pp. 801–819

AN ANALOGUE OF WIENER MEASURE AND ITS APPLICATIONS

Man Kyu Im and Kun Sik Ryu

Dedicated to Professor Kun Soo Chang on his sixtieth birthday

Abstract. In this note, we establish a translation theorem in an analogue of Wiener space $(C[0,t],\omega_\varphi)$ and find formulas for the conditional ω_φ-integral given by the condition $X(x)=(x(t_0),x(t_1),\cdots,x(t_n))$ which is the generalization of Chang and Chang's results in 1984. Moreover, we prove a translation theorem for the conditional ω_φ-integral.

1. Introduction

Since the Brownian motion was found by the British botanist Robert Brown in 1827, the theory of this motion was developed extensively and deeply by many scientists including Albert Einstein. Specially, for the probabilistic approach to the theory of Brownian motion, Wiener suggested a measure space $(C_0[0,t],m_w)$ where $C_0[0,t]$ is the space of all continuous functions on a closed interval $[0,t]$ which vanish at origin, the so called Wiener space in 1923 [7]. But, through the Wiener measure theory, one could obtain theories for a single small particle, merely.

Recently, the authors introduced a new definition of an analogue of Wiener measure space $(C[0,t],\omega_\varphi)$ and investigated some theories on many small particles moving along the law of diffusion [6]. In other word, we assume that space is filled by a solvent, that particles reacting with the solvent spreads through it according to the laws of diffusion and that the distribution of the substance at the beginning is a measure φ, our theory tell us what distribution would be at any time afterwards. In

Received March 14, 2002.

2000 Mathematics Subject Classification: 28C20, 28D05, 60A99, 60J65.

Key words and phrases: analogue of Wiener measure, translation theorem, conditional ω_φ-integral.

This paper was supported by the Research Fund in Hannam University in 2001.

this case, if φ is the Dirac measure δ_0, the total mass one concentrated at origin then ω_φ is the Wiener measure.

In the theory of infinite dimensional analysis, translations are difficult problems to deal with and Cameron and Martin proved a translation theorem on the Wiener space $C_0[0,t]$, under some conditions, in [1]. In Section 3, we will treat a translation theorem on the analogue of Wiener space $(C[0,t],\omega_\varphi)$. In [10, 11], Yeh presented the definition and some examples of the conditional Wiener integral and he proved a translation theorem for the conditional Wiener integral. In section 4, we will find formulas for the conditional ω_φ-integral under the condition $X(x) = (x(s_0), x(s_1), \cdots, x(s_n))$ which is the generalization of Chang and Chang's results in [3]. In the last section, we will establish a translation theorem for the conditional ω_φ-integral $X(x) = x(t)$.

2. Preliminaries

In this section, we will introduce some notations, definitions and facts which are needed in the subsequent sections.

(A) Let $\mathbb{R}$ be the real number system and let $\mathbb{C}$ be the complex number system. For a natural number n, let $\mathbb{R}^n$ be the n-times product space of $\mathbb{R}$. Let $\mathcal{B}(\mathbb{R}^n)$ be the set of all Borel measurable subsets of $\mathbb{R}^n$. Let m_L be the Lebesgue measure on $\mathbb{R}$.

(B) For a positive real number t, let $C[0,t]$ be the space of all real-valued continuous functions on a closed bounded interval $[0,t]$ with the supremum norm $\|\cdot\|_\infty$.

(C) Let t be a positive real number and let n be a non-negative integer. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$, let $J_{\vec{t}} : C[0,t] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n)) . \tag{2.1}$$

For B_j $(j = 0, 1, 2, \cdots, n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^n B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all intervals. For a probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we let

$$\begin{aligned} &m_\varphi\Big(J_{\vec{t}}^{-1}\Big(\prod_{j=0}^n B_j\Big)\Big) \\ &= \int_{B_0}\Big[\int_{\prod_{j=1}^n B_j} W(n+1;\vec{t};u_0,u_1,\cdots,u_n) d\prod_{j=1}^n m_L(u_1,\cdots,u_n)\Big] d\varphi(u_0), \end{aligned} \tag{2.2}$$

where

$$W(n+1;\vec{t};u_0,u_1,\cdots,u_n) = \Big(\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi(t_j-t_{j-1})}}\Big)\exp\Big\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\Big\}. \tag{2.3}$$

By [5, Theorem 5.1, p.144] and [5, Theorem 2.1, p.212], $\mathcal{B}(C[0,t])$, the set of all Borel subsets in $C[0,t]$, coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure ω_φ on $(C[0,t],\mathcal{B}(C[0,t]))$ such that $\omega_\varphi(I)=m_\varphi(I)$ for all I in $\mathcal{I}$. This measure ω_φ is called an analogue of Wiener measure associated with the probability measure φ.

By the change of variable formula, we can easily prove the following lemma.

LEMMA 2.1. (The Wiener integration formula) *If $f:\mathbb{R}^{n+1}\to\mathbb{C}$ is a Borel measurable function then the following equality holds.*

$$\begin{aligned}&\int_{C[0,t]} f(x(t_0),x(t_1),\cdots,x(t_n))\,d\omega_\varphi(x)\\ &\overset{*}{=}\int_{\mathbb{R}}\Big[\int_{\mathbb{R}^n} f(u_0,u_1,\cdots,u_n)W(n+1;\vec{t};u_0,u_1,\cdots,u_n)\\ &\quad\times d\prod_{j=1}^{n}m_L((u_1,u_2,\cdots,u_n)\Big]d\varphi(u_0),\end{aligned} \tag{2.4}$$

where $\overset{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

(D) Let φ be a probability measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ and let n be a non-negative integer. Let X be a $\mathbb{R}^{n+1}$-valued measurable function on $(C[0,t],\mathcal{B}(C[0,t]),\omega_\varphi)$. We write P_X for a measure on $(\mathbb{R}^{n+1},\mathcal{B}(\mathbb{R}^{n+1}))$ determined by X, that is,

$$P_X(B) = \omega_\varphi(X^{-1}(B)) \tag{2.5}$$

for B in $\mathcal{B}(\mathbb{R}^{n+1})$. If $X(x)=x(t)$ then

$$P_X(B) = \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}\int_B \exp\Big\{-\frac{(\xi-u_0)^2}{2}\Big\}\,dm_L(\xi)d\varphi(u_0). \tag{2.6}$$

Let Z be an integrable function on $(C[0,t],\mathcal{B}(C[0,t]),\omega_\varphi)$. The conditional ω_φ-integral of Z given X, written $E^{\omega_\varphi}(Z|X)$, is defined to be

any real-valued Borel measurable and P_X-integrable function ψ on $\mathbb{R}^{n+1}$ such that

$$\int_{X^{-1}(H)} Z(x)d\omega_\varphi(x) = \int_H \psi(\xi)dP_X(\xi) \tag{2.7}$$

for H in $\mathcal{B}(\mathbb{R}^{n+1})$. By the Radon-Nikodym theorem, we know that such a function ψ always exists.

According to the similar method as in the proof of Lemma 1 in [11, p.627], we can obtain the following lemma, so we state the lemma without the proof.

LEMMA 2.2. *Let Z be a real-valued integrable function on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$ and let g be a measurable function of $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$ into $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then*

$$\int_{C[0,t]} g(X(x))Z(x)\ d\omega_\varphi(x) \stackrel{*}{=} \int_{\mathbb{R}^{n+1}} g(\xi)E^{\omega_\varphi}(Z|X)(\xi)\ dP_X(\xi), \tag{2.8}$$

where $\stackrel{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

3. A translation theorem on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$ and the Paley-Wiener-Zygmund integral

It is well-known fact that there is no quasi-invariant probability measure on the infinite dimensional vector space [8]. So, there is no quasi-invariant probability measure on $C_0[0,t]$ or $C[0,t]$. In 1944, under the some assumptions, Cameron and Martin established a translation theorem on $(C_0[0,t], m_w)$ in [1]. In this section, we will prove a translation theorem on $(C[0,t], \omega_\varphi)$ under the similar assumptions to Cameron's assumptions. From these concepts, we will show that the Paley-Wiener-Zygmund integral is well-defined $\omega_\varphi - a.e.$.

By the similar method as in the proof of Cameron and Martin's translation theorem on $C_0[0,t]$ in [1], we can prove the following theorem, so we give only the sketch of proof of it.

THEOREM 3.1. (The translation theorem on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$) *Let h be in $C[0,t]$ and of bounded variation. Let α be in $\mathbb{R}$ and let $x_0(s) = \int_0^s h(u)dm_L(u) + \alpha$ for $0 \le s \le t$. Let $L : C[0,t] \to C[0,t]$ be a function with $L(x) = x + x_0$ and let φ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let φ_α be a measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\varphi_\alpha(B) =$*

$\varphi(B+\alpha)$ for B in $\mathcal{B}(\mathbb{R})$. Then if F is ω_φ-integrable then $F(x+x_0)$ is ω_φ-integrable of x and

$$\begin{aligned}(3.1)\qquad &\int_{C[0,t]} F(y)\ d\omega_\varphi(y)\\ &= e^{-\frac{1}{2}\|h\|_2^2}\int_{C[0,t]} F(x+x_0)\ e^{-\int_0^t h(u)\ dx(u)}\ d\omega_{\varphi_\alpha}(x).\end{aligned}$$

Proof. Suppose F is bounded and continuous, and vanishes on $\{y \in C[0,t]|\ \|\ y\ \|_\infty > M\}$ for some real number M. Then $F \circ L$ is ω_φ-integrable and F is bounded by K for some real number K. For a natural number n, we consider two functions $P_n : C[0,t] \to C[0,t]$ and $G_n : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$(3.2)\qquad P_n(y)(t) = \frac{y(t_i)-y(t_{i-1})}{t_i - t_{i-1}}(t-t_{i-1}) + y(t_{i-1})$$

for $t \in [t_{i-1}, t_i]$ and

$$(3.3)\qquad G_n(y) = F(y(t_0), y(t_1), y(t_2), \cdots, y(t_n)),$$

where $t_i = \frac{i}{n}t$ for $i = 0, 1, 2, \cdots, n$. Then by Lemma 2.1 and the change of variable theorem, we have

$$\begin{aligned}(3.4)\qquad &\int_{C[0,t]} F(P_n(y))\ d\omega_\varphi(y)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}^n} G_n(v_0, v_1, \cdots, v_n) W(n+1; \vec{t}; v_0, v_1, \cdots, v_n)\\ &\quad \times d\prod_{j=1}^{n} m_L(v_1, v_2, \cdots, v_n) d\varphi(v_0)\\ &= \exp\Big\{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_0(t_i)-x_0(t_{i-1}))^2}{t_i - t_{i-1}}\Big\}\int_{C[0,t]} F(P_n(x+x_0))\\ &\quad \times \exp\Big\{-\sum_{i=1}^{n}\frac{(x(t_i)-x(t_{i-1}))(x_0(t_i)-x_0(t_{i-1}))}{t_i - t_{i-1}}\Big\}\ d\omega_{\varphi_\alpha}(x),\end{aligned}$$

where $\vec{t} = (t_0, t_1, \cdots, t_n)$. From the mean-value theorem, for each $i = 1, 2, \cdots, n$ there is a real number τ_i in $[t_{i-1}, t_i]$ such that $\frac{x_0(t_i)-x_0(t_{i-1})}{t_i - t_{i-1}} = x_0'(\tau_i) = h(\tau_i)$. Since $\langle P_n \rangle$ converges uniformly to $I_{C[0,t]}$, the identity function, $\langle FP_n(x+x_0)\rangle$ converges to $F(x+x_0)$ for each x in $C[0,t]$. Moreover, if $\|\ P_n(x+x_0)\ \|_\infty \leq M$ then $|x(t_i)| \leq M + |x_0(t_i)| \leq M + \|$

$x_0 \|_\infty$ for $i = 0, 1, \cdots, n$. If F is a non-negative, bounded and continuous function, let

$$M_n(u) = \begin{cases} 1 & \text{if } 0 \le u \le n \\ n+1-u & \text{if } 0 \le u \le n \\ 0 & \text{if } n+1 \le u \end{cases} \tag{3.5}$$

and $F_n(x) = F(x)M_n(\| x_0 \|_\infty)$ for each natural number n. For each n, F_n is bounded, continuous and vanishes on $\| x_0 \|_\infty > M(\| x_0 \|_\infty)$, so the equality (3.1) holds for F_n, and by the monotone convergence theorem, the equality (3.1) holds for a non-negative bounded continuous function. By the properties of integral and $F(x) = \max\{F(x), 0\} - \max\{-F(x), 0\}$, we know that the equality (3.1) holds for a bounded continuous function F. Appling Lusin's theorem, we obtain that the equality (3.1) holds for ω_φ-integrable function F. □

Putting $F \equiv 1$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Under the assumptions in Theorem 3.1,*

$$\int_{C[0,t]} \exp\Big\{ -\int_0^t h(u)\, dx(u) \Big\}\, d\omega_{\varphi_\alpha}(x) \;=\; \exp\Big\{ -\frac{1}{2} \| h \|_2^2 \Big\} \tag{3.6}$$

for any α in $\mathbb{R}$.

Replacing h by λh in the above Corollary 3.2, by the uniqueness theorem for analytic extension in the theory of complex analysis, we have the following corollary.

Corollary 3.3. *Under the assumptions in Theorem 3.1, for all λ in $\mathbb{C}$,*

$$\int_{C[0,t]} \exp\Big\{ -\lambda \int_0^t h(u) dx(u) \Big\}\, d\omega_{\varphi_\alpha}(x) = \exp\Big\{ -\frac{\lambda^2}{2} \| h \|_2^2 \Big\} \tag{3.7}$$

for any α in $\mathbb{R}$.

Theorem 3.4. *Under the assumptions in Theorem 3.1, consider a random variable $X : C[0,t] \to \mathbb{R}$ with $X(x) = \int_0^t h(u)\, dx(u)$. Then X has a normal distribution with the mean zero and the variation $\| h \|_2^2$.*

Proof. By Corollary 3.3, taking $\lambda = i$ and $\alpha = 0$, the Fourier transform of X is given by

$$(3.8) \qquad \begin{aligned} [\mathcal{F}(X)](\xi) &= \int_{C[0,t]} \exp\Big\{ i\xi \int_0^t h(u)\, dx(u) \Big\}\, d\omega_{\varphi_\alpha}(x) \\ &= \exp\Big\{ -\frac{\xi^2}{2} \parallel h \parallel_2^2 \Big\}. \end{aligned}$$

Hence X has a normal distribution with the mean zero and the variation $\parallel h \parallel_2^2$. □

By the same method as in the proof of Theorem 29.7 in [9, p.447], we can prove the following theorem.

THEOREM 3.5. *Let $\{h_1, h_2, \cdots, h_n\}$ be an orthonormal system such that each h_i is of bounded variation. For $i = 1, 2, \cdots, n$, let $X_i(x) = \int_0^t h_i(s)\, dx(s)$. Then $X_1, X_2, \cdots, X_n$ are independent, each X_i has the standard normal distribution. Moreover, if $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable,*

$$(3.9) \qquad \begin{aligned} &\int_{C[0,t]} f(X_1(x), X_2(x), \cdots, X_n(x))\, d\omega_\varphi(x) \\ \overset{*}{=}\ & (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(u_1, u_2, \cdots, u_n) \exp\Big\{ -\frac{1}{2} \sum_{j=1}^n u_j \Big\} \\ & \times d \prod_{i=1}^n m_L(u_1, u_2, \cdots, u_n), \end{aligned}$$

where $\overset{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

Let $\{h_k | k = 1, 2, \cdots\}$ be a complete orthonormal set in $L^2([0,t], m_L)$ such that each h_k is of bounded variation. For f in $L^2([0,t], m_L)$ and x in $C[0,t]$, we let

$$(3.10) \int_0^t f(s) \hat{dx}(s) = \lim_{n\to\infty} \int_0^t \Big[\sum_{k=1}^n \int_0^t f(u) e_k(u)\, dm_L(u)\, e_k(v) \Big] dx(v)$$

if the limit exists. $\int_0^t f(s)\hat{dx}(s)$ is called the Paley-Wiener-Zygmund integral of f according to x. By the routine method in the theory of Wiener space, we can prove that the integral $\int_0^t f(s)\hat{dx}(s)$ is independent on the

orthonormal set $\{e_k|k=1,2,\cdots\}$ and the Paley-Wiener-Zygmund integral exists $\omega_\varphi - a.e.\ x$ in $C[0,t]$.

REMARK 3.6. In 1980, Cameron and Storvick introduced the definitions and some related theories of the spaces S, S' and S'' of Wiener functionals. If we replace $(C_0[0,t], m_w)$ by $(C[0,t], \omega_\varphi)$ in their paper, we can prove various results on $(C[0,t], \omega_\varphi)$ which are similar to Cameron and Storvick's results in [2].

4. The conditional ω_φ-integral

In 1975, Yeh introduced the definition and some related theories of conditional Wiener integral. In 1984, Chang and Chang found some formulas for conditional Wiener integral $E^w(Z|X)$, given by $X(x) = (x(s_1), x(s_2), \cdots, x(s_n))$ [3]. In this section, we will establish some formulas for conditional ω_φ-integral which generalize Chang and Chang's the results.

From [4, Lemma 1, p.67], we have following lemma.

LEMMA 4.1. *For each measurable partition π of $\mathbb{R}^{n+1}$, define the linear operator*

$$T_\pi : L_\infty\Big(\mathbb{R}^{n+1}, \prod_{i=1}^n m_L \times \varphi\Big) \to L_\infty\Big(\mathbb{R}^{n+1}, \prod_{i=1}^n m_L \times \varphi\Big) \tag{4.1}$$

by

$$T_\pi(f) = \sum_{A\in\pi} \frac{\int_A f \; d\prod_{i=1}^n m_L \times \varphi}{(\prod_{i=1}^n m_L \times \varphi)(A)} \chi_A \tag{4.2}$$

for f in $L_\infty(\mathbb{R}^{n+1}, \prod_{i=1}^n m_L \times \varphi)$. Here if $(\prod_{i=1}^n m_L \times \varphi)(A) = 0$ then we take $\frac{\int_A f \; d\prod_{i=1}^n m_L \times \varphi}{(\prod_{i=1}^n m_L \times \varphi)(A)} \equiv 0$. Then if the partitions are directed by refinement, then $\lim_\pi \| T_\pi(f) - f \|_\infty = 0$ for f in $L_\infty(\mathbb{R}^{n+1}, \prod_{i=1}^n m_L \times \varphi)$.

LEMMA 4.2. *Let X and Z be as in Lemma 2.2. Assume that $P_X \ll \prod_{i=1}^n m_L \times \varphi$ on $(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. For $(u_0, u_1, \cdots, u_n) \in \mathbb{R}^{n+1}$ let $J_\pi(u_0, u_1, \cdots, u_n; \cdot)$ be the function on $\mathbb{R}^{n+1}$ defined by*

$$\begin{aligned} &J_\pi(u_0, \cdots, u_n; s_0, \cdots, s_n) \\ &= \sum_{A\in\pi} \Big[\prod_{i=1}^n m_L \times \varphi(A)\Big]^{-1} \chi_A(s_0, \cdots, s_n)\ \chi_A(u_0, \cdots, u_n), \end{aligned} \tag{4.3}$$

where π is a partition of measurable sets on $\mathbb{R}^{n+1}$ and if $(\prod_{i=1}^n m_L \times \varphi)(A) = 0$ then we take $[(\prod_{i=1}^n m_L \times \varphi)(A)]^{-1} \equiv 0$. Then there is a version of $E^{\omega_\varphi}(Z|X)\frac{dP_X}{d\prod_{i=1}^n m_L \times \varphi}$ such that

$$E^{\omega_\varphi}(Z|X)(u_0, \cdots, u_n)\frac{dP_X}{d\prod_{i=1}^n m_L \times \varphi}(u_0, \cdots, u_n) \tag{4.4}$$
$$= \lim_\pi \int_{C[0,t]} (J_\pi(u_0, \cdots, u_n; X(x))Z(x)\ d\omega_\varphi(x)$$

for $\prod_{i=1}^n m_L \times \varphi - a.e.\ ((u_1, \cdots, u_n), u_0)$.

Proof. Substituting $J_\pi(u_0, u_1, \cdots, u_n; \cdot)$ in the place of g in Lemma 2.2, we have

$$\lim_\pi \int_{C[0,t]} (J_\pi(u_0, \cdots, u_n; X(x))Z(x)\ d\omega_\varphi(x) \tag{4.5}$$
$$= \lim_\pi \int_{\mathbb{R}} \int_{\mathbb{R}^n} (J_\pi(u_0, \cdots, u_n; s_0, s_1, \cdots, s_n)E^{\omega_\varphi}(Z|X)(s_0, \cdots, s_n)$$
$$\times \frac{dP_X}{d\prod_{i=1}^n m_L \times \varphi}(s_0, \cdots, s_n) d\prod_{i=1}^n m_L(s_1, \cdots, s_n)d\varphi(s_0)$$

for $\prod_{i=1}^n m_L \times \varphi - a.e.\ ((u_1, \cdots, u_n), u_0)$. Let $f = E^{\omega_\varphi}(Z|X)\frac{dP_X}{d\prod_{i=1}^n m_L \times \varphi}$. Then f is $\prod_{i=1}^n m_L \times \varphi$-integrable since $P_X \ll \prod_{i=1}^n m_L \times \varphi$ and $E^{\omega_\varphi}(Z|X)$ is P_X-integrable. So, from Lemma 4.1 we have the result. □

The following Theorem 4.3 and Theorem 4.6 are the generalization of Chang and Chang's 1984 results in [3].

THEOREM 4.3. *For $x \in C[0,t]$, let $X(x) = (x(s_0), x(s_1), \cdots, x(s_n))$ and let $Z(x) = \frac{1}{t}\int_0^t x(s)ds$ where $0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = t$ for a non-negative integer n. Suppose $f(u) = u$ is φ-integrable. Then the conditional ω_φ-integral of Z given X is*

$$E^{\omega_\varphi}(Z|X)(u_0, u_1, \cdots, u_n) \tag{4.6}$$
$$= \frac{1}{2t}\sum_{i=1}^n (s_i - s_{i-1})(u_{i-1} + u_i) + \frac{1}{t}(t - s_n)u_n.$$

Proof. Since $|Z(x)| \le \frac{1}{t}\int_0^t |x(s)|\ ds$ and $\int_{\mathbb{R}} |w|\ d\varphi(w)$ is finite, $\int_{C[0,t]} |x(s)|\ d\omega_\varphi(x)$ is finite. So, $\int_{C[0,t]} \frac{1}{t}\int_0^t |x(s)|\ ds\ d\omega_\varphi(x) = \frac{1}{t}\int_0^t \int_{C[0,t]} |x(s)|\ d\omega_\varphi(x)\ ds$ is finite. Therefore $\int_{C[0,t]} |Z| d\omega_\varphi(x)$ is finite and so $E^{\omega_\varphi}(Z|X)$

exists. According to Lemma 4.2, a version of $E^{\omega_\varphi}(Z|X)\frac{dP_X}{d\prod_{i=1}^n m_L\times\varphi}$ is given by

$$
\begin{aligned}
(4.7)\qquad & E^{\omega_\varphi}(Z|X)(u_0,u_1,\cdots,u_n)\frac{dP_X}{d\prod_{i=1}^n m_L\times\varphi}(u_0,u_1,\cdots,u_n)\\
&=\lim_{\pi}\int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;X(x))Z(x)\ d\omega_\varphi(x),
\end{aligned}
$$

where $(u_0,u_1,\cdots,u_n)\in\mathbb{R}^{n+1}$ and π is a partition of measurable sets on $\mathbb{R}^{n+1}$. With our Z and X, we have

$$
\begin{aligned}
(4.8)\quad & \int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;X(x))Z(x)\ d\omega_\varphi(x)\\
&=\sum_{i=1}^{n+1}\int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;x(s_0),\cdots,x(s_n))\frac{1}{t}\int_{s_{i-1}}^{s_i}x(v)dv\ d\omega_\varphi(x).
\end{aligned}
$$

By Lemma 2.1, if $v=0$,

$$
\begin{aligned}
(4.9)\qquad & \int_0^t\int_{C[0,t]}\sum_{A\in\pi}\Big[(\prod_{i=1}^n m_L\times\varphi)(A)\Big]^{-1}|x(v)|\ d\omega_\varphi(x)\ dm_L(v)\\
&=\sum_{A\in\pi}\Big[(\prod_{i=1}^n m_L\times\varphi)(A)\Big]^{-1}\int_0^t\Big(\int_{\mathbb{R}}|u_0|\ d\varphi(u_0)\Big)dm_L(v)
\end{aligned}
$$

is finite, and if $0<v\le t$,

$$
\begin{aligned}
(4.10)\qquad & \int_0^t\int_{C[0,t]}\sum_{A\in\pi}\Big[(\prod_{i=1}^n m_L\times\varphi)(A)\Big]^{-1}|x(v)|\ d\omega_\varphi(x)\ dm_L(v)\\
&=\sum_{A\in\pi}\Big[(\prod_{i=1}^n m_L\times\varphi)(A)\Big]^{-1}\int_0^t\Big(\frac{1}{\sqrt{2\pi v}}\int_{\mathbb{R}}\int_{\mathbb{R}}|u_1|\\
&\quad\times\ \exp\Big\{-\frac{(u_1-u_0)^2}{2v}\Big\}dm_L(u_1)\ d\varphi(u_0)\Big)dm_L(v)
\end{aligned}
$$

is finite. So, by the Fubini's Theorem and Lemma 2.1, we have

$$
\begin{aligned}
(4.11)\quad & \sum_{i=1}^{n+1}\int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;x(s_0),x(s_1),\cdots,x(s_n))\frac{1}{t}\int_{s_{i-1}}^{s_i}x(v)dv\\
&\times\ d\omega_\varphi(x)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \int_{C[0,t]} (J_\pi(u_0, \cdots, u_n; x(s_0), x(s_1), \cdots, x(s_n)) \\
&\quad \times \frac{1}{t} \int_{s_{i-1}}^{s_i} x(v)dv \; d\omega_\varphi(x) \\
&+ \int_{C[0,t]} (J_\pi(u_0, \cdots, u_n; x(s_0), x(s_1), \cdots, x(s_n)) \\
&\quad \times \frac{1}{t} \int_{s_n}^{s_{n+1}} x(v)dv \; d\omega_\varphi(x) \\
&= \sum_{i=1}^{n} \frac{1}{2t}(s_i - s_{i-1}) \Big[(2\pi)^n \prod_{j=1}^{n} (s_j - s_{j-1}) \Big]^{-\frac{1}{2}} \sum_{A \in \pi} \frac{\chi_A(u_0, \cdots, u_n)}{(\prod_{i=1}^{n} m_L \times \varphi)(A)} \\
&\quad \times \int_A (\alpha_{i-1} + \alpha_i) \exp\Big\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(\alpha_j - \alpha_{j-1})^2}{s_j - s_{j-1}} \Big\} \\
&\quad \times d\Big(\prod_{i=1}^{n} m_L \times \varphi \Big)((\alpha_1, \cdots, \alpha_n), \alpha_0) \\
&+ \frac{1}{t}(t - s_n) \Big[(2\pi)^n \prod_{j=1}^{n} (s_j - s_{j-1}) \Big]^{-\frac{1}{2}} \sum_{A \in \pi} \frac{\chi_A(u_0, \cdots, u_n)}{(\prod_{i=1}^{n} m_L \times \varphi)(A)} \\
&\quad \times \int_A \alpha_n \exp\Big\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(\alpha_j - \alpha_{j-1})^2}{s_j - s_{j-1}} \Big\} \\
&\quad \times d\Big(\prod_{i=1}^{n} m_L \times \varphi \Big)((\alpha_1, \cdots, \alpha_n), \alpha_0),
\end{aligned}
$$

where $x(s_i) = \alpha_i$ for $i = 0, 1, 2, \cdots, n$. By (4.7) and Lemma 4.1,

$$
\begin{aligned}
(4.12) \qquad & E^{\omega_\varphi}(Z|X)(u_0, u_1, \cdots, u_n) \frac{dP_X}{d\prod_{i=1}^{n} m_L \times \varphi}(u_0, u_1, \cdots, u_n) \\
&= \Big[\frac{1}{2t} \sum_{i=1}^{n} (s_i - s_{i-1})(u_{i-1} + u_i) + \frac{1}{t}(t - s_n)u_n \Big] \\
&\quad \times \Big[(2\pi)^n \prod_{j=1}^{n} (s_j - s_{j-1}) \Big]^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{s_j - s_{j-1}} \Big\}.
\end{aligned}
$$

Therefore, for a non-negative integer n we obtain

$$E^{\omega_\varphi}(Z|X)(u_0,u_1,\cdots,u_n) = \frac{1}{2t}\sum_{i=1}^{n}(s_i-s_{i-1})(u_{i-1}+u_i)+\frac{1}{t}(t-s_n)u_n. \tag{4.13}$$

□

From Theorem 4.3, we have the following two corollaries.

COROLLARY 4.4. *For $x \in C[0,t]$, let $Z(x)$ be as in Theorem 4.3. And let $X(x) = (x(0))$. Suppose $f(u) = u$ is φ-integrable. Then the conditional ω_φ-integral of Z given X is*

$$E^{\omega_\varphi}(Z|X)(u_0) = u_0. \tag{4.14}$$

COROLARY 4.5. *For $x \in C[0,t]$, let $Z(x)$ be as in Theorem 4.3. And let $X(x) = (x(s_0), x(s_1), \cdots, x(s_{n+1}))$ where $0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = t$ for a non-negative integer n. Suppose $f(u) = u$ is φ-integrable. Then the conditional ω_φ-integral of Z given X is*

$$E^{\omega_\varphi}(Z|X)(u_0,u_1,\cdots,u_{n+1}) = \frac{1}{2t}\sum_{i=1}^{n+1}(s_i-s_{i-1})(u_{i-1}+u_i). \tag{4.15}$$

THEOREM 4.6. *For $x \in C[0,t]$, let $X(x) = (x(s_0), x(s_1), \cdots, x(s_n))$ and let $Z(x) = \sum_{i=1}^{n+1} \frac{1}{s_i - s_{i-1}} \int_{(s_{i-1},s_i)} x(v)dm_L(v)$ where $0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = t$ for a non-negative integer n. Suppose $f(u) = u$ is φ-integrable. Then the conditional ω_φ-integral of Z given X is*

$$E^{\omega_\varphi}(Z|X)(u_0,u_1,\cdots,u_n) = \frac{1}{2}\sum_{i=1}^{n}(u_{i-1}+u_i)+u_n. \tag{4.16}$$

Proof. It is obvious that $E^{\omega_\varphi}(Z|X)$ exists. By Lemma 4.2, a version of $E^{\omega_\varphi}(Z|X)\frac{dP_X}{d\prod_{j=1}^{n} m_L\times\varphi}$ is given by

$$\begin{aligned} E^{\omega_\varphi}(Z|X)(u_0,u_1,\cdots,u_n)&\frac{dP_X}{d\prod_{j=1}^{n} m_L\times\varphi}(u_0,u_1,\cdots,u_n) \\ &= \lim_{\pi}\int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;x(s_0),x(s_1),\cdots,x(s_n)) \\ &\quad\times\sum_{i=1}^{n+1}\frac{1}{s_i-s_{i-1}}\int_{s_{i-1}}^{s_i}x(v)dv\ d\omega_\varphi(x). \end{aligned} \tag{4.17}$$

Then by Lemma 4.1, we have

$$\begin{aligned}(4.18)\qquad &\lim_{\pi}\int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;x(s_0),x(s_1),\cdots,x(s_n))\\ &\quad\times\sum_{i=1}^{n+1}\frac{1}{s_i-s_{i-1}}\int_{s_{i-1}}^{s_i}x(v)dv\ d\omega_\varphi(x)\\ =\ &\frac{1}{2}\sum_{i=1}^{n}(u_{i-1}+u_i)+u_n\end{aligned}$$

as desired. □

From Theorem 4.6, we have the following a corollary.

COROLLARY 4.7. *For $x\in C[0,t]$, let $Z(x)$ be as in Theorem 4.6. And let $X(x)=(x(s_0),x(s_1),\cdots,x(s_{n+1}))$ where $0=s_0<s_1<\cdots<s_n<s_{n+1}=t$ for a non-negative integer n. Suppose $f(u)=u$ is φ-integrable. Then the conditional ω_φ-integral of Z given X is*

$$(4.19)\qquad E^{\omega_\varphi}(Z|X)(u_0,u_1,\cdots,u_{n+1})=\frac{1}{2}\sum_{i=1}^{n+1}(u_{i-1}+u_i).$$

THEOREM 4.8. *For $x\in C[0,t]$, let $X(x)=(x(s_0),x(s_1),\cdots,x(s_n))$ and let $Z(x)=\int_0^t(x(v))^2dm_L(v)$ where $0=s_0<s_1<\cdots<s_n<s_{n+1}=t$ for a non-negative integer n. Suppose $f(u)=u^2$ is φ-integrable. Then the conditional ω_φ-integral of Z given X is*

$$\begin{aligned}(4.20)\qquad &E^{\omega_\varphi}(Z|X)(u_0,u_1,\cdots,u_n)\\ =\ &\frac{1}{6}\sum_{i=1}^{n}(s_i-s_{i-1})[(s_i-s_{i-1})\\ &\qquad+2(u_i^2+u_{i-1}u_i+u_{i-1}^2)]+\frac{1}{2}(t-s_n)^2.\end{aligned}$$

Proof. It is trivial that $E^{\omega_\varphi}(Z|X)$ exists. By Lemma 4.2, a version of $E^{\omega_\varphi}(Z|X)\ \frac{dP_X}{d\prod_{j=1}^n m_L\times\varphi}$ is given by

$$E^{\omega_\varphi}(Z|X)(u_0,u_1,\cdots,u_n)\frac{dP_X}{d\prod_{j=1}^n m_L\times\varphi}(u_0,u_1,\cdots,u_n) \tag{4.21}$$
$$=\lim_\pi \sum_{i=1}^{n+1}\int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;x(s_0),x(s_1),\cdots,x(s_n))$$
$$\times\int_{s_{i-1}}^{s_i}(x(v))^2dm_L(v)\ d\omega_\varphi(x).$$

Thus by (2.4) and the elementary calculus, we have

$$\sum_{i=1}^{n+1}\int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;x(s_0),x(s_1),\cdots,x(s_n)) \tag{4.22}$$
$$\int_{s_{i-1}}^{s_i}(x(v))^2dm_L(v)\ d\omega_\varphi(x)$$
$$=\sum_{i=1}^n\int_{s_{i-1}}^{s_i}\int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;x(s_0),x(s_1),\cdots,x(s_n))(x(v))^2$$
$$\times dm_L(v)d\omega_\varphi(x)$$
$$+\int_{s_n}^{s_{n+1}}\int_{C[0,t]}(J_\pi(u_0,\cdots,u_n;x(s_0),x(s_1),\cdots,x(s_n))$$
$$\times(x(v))^2\ dm_L(v)d\omega_\varphi(x)$$
$$=\sum_{i=1}^n\sum_{A\in\pi}\frac{\chi_A(u_0,\cdots,u_n)}{(\prod_{i=1}^n m_L\times\varphi)(A)}\int_A\Big[(2\pi)^n\prod_{j=1}^n(s_j-s_{j-1})\Big]^{-\frac12}$$
$$\times\Big[\frac16(s_i-s_{i-1})^2+\frac13(s_i-s_{i-1})(\alpha_i^2+\alpha_{i-1}\alpha_i+\alpha_{i-1}^2)\Big]$$
$$\times\exp\Big\{-\frac12\sum_{j=1}^n\frac{(\alpha_j-\alpha_{j-1})^2}{s_j-s_{j-1}}\Big\}d\Big(\prod_{i=1}^n m_L\times\varphi\Big)((\alpha_1,\cdots,\alpha_n),\alpha_0)$$
$$+\sum_{A\in\pi}\frac{\chi_A(u_0,\cdots,u_n)}{(\prod_{i=1}^n m_L\times\varphi)(A)}\int_A\frac12(s_{n+1}-s_n)^2\Big[(2\pi)^n\prod_{j=1}^n(s_j-s_{j-1})\Big]^{-\frac12}$$
$$\times\exp\Big\{-\frac12\sum_{j=1}^n\frac{(\alpha_j-\alpha_{j-1})^2}{s_j-s_{j-1}}\Big\}\ d\Big(\prod_{i=1}^n m_L\times\varphi\Big)((\alpha_1,\cdots,\alpha_n),\alpha_0),$$

where $x(s_i) = \alpha_i$ for $i = 0, 1, 2, \cdots, n$. So, by Lemma 4.1,

$$
\begin{aligned}
(4.23) \qquad & \lim_{\pi} \sum_{i=1}^{n+1} \int_{C[0,t]} (J_\pi(u_0, \cdots, u_n; x(s_0), x(s_1), \cdots, x(s_n)) \\
& \quad \times \int_{s_{i-1}}^{s_i} (x(v))^2 dm_L(v)\ d\omega_\varphi(x) \\
= \quad & \left(\frac{1}{6} \sum_{i=1}^{n} (s_i - s_{i-1})[(s_i - s_{i-1}) + 2(u_i^2 + u_{i-1}u_i + u_{i-1}^2)] \right. \\
& \quad \left. + \frac{1}{2}(t - s_n)^2 \right) \Big[(2\pi)^n \prod_{j=1}^{n} (s_j - s_{j-1}) \Big]^{-\frac{1}{2}} \\
& \quad \times \exp\Big\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{(u_j - u_{j-1})^2}{s_j - s_{j-1}} \Big\}.
\end{aligned}
$$

Hence we have the desired result. □

By Theorem 4.8, we have the following two corollaries.

COROLLARY 4.9. *For $x \in C[0,t]$, let $Z(x)$ be as in Theorem 4.8 and let $X(x) = (x(0))$. Suppose $f(u) = u^2$ is φ-integrable. Then the conditional ω_φ-integral of Z given X is*

$$
(4.24) \qquad E^{\omega_\varphi}(Z|X)(u_0) = \frac{1}{2}t^2 \ .
$$

COROLLARY 4.10. *For $x \in C[0,t]$, let $Z(x)$ be as in Theorem 4.8 and let $X(x) = (x(s_0), x(s_1), \cdots, x(s_{n+1}))$, where $0 = s_0 < s_1 < \cdots < s_n < s_{n+1} = t$ for a non-negative integer n. Suppose $f(u) = u^2$ is φ-integrable. Then the conditional ω_φ-integral of Z given X is*

$$
\begin{aligned}
(4.25) \qquad & E^{\omega_\varphi}(Z|X)(u_0, u_1, \cdots, u_{n+1}) \\
= \quad & \frac{1}{6} \sum_{i=1}^{n+1} (s_i - s_{i-1})[(s_i - s_{i-1}) + 2(u_i^2 + u_{i-1}u_i + u_{i-1}^2)].
\end{aligned}
$$

5. A translation theorem of conditional ω_φ-integral

In 1978, Yeh found a formula for the translation of conditional Wiener integral [12]. In this section, we will prove a formula, similar to Yeh's result, for the translation of conditional ω_φ-integral.

Using the similar method as in the proofs of Lemma 1, Lemma 2 and Theorem 1 in [12], we can prove the following three lemmas, so we state the lemmas without the proofs.

LEMMA 5.1. *Suppose $f(u) = u$ is φ-integrable. Let Z be a real-valued integrable random variable on $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$ and let $X : (C[0,t], \mathcal{B}(C[0,t])) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable function. Let $T : (C[0,t], \mathcal{B}(C[0,t])) \to (C[0,t], \mathcal{B}(C[0,t]))$ be measurable function. Suppose there is a bijective function h from $\mathbb{R}$ to $\mathbb{R}$ such that h and h^{-1} are measurable, and $X \circ T = h \circ X$ for $\omega_\varphi - a.e.\ x$ in $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$. Then*

$$E^{\omega_\varphi}(Z|X \circ T)(\xi) = E^{\omega_\varphi}(Z|X)(h^{-1}(\xi)) \tag{5.1}$$

for $P_{X\circ T} - a.e.\ \xi$.

LEMMA 5.2. *Under the assumptions in Lemma 5.1, if there is a real-valued $\mathcal{B}(C[0,t])$-measurable function J on $C[0,t]$ such that $\omega_\varphi(B) = \int_{T^{-1}(B)} J(x)\, d\omega_\varphi(x)$ for B in $\mathcal{B}(C[0,t])$, then for any real-valued $\mathcal{B}(C[0,t])$ -measurable function T on $C[0,t]$, we have*

$$\int_{C[0,t]} T(x)\, d\omega_\varphi(x) \stackrel{*}{=} \int_{C[0,t]} (Z \circ T)(x) J(x)\, d\omega_\varphi(x), \tag{5.2}$$

where $\stackrel{*}{=}$ means that if one side exists then both sides exist and the two values are equal.

LEMMA 5.3. *Under the assumptions in Lemma 5.1 and Lemma 5.2, for every measurable function g of $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ to itself, we have*

$$\begin{aligned} &\int_{\mathbb{R}} g(\xi) E^{\omega_\varphi}(Z|X)(\xi)\, dP_X(\xi) \\ \stackrel{*}{=}\ &\int_{\mathbb{R}} g(\xi) E^{\omega_\varphi}((Z \circ T)J|X)(h^{-1}(\xi))\, dP_{X\circ T}(\xi), \end{aligned} \tag{5.3}$$

where $\stackrel{*}{=}$ *means that if one side exists then both sides exist and the two values are equal. Moreover, if* $P_{X\circ T} \ll P_X$ *then*

$$E^{\omega_\varphi}(Z|X)(\xi) = E^{\omega_\varphi}((Z\circ T)J|X)(h^{-1}(\xi))\frac{dP_{X\circ T}}{dP_X}(\xi) \tag{5.4}$$

for $P_X - a.e.\ \xi$.

Now, we will establish a translation theorem for the conditional ω_φ-integral.

THEOREM 5.4. *Let* Y *be a real-valued integrable function on* $(C[0,t], \mathcal{B}(C[0,t]), \omega_\varphi)$ *and let* $X(x) = x(t)$ *for* x *in* $C[0,t]$. *Let* x_0 *be given as in Theorem 3.1. Then we have*

$$\begin{aligned} & E^{\omega_\varphi}(Y|X)(\xi) \\ = \ & E^{\omega_\varphi}(Y(\cdot + x_0)J|X)(\xi - x_0(t)) \\ & \times \int_{\mathbb{R}} \exp\left\{ -\frac{(\xi - x_0(t) - u_0)^2}{2t} \right\} d\varphi(u_0) \\ & \times \left[\int_{\mathbb{R}} \exp\left\{ -\frac{(\xi - u_0)^2}{2t} \right\} d\varphi(u_0) \right]^{-1} \end{aligned} \tag{5.5}$$

for $m_L - a.e.\ \xi$, *where*

$$J(x) = \exp\left\{ -\frac{1}{2}\int_0^t (h(s))^2\ dm_L(s) \right\} \exp\left\{ -\int_0^t h(s)\ dx(s) \right\}. \tag{5.6}$$

Proof. Let T be the function of $C[0,t]$ into itself with $T(x) = x + x_0$ for x in $C[0,t]$ and h a function of $\mathbb{R}$ into itself with $h(x) = \xi + x_0(t)$ for ξ in $\mathbb{R}$. Then $(X\circ T)(x) = h\circ X(x)$, h and h^{-1} are bijective measurable. By Theorem 3.1, letting $J(x) = e^{-\frac{1}{2}\|h(s)\|_2^2} e^{-\int_0^t h(s)\ dx(s)}$,

$$\omega_\varphi(B) = \int_{T^{-1}(B)} J(x)\ d\omega_\varphi(x) \tag{5.7}$$

for B in $\mathcal{B}(C[0,t])$, that is, $P_{X\circ T} \ll P_X$. By (2.7), we have

$$\begin{aligned}(5.8)\qquad & \frac{dP_{X\circ T}}{dP_X}(\xi) \\ &= \frac{dP_{X\circ T}}{dm_L}(\xi)\left[\frac{dP_X}{dm_L}(\xi)\right]^{-1} \\ &= \int_{\mathbb{R}} \exp\left\{-\frac{(\xi - x_0(t) - u_0)^2}{2t}\right\} d\varphi(u_0) \\ &\qquad \times\left[\int_{\mathbb{R}} \exp\left\{-\frac{(\xi-u_0)^2}{2t}\right\} d\varphi(u_0)\right]^{-1}.\end{aligned}$$

So, from (5.4) we have

$$\begin{aligned}(5.9)\qquad & E^{\omega_\varphi}(Y|X)(\xi) \\ &= E^{\omega_\varphi}((Y\circ T)J|X)(h^{-1}(\xi))\frac{dP_{X\circ T}}{dP_X}(\xi) \\ &= E^{\omega_\varphi}(Y(\cdot + x_0)J|X)(\xi - x_0(t)) \\ &\qquad \times \int_{\mathbb{R}} \exp\left\{-\frac{(\xi - x_0(t) - u_0)^2}{2t}\right\} d\varphi(u_0) \\ &\qquad\qquad \times\left[\int_{\mathbb{R}} \exp\left\{-\frac{(\xi-u_0)^2}{2t}\right\} d\varphi(u_0)\right]^{-1},\end{aligned}$$

where

$$(5.10)\quad J(x) = \exp\left\{-\frac{1}{2}\int_0^t (h(s))^2\, dm_L(s)\right\}\exp\left\{-\int_0^t h(s)\, dx(s)\right\}.$$

□

REMARK 5.5. In Theorem 5.4, if we take $\varphi = \delta_0$, the Dirac measure at the origin, then we obtain Yeh's 1978 result.

References

[1] R. H. Cameron and W. T. Martin, *Transformations of Wiener integrals under translations*, Ann. of Math. **45** (1944), 386–396.

[2] R. H. Cameron and D. A. Storvick, *Some Banach algebras of analytic Feynman integrable functionals*, Lecture Notes in Math. **798**, Springer, Berlin-New York, 1980, 18–67.

[3] K. S. Chang and J. S. Chang, *Evaluation of some conditional Wiener integrals*, Bull. Korean Math. Soc. **21** (1984), 99–106.

[4] J. Diestel and J. J. Uhl, *Vector measures*, Mathematical Surveys, No. 15, American Mathematical Society, Providence, R.I., 1977.

[5] K. R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York, 1967.
[6] K. S. Ryu and M. K. Im, *A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula*, Trans. Amer. Math. Soc. (to appear).
[7] N. Wiener, *Differential space*, J. Math. Phys. **2** (1923), 131–174.
[8] Y. Yamasaki, *Measures on infinite dimensional space*, Series in Pure Mathematics **5**, World Scientific Publishing Co., Singapore, 1985.
[9] J. Yeh, *Stochastic processes and the Wiener integral*, Pure and Applied Mathematics, Vol. **13**, Marcel Deckker, Inc., New York, 1973.
[10] ———, *Inversion of conditional expectations*, Pacific J. Math. **52** (1974), no. 2, 631–640.
[11] ———, *Inversion of conditional Wiener integrals*, Pacific J. Math. **59** (1975), no. 2, 631–640.
[12] ———, *Transformation of conditional Wiener integrals under translation and the Cameron-Martin translation theorem*, Tohoku Math. J. **30** (1978), 505–515.

Department of Mathematics
Hannam University
Taejon 306-791, Korea
E-mail: mki@mail.hannam.ac.kr
ksr@math.hannam.ac.kr