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An Application of the Umbral Calculus

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The partial difference equation

$$r(i,j)=r(i,j-1)+r(i-1,j)+r(i-1,j+1),$$

where $r(i,j)$ are defined for integer numbers i and j, $i\geqslant 0$, by the conditions $r(0,j)=1$ for all j and $r(i,-1)=0$ for $i\geqslant 1$ is solved. For $i\geqslant 0$ and $j\geqslant 0$ a combinatorial meaning of numbers $r(i,j)$ is given. The solution is obtained by the modern classical umbral calculus.

1. INTRODUCTION

PROBLEM. *Let $S=\{(i,j): i,j=0,1,2,...,\}$. Define in the set S the relation ρ by*

> *$(i,j)\,\rho(p,q)$ if and only if $(p=i, q=j-1)$ or $(p=i-1, q=j)$ or $(p=i-1, q=j+1)$.*

The point $(i,j)\in S$ is said to be *connected* with the origin $(0,0)\in S$ if and only if there exist points (i_1,j_1), (i_2,j_2), ..., (i_n,j_n) in S, where $(i_1,j_1)\,\rho(0,0)$, $(i_2,j_2)\,\rho(i_1,j_1)$, ..., $(i,j)\,\rho(i_n,j_n)$. Our aim is to compute the number $r(i,j)$ of different connections of the point $(i,j)\in S$ with the origin $(0,0)$. If we put it in the language of the graph theory, our problem is to determine the number of linearly connected graphs with vertices in the set S and with edges oriented parallel to the vectors $(1,0)$, $(0,1)$, and $(1,-1)$. Figure 1 shows one of the possible connections of the point $(3,2)$ with the origin.

It is clear that $r(0,j)=1$ for $j\geqslant 1$. Define $r(0,0)=1$. By an easy combinatorial argument we get the partial difference equation

$$\begin{aligned} &r(i,j)=r(i,j-1)+r(i-1,j)+r(i-1,j+1), \qquad i\geqslant 1, j\geqslant 0;\\ &r(0,j)=1, j>0; \qquad r(i,-1)=0, i>0. \end{aligned} \tag{1}$$

A simple computation gives us the numbers $r(i,j)$ in Table I.

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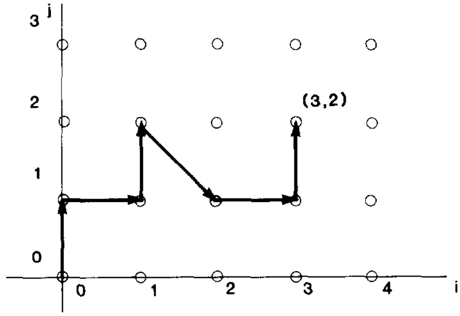

FIGURE 1

In the sequel we shall derive the formula for our numbers $r(i, j)$, the generating functions for the rows $r(\cdot, j)$, and the columns $r(i, \cdot)$.

The umbral calculus. We repeat the basic facts following Niven and Roman (see [1, 2]). Let F denote the algebra of formal power series in the variable t over the field $\mathbb{C}$. An element in F has the form

$$f(t)=\sum_{k=0}^{\infty} a_k t^k, \qquad a_k \in \mathbb{C}. \tag{2}$$

The addition and multiplication are defined formally by

$$\sum_{k=0}^{\infty} a_k t^k + \sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} (a_k + b_k)\, t^k$$

$$\left(\sum_{k=0}^{\infty} a_k t^k\right)\left(\sum_{k=0}^{\infty} b_k t^k\right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j}\right) t^k.$$

TABLE I

The Numbers $r(i, j)$ for $i \geqslant 0, j \geqslant 0$

j \ i	0	1	2	3	4	5	6
7	1	16	160	...	...	...	...
6	1	14	126	938	...	...	...
5	1	12	96	652	...	...	...
4	1	10	70	430	...	...	...
3	1	8	48	264	1408	...	...
2	1	6	30	146	714	3534	...
1	1	4	16	68	304	1412	...
0	1	2	6	22	90	394	1806

Two formal power series are equal if and only if $a_k = b_k$ for all k. Let F_0 denote the set of all formal power series (2) where $a_0 \neq 0$ and F_1 the set of all formal power series (2) where $a_0 = 0$ and $a_1 \neq 0$. If $f(t) \in F_0$ then $f(t)$ is invertible, and the formal inverse will be denoted by $f(t)^{-1}$. The coefficients of the inverse can be computed solving a simple triangular system of equations. If f belong to the set F_1, then a compositional inverse $\bar{f}(t)$ exists, such that $\bar{f}(f(t)) = t$.

The formal derivative of the series (2) is defined as

$$D_t f(t) = \sum_{k=1}^{\infty} k a_k t^{k-1}.$$

Let P denote the algebra of polynomials in the single variable x over the field $\mathbb{C}$. Let P^* be the vector space of all linear functionals on P. The action of the functional $L \in P^*$ on the polynomial $p(x) \in P$ will be denoted by

$$\langle L \mid p(x) \rangle.$$

Each formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \tag{3}$$

defines a linear functional on P if we set

$$\langle f(t) \mid x^n \rangle = a_n \qquad \text{for} \quad n \geq 0.$$

For any linear functional $L \in P^*$ we have a formal power series

$$f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L \mid x^k \rangle}{k!} t^k$$

which has the form (3) and satisfies the relation

$$\langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle \qquad \text{for} \quad n \geq 0.$$

The map $L \to f_L(t)$ is a vector space isomorphism from P^* to F.

In the sequel we shall need the formulas

$$\langle t^k \mid p(x) \rangle = p^{(k)}(0), \qquad k \geq 0, p(x) \in P \tag{4}$$

$$\langle f(t)\, g(t) \mid x^n \rangle = \sum_{k=0}^{n} \binom{n}{k} \langle f(t) \mid x^k \rangle \langle g(t) \mid x^{n-k} \rangle \tag{5}$$

$$\langle f(t) \mid x p(x) \rangle = \langle D_t f(t) \mid p(x) \rangle. \tag{6}$$

Any power series defines a linear operator on P. If $f(t)$ has the form (3), then we define

$$f(t)\, x^n = \sum_{k=0}^{n} \binom{a}{k} a_k x^{n-k} \qquad \text{for} \quad n \geqslant 0. \tag{7}$$

Especially, for $f(t) = t^k$ we get

$$t^k x^n = k! \binom{n}{k} x^{n-k},$$

the kth derivative of the power x^n. Using the relation (5) we obtain

$$\langle f(t)\, g(t) | \, p(x) \rangle = \langle g(t) | \, f(t)\, p(x) \rangle. \tag{8}$$

Sheffer sequences. For each series $f(t) \in F_1$ and each series $g(t) \in F_0$ there exists a unique sequence of polynomials $s_n(x)$ such that

$$\langle g(t) f(t)^k \, | \, s_n(x) \rangle = n! \delta_{n,k},$$

where $\delta_{n,k}$ denotes the Kronecker delta function and the polynomial $s_n(x)$ has degree n. We say that the sequence $s_n(x)$ is *Sheffer* for the pair $(g(t), f(t))$. If $s_n(x)$ is Sheffer for the pair $(1, f(t))$ then $s_n(x)$ is *associated* to $f(t)$. The Sheffer sequence $s_n(x)$ of the pair $(g(t), f(t))$ admits the generating function

$$g(\bar{f}(t))^{-1} \, e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} \, t^k, \tag{9}$$

where $y \in \mathbb{C}$.

From (8) it follows that the sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if the sequence $g(t)\, s_n(x)$ is associated to $f(t)$.

A sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ for some $g(t) \in F_0$ if and only if the relation

$$f(t)\, s_n(x) = n s_{n-1}(x) \tag{10}$$

holds for all $n \geqslant 0$.

The sequence $s_n(x)$ is associated to $f(t)$ if and only if $\langle t^0 \, | \, s_n(x) \rangle = \delta_{n,0}$ and $f(t)\, s_n(x) = n s_{n-1}(x)$.

For the series $f(t)=a_1 t+a_2 t^2+\dots$, $a_1 \neq 0$, denote

$$\frac{f(t)}{t}=a_1+a_2 t+\dots .$$

It is clear that $f(t)\in F_1$ and $f(t)/t \in F_0$. The inverse of the series $f(t)/t$ will be denoted by $t/f(t)$.

We compute the associated sequence of the series $f(t)\in F_1$ by the transfer formula

$$s_n(x)=x\left(\frac{t}{f(t)}\right)^n x^{n-1} \tag{11}$$

for $n\geqslant 1$. Note that $s_0(x)=1$.

These are the results of the excellent monograph [2]. We return now to our problem.

2. Main Results

Since the simple power series $1+t$ and $2+t$ are formally invertible the formal power series

$$f(t)=t(1+t)^{-1}(2+t)^{-1} \tag{12}$$

belongs to the set F_1. For each series $g(t)\in F_0$ we have the unique sequence of polynomials $s_n(x)$ which are Sheffer for $(g(t), f(t))$. Denote $p_n(x)$ as the associated sequence for $f(t)$. It is clear that

$$s_n(x)=g(t)^{-1}p_n(x) \tag{13}$$

for all $n\geqslant 0$.

Lemma 1. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$, where $f(t)$ is given by (12) and $g(t)$ is an arbitrary invertible formal power series. Then the double sequence*

$$q(i,j)=\frac{1}{i!}\langle (1+t)^j \mid s_i(x)\rangle, \qquad i\geqslant 0, j\in\mathbb{Z}, \tag{14}$$

satisfies the partial difference equation

$$q(i,j)=q(i,j-1)+q(i-1,j)+q(i-1,j+1) \tag{15}$$

for $i\geqslant 1$ and $j\in\mathbb{Z}$.

Proof. For every $j \in \mathbb{Z}$ and $i \geqslant 1$ we have

$$
\begin{aligned}
&q(i,j)-q(i,j-1)-q(i-1,j)-q(i-1,j+1)\\
&\quad=\frac{1}{i!}\langle (1+t)^j \mid s_i(x)\rangle-\frac{1}{i!}\langle (1+t)^{j-1}\mid s_i(x)\rangle\\
&\qquad-\frac{1}{(i-1)!}\langle (1+t)^j\mid s_{i-1}(x)\rangle-\frac{1}{(i-1)!}\langle (1+t)^{j+1}\mid s_{i-1}(x)\rangle\\
&\quad=\frac{1}{i!}\langle (1+t)^{j-1}\, t\mid s_i(x)\rangle-\frac{1}{(i-1)!}\langle (1+t)^j\,(2+t)\mid s_{i-1}(x)\rangle.
\end{aligned}
$$

Using the relation (10) we obtain

$$
\begin{aligned}
&q(i,j)-q(i,j-1)-q(i-1,j)-q(i-1,j+1)\\
&\quad=\frac{1}{i!}(\langle (1+t)^{j-1}\,(t-(1+t)(2+t)f(t))\mid s_i(x)\rangle=0
\end{aligned}
$$

because of (12).

LEMMA 2. *If the invertible series $g(t)$ in Lemma 1 has the form*

$$g(t)=1+a_1t+a_2t^2+\cdots$$

then the sequence $q(i,j)$ has the property

$$q(0,j)=1$$

for all $j \in \mathbb{Z}$.

Proof. By (13) we have

$$
\begin{aligned}
q(0,j)&=\langle (1+t)^j\mid s_0(x)\rangle=\langle (1+t)^j\mid g(t)^{-1}\,p_0(t)\rangle\\
&=\langle (1+t)^j\,g(t)^{-1}\mid 1\rangle=\langle h(t)\mid 1\rangle,
\end{aligned}
$$

where the formal power series $h(t)$ has the form

$$h(t)=1+b_1t+b_2t^2+\cdots.$$

By the definition of the power series as a linear functional on the vector space P we get $q(0,j)=1$.

LEMMA 3. *The unique invertible series $g(t)$, such that the double sequence $q(i,j)$ in Lemma 1 has properties*

(i) $q(0, j) = 1$ *for all* $j \in \mathbb{Z}$,

(ii) $q(i, -1) = 0$ *for all* $i \in \mathbb{N}$

is the series $g(t) = (1+t)^{-1}$.

Proof. According to Lemma 2 we must prove only (ii). Let

$$g(t)^{-1} = 1 + c_1 t + c_2 t^2 + \cdots.$$

We have

$$q(i, -1) = \frac{1}{i!} \langle (1+t)^{-1} g(t)^{-1} \mid p_i(x) \rangle$$

$$= \frac{1}{i!} \langle h(t) \mid p_i(x) \rangle,$$

where

$$h(t) = 1 + b_1 t + b_2 t^2 + \cdots$$

with

$$b_n = \sum_{k=0}^{n} (-1)^k c_{n-k}, \qquad b_0 = c_0 = 1.$$

For $i = 1, 2, 3, \ldots$, we deduce, using the relation (4), that

$$q(i, -1) = \frac{1}{i!} \sum_{k=0}^{i} b_k p_i^{(k)}(0).$$

Note that $p_i(x)$ is a polynomial of degree i, thus $p_i^{(i)}(0) \neq 0$. The relation $\langle t^0 \mid p_i(x) \rangle = p_i(0) = \delta_{i,0}$ implies, according to (ii), the system of equations for b_k:

$$b_1 p_1'(0) = 0$$
$$b_1 p_2'(0) + b_2 p_2''(0) = 0$$
$$b_1 p_3'(0) + b_2 p_3''(0) + b_3 p_3'''(0) = 0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Step by step we conclude that $b_1 = b_2 = b_3 = \cdots = 0$. From the other system

$$b_1 = c_1 - 1$$
$$b_2 = c_2 - c_1 + 1$$
$$b_3 = c_3 - c_2 + c_1 - 1$$
$$\cdots\cdots\cdots\cdots$$

we get $c_1=1$, $c_2=c_3=c_4=\cdots=0$. Thus $g(t)^{-1}=1+t$ respectively $g(t)=(1+t)^{-1}$.

Lemmas 1, 2, and 3 imply the following result:

THEOREM 1. *The unique solution of the partial difference equation*

$$r(i,j)=r(i,j-1)+r(i-1,j)+r(i-1,j+1)$$

with conditions

$$r(0,j)=1 \quad \text{for all } j \quad \text{and} \quad r(i,-1)=0 \quad \text{for} \quad i\geqslant 1$$

is given by the formula

$$r(i,j)=\frac{j+1}{i!}\langle (1+t)^{i+j}(2+t)^i \mid x^{i-1}\rangle \tag{16}$$

for all j and $i\geqslant 1$.

Proof. It is clear that $r(i,j)=q(i,j)$ in Lemma 1 for $g(t)=(1+t)^{-1}$. For $i\geqslant 0$ and every $j\in\mathbb{Z}$ we have

$$r(i,j)=\frac{1}{i!}\langle (1+t)^j \mid s_i(x)\rangle=\frac{1}{i!}\langle (1+t)^{j+1}\mid p_i(x)\rangle.$$

By the transfer formula (11) we find an explicit form for the associated sequence $p_n(x)$ of the series (12):

$$p_n(x)=x(1+t)^n(2+t)^n x^{n-1}, \qquad n\geqslant 1.$$

Using formula (6) we obtain

$$\begin{aligned} r(i,j)&=\frac{1}{i!}\langle (1+t)^{j+1}\mid x(1+t)^i(2+t)^i x^{i-1}\rangle \\ &=\frac{j+1}{i!}\langle (1+t)^j \mid (1+t)^i(2+t)^i x^{i-1}\rangle \\ &=\frac{j+1}{i!}\langle (1+t)^{i+j}(2+t)^i\mid x^{i-1}\rangle \end{aligned}$$

for all $i\geqslant 1$ and j. This concludes the proof.

THEOREM 2. *The explicit form for the numbers $r(i,j)$ for $i\geqslant 1$ and $j\in\mathbb{Z}$ is*

$$r(i,j)=\frac{j+1}{i}\sum_{k=0}^{i-1}\binom{i+j}{k}\binom{i}{k+1}2^{k+1}.$$

Proof. Formula (4) implies $\langle t^k | x^n \rangle = n!\delta_{k,n}$. Using formula (5) we get from (16)

$$r(i,j) = \frac{j+1}{i!} \sum_{k=0}^{i-1} \binom{i-1}{k} \langle (1+t)^{i+j} | x^k \rangle \langle (2+t)^i | x^{i-1-k} \rangle.$$

Since

$$\langle (1+t)^{i+j} | x^k \rangle = \binom{i+j}{k} k! \qquad \text{and} \qquad \langle (2+t)^i | x^{i-1-k} \rangle$$
$$= \binom{i}{k+1} 2^{k+1} (i-1-k)!$$

the desired result follows from a simple computation.

In our case formula (10) gives the recurrence formula for the associated polynomials

$$p_n'(x) = n(p_{n-1}''(x) + 3p_{n-1}'(x) + 2p_{n-1}(x))$$

for $n \geqslant 1$ and the initial conditions $p_n(0) = \delta_{n,0}$. We find

$$p_0(x) = 1, p_1(x) = 2x, p_2(x) = 4x^2 + 12x, p_3(x) = 8x^3 + 72x^2 + 132x.$$

3. Generating Functions

The generating function for the sequence of polynomials $p_n(x)$ follows immediately from the expansion (9)

$$e^{yf(t)} = \sum_{n=0}^{\infty} \frac{p_n(y)}{n!} t^n. \tag{17}$$

If we differentiate this relation with respect to y, we obtain after setting $y = 0$

$$f(t) = \sum_{n=1}^{\infty} \frac{p_n'(0)}{n!} t^n. \tag{18}$$

For $n > 0$ we have from (14)

$$r(n, 0) = \frac{1}{n!} \langle (1+t) | p_n(x) \rangle = \frac{1}{n!} (p_n(0) + p_n'(0)) = \frac{1}{n!} p_n'(0)$$

and so

$$\bar{f}(t)=\sum_{n=1}^{\infty} r(n,0)\,t^n. \tag{19}$$

Since $r(0,0)=1$ we have the generating function for the row $r(\cdot,0)$:

$$1+\bar{f}(t)=\sum_{n=0}^{\infty} r(n,0)\,t^n. \tag{20}$$

Recall that for every formal power series (see [1])

$$h(t)=1+a_1t+a_2t^2+\cdots$$

there is a unique formal power series $h(t)^{1/2}$ of the form

$$h(t)^{1/2}=1+b_1t+b_2t^2+\cdots$$

such that $(h(t)^{1/2})^2=h(t)$. From (12) we obtain the candidate for the series $\bar{f}(t)$:

$$\bar{f}(t)=\frac{1-3t-(1-6t+t^2)^{1/2}}{2t}. \tag{21}$$

We must show that the numerator in (21) has the correct form. Let

$$(1-6t+t^2)^{1/2}=1+b_1t+b_2t^2+\cdots.$$

We get the system of equations for the coefficients b_n:

$$\begin{aligned} &2b_1=-6\\ &2b_2+b_1^2=1\\ &2b_3+2b_1b_2=0\\ &2b_4+2b_1b_3+b_2^2=0\\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{22}$$

Successively we compute:
$b_1=-3$, $b_2=-4$, $b_3=-12$, $b_4=-44$, ... The numerator in (21) is the formal power series

$$4t^2+12t^4+44t^4+\cdots$$

and the compositional inverse $\bar{f}(t)$ of the series $f(t)$ should be

$$\bar{f}(t)=2t+6t^2+22t^3+\cdots .$$

A straightforward computation shows that the right side in (21) is really $\bar{f}(t)$ in the sense of the formal power series theory. We omit the proof.

The solution of the system (22) is connected with the numbers $r(i, 0)$, namely,

$$r(0, 0) = 1, r(1, 0) = -b_2/2, r(2, 0) = -b_3/2, \ldots .$$

We can find the row $r(\cdot, 0)$ independently of the other rows and columns.

Denote by $G_n(t)$ the generating functions of the nth row in Table I

$$G_n(t) = \sum_{i=0}^{\infty} r(i, n)\, t^i, \qquad n \in \mathbb{Z}. \tag{23}$$

We have the result

$$G_0(t) = 1 + \bar{f}(t) = \frac{1 - t - (1 - 6t + t^2)^{1/2}}{2t}. \tag{24}$$

THEOREM 3. *The generating functions $G_n(t)$ of the nth row of the numbers $r(i, j)$ are given by*

$$G_n(t) = (1 + \bar{f}(t))^{n+1} = \left(\frac{1 - t - (1 - 6t + t^2)^{1/2}}{2t}\right)^{n+1}. \tag{25}$$

Proof. If we differentiate the relation (17) k times with respect to y, we get

$$\bar{f}(t)^k = \sum_{n=0}^{\infty} \frac{p_n^{(k)}(0)}{n!}\, t^n.$$

By the binomial formula we have

$$\begin{aligned}(1 + \bar{f}(t))^{m+1} &= \sum_{k=0}^{m+1} \binom{m+1}{k} \bar{f}(t)^k = \sum_{k=0}^{m+1} \binom{m+1}{k} \sum_{n=0}^{\infty} \frac{p_n^{(k)}(0)}{n!}\, t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{m+1} \binom{m+1}{k} \langle t^k \mid p_n(x) \rangle \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \langle (1+t)^{m+1} \mid p_n(x) \rangle \frac{t^n}{n!} = \sum_{n=0}^{\infty} r(n, m)\, t^n.\end{aligned}$$

COROLLARY. *For every integer p the relation*

$$r(n, m) = \sum_{k=0}^{n} r(k, p-1)\, r(n-k, m-p) \tag{26}$$

holds. Especially

$$r(n, m) = \sum_{k=0}^{n} r(k, m-1)\, r(n-k, 0).$$

In other words, the convolution product of the pth and qth rows gives the $(p+q+1)$ th row in the table of the numbers $r(i, j)$.

Similarly, we define the generating functions for the columns. Let

$$H_i(t) = \sum_{j=0}^{\infty} \frac{r(i, j)}{j!}\, t^j.$$

Note that only the numbers $r(i, j)$, $i \geqslant 0, j \geqslant 0$, enter in this formal power series.

THEOREM 4. *For every non-negative number i the generating function for columns of the numbers* $r(i, j)$ *can be written in the form*

$$H_i(t) = \frac{1}{i!}\, e^t s_i(t). \tag{27}$$

Proof. The definition of a formal power series as a linear functional on P implies that

$$\langle H_i(t) \mid x^n \rangle = r(i, n)$$

for $i \geqslant 0$ and $n \geqslant 0$.

Define

$$f_i(t) = \frac{1}{i!}\, e^t s_i(t).$$

We have

$$\begin{aligned} \langle f_i(t) \mid x^n \rangle &= \frac{1}{i!} \langle e^t s_i(t) \mid x^n \rangle = \frac{1}{i!} \langle s_i(t) \mid e^t x^n \rangle \\ &= \frac{1}{i!} \langle (t+1)^n \mid s_i(x) \rangle = r(i, n). \end{aligned}$$

It follows that $\langle H_i(t) \mid x^n \rangle = \langle f_i(t) \mid x^n \rangle$ which implies $H_i(t) = f_i(t)$. Note that the series

$$e^{yt} = 1 + \frac{yt}{1!} + \frac{y^2 t^2}{2!} + \cdots$$

implies $\langle e^{yt} | p(x) \rangle = p(y)$ and $e^{yt}p(x) = p(x+y)$ for every $y \in \mathbb{C}$ and every polynomial $p(x) \in P$. It is also easy to see that $\langle p(t) | q(x) \rangle = \langle q(t) | p(x) \rangle$ for any two polynomials $p(x)$ and $q(x)$. The proof is complete.

We now go a step further. It is possible to construct a generating function for $H_i(t)$. For a fixed $s \in \mathbb{C}$ we define

$$\mathscr{G}(s, t) = \sum_{i=0}^{\infty} H_i(s)\, t^i.$$

The function $\mathscr{G}(s, t)$ can be written in the closed form. Recall that

$$(1 + \bar{f}(t))\, e^{s\bar{f}(t)} = \sum_{i=0}^{\infty} \frac{s_i(s)}{i!}\, t^i$$

because of the expansion (9). We obtain

$$\begin{aligned}\mathscr{G}(s, t) &= \sum_{i=0}^{\infty} e^s \frac{s_i(s)}{i!}\, t^i = e^s(1 + \bar{f}(t))\, e^{s\bar{f}(t)} \\ &= e^s G_0(t)\, e^{s\bar{f}(t)}.\end{aligned}$$

Note that the form $e^{s(1+\bar{f}(t))}$ is not correct because the series $1 + \bar{f}(t)$ has the zeroth coefficient different from 0.

Differentiating with respect to s we get

$$\begin{aligned} D_s \mathscr{G}(s, t) &= e^s G_0(t)^2\, e^{s\bar{f}(t)}, \\ D_s^2 \mathscr{G}(s, t) &= e^s G_0(t)^3\, e^{s\bar{f}(t)}. \end{aligned}$$

Since $f(\bar{f}(t)) = t$ we have an equation for the function $G_0(t)$:

$$tG_0(t)(1 + G_0(t)) = G_0(t) - 1.$$

THEOREM 5. *The function $\mathscr{G}(s, t)$ is a formal solution of the equation*

$$(tD_s^2 + (t-1)\, D_s + 1)\, \mathscr{G}(s, t) = 0$$

with the boundary condition

$$D_s^2 \mathscr{G}(0, t) = \mathscr{G}(0, t)^2 \neq 0.$$

Proof. A simple verification.

4. The Group Structure

Table I contains the numbers $r(i, j)$ for $j \geqslant 0$ only. But we also can write these for $j < 0$. One method is by using generating functions $G_i(t)$. The other, simplest, way to compute $r(i, j)$ is with the recurrence relation

$$r(i, j-1) = r(i, j) - r(i-1, j) - r(i-1, j+1).$$

Table II the central part of the extended table for numbers $r(i, j)$. Denote $r(\cdot, j) = a_{j+1}$. Define the convolution product $x * y$ of sequences $x = (x_0, x_1, x_2, ...,)$ and $y = (y_0, y_1, y_2, ...,)$:

$$(x * y)_n = \sum_{k=0}^{n} x_k y_{n-k}.$$

It is easy to see that $(x * y) * z = x * (y * z)$ for all sequences x, y, and z. For our sequences a_k we find the following properties:

$$a_i * a_j = a_{i+j}$$

$$a_i * a_0 = a_i, \qquad a_i * a_{-i} = a_0.$$

We have

Theorem 6. *The set* $\{\cdots, a_{-2}, a_{-1}, a_0, a_1, a_2, ...,\}$ *with the convolution product is a infinite cyclic group. The unit in this group is the sequence* $a_0 = (1, 0, 0, ...,)$. *The convolutional inverse of the sequence* a_i *is the sequence* a_{-i}. *For any sequence* y *the equation* $a_i * x = y$ *has the unique solution* $x = a_{-i} * y$. *The group generator is the sequence* a_1.

TABLE II

....							
$r(\cdot, 2)$	1	6	30	146	714	...	a_3
$r(\cdot, 1)$	1	4	16	68	304	...	a_2
$r(\cdot, 0)$	1	2	6	22	90	...	a_1
$r(\cdot, -1)$	1	0	0	0	0	...	a_0
$r(\cdot, -2)$	1	−2	−2	−2	−22	...	a_{-1}
$r(\cdot, -3)$	1	−4	0	−4	−16	...	a_{-2}
$r(\cdot, -4)$	1	−6	6	−2	−6	...	a_{-3}
....							

The generating function of numbers in a_1 is given by (24) and (20). From the relation

$$1-t-(1-6t+t^2)^{1/2}=2\sum_{n=1}^{\infty} r(n-1,0)\,t^n \tag{28}$$

we obtain after formal derivation

$$-1-(t-3)(1-6t+t^2)^{-1/2}=2\sum_{n=0}^{\infty}(n+1)\,r(n,0)\,t^n. \tag{29}$$

Multiply (29) by $1-6t+t^2$. We get, again using (28), the relation

$$1+t+(t-3)\sum_{n=1}^{\infty} r(n-1,0)\,t^n$$
$$=(1-6t+t^2)\sum_{n=0}^{\infty}(n+1)\,r(n,0)\,t^n.$$

The equality principle of formal power series gives a new result:

THEOREM 7. *The numbers $r(n,0)$ admit a three-term recurrent formula*

$$(n+1)\,r(n,0)-3(2n-1)\,r(n-1,0)+(n-2)\,r(n-2,0)=0, \qquad n\geqslant 2 \tag{30}$$

with the initial conditions $r(0,0)=1$ and $r(1,0)=2$.

We can now get the numbers $r(n,0)$ very quickly using (30): $r(6,0)=1806$, $r(7,0)=8558$, $r(8,0)=41586$, $r(9,0)=206098$, $r(10,0)=1037718$, $r(11,0)=5293446$, $r(12,0)=27297738$.

We also can express the numbers $r(n,0)$ by Legendre polynomials $P_k(x)$. The formal power series

$$(1-2xt+t^2)^{-1/2}=\sum_{k=0}^{\infty} P_k(x)\,t^k$$

gives us the numbers $r(n,0)$ in a closed form. It is easy to see that

$$2\sum_{n=0}^{\infty} r(n,0)\,t^{n+1}=1-t-\sum_{n=0}^{\infty} P_n(3)\,t^{n+2}$$
$$+6\sum_{n=0}^{\infty} P_n(3)\,t^{n+1}-\sum_{n=1}^{\infty} P_n(3)\,t^n.$$

It follows that

$$2r(n,0)=-P_{n-1}(3)+6P_n(3)-P_{n+1}(3) \qquad \text{for} \quad n\geqslant 1.$$

THEOREM 8. *The numbers $r(n, 0)$ can be written in the form*

$$r(n, 0) = -\tfrac{1}{2}(P_{n-1}(3) - 6P_n(3) + P_{n+1}(3))$$

for every $n \geq 1$, where $P_k(x)$ denote the Legendre polynomials.

REFERENCES

1. I. NIVEN, Formal power series, *Amer. Math. Monthly* **76** (1969), 871–889.
2. S. ROMAN, "The Umbral Calculus," Academic Press, Orlando, FL, 1984.