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# Some results for Apostol-type polynomials associated with umbral algebra

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**Abstract**

A family of the Apostol-type polynomials was introduced and investigated recently by Luo and Srivastava (see (Appl. Math. Comput. 217:5702-5728, 2011)). In this paper, we study this polynomial family on *P*, the algebra of polynomials in a single variable *x* over all linear functional on *P*. By using the way of the umbral algebra, we obtain some fundamental properties of the generalized Apostol-type polynomials. We also show some special cases which include the corresponding results of Dere and Simsek *etc.*

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## 1 Introduction, definitions and motivation

Throughout this paper, we make use of the following conventional notations: $\mathbb{N} = \{1,2,3,\ldots\}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the set of complex numbers.

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of order $\alpha$, are usually defined by means of the following generating functions (see, for details, [1, pp.532-533] and [2]):

$$\left(\frac{z}{e^z-1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} \quad \left(|z| < 2\pi\right), \tag{1.1}$$

$$\left(\frac{2}{e^z+1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!} \quad \left(|z| < \pi\right) \tag{1.2}$$

and

$$\left(\frac{2z}{e^z+1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x)\frac{z^n}{n!} \quad \left(|z| < \pi\right). \tag{1.3}$$

It is easy to see that $B_n(x)$, $E_n(x)$ and $G_n(x)$ are given, respectively, by

$$B_n(x) := B_n^{(1)}(x), \qquad E_n(x) := E_n^{(1)}(x) \quad \text{and} \\ G_n(x) := G_n^{(1)}(x) \quad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right). \tag{1.4}$$

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For the classical Bernoulli numbers $B_n$, the classical Euler numbers $E_n$ and the classical Genocchi numbers $G_n$ of order $n$, we have

$$B_n := B_n(0) = B_n^{(1)}(0), \qquad E_n := E_n(0) = E_n^{(1)}(0) \quad \text{and} \quad G_n := G_n(0) = G_n^{(1)}(0), \tag{1.5}$$

respectively.

Some interesting analogues of the classical Bernoulli polynomials and numbers were first investigated by Apostol (see [3, p.165, Eq. (3.1)]) and (more recently) by Srivastava (see [4, pp.83-84]). We begin by recalling Apostol's definitions as follows.

**Definition 1.1** (Apostol [3]; see also Srivastava [4]) The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ ($\lambda \in \mathbb{C}$) are defined by means of the following generating function:

$$\frac{ze^{xz}}{\lambda e^z - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x;\lambda)\frac{z^n}{n!}$$
$$\bigl(|z| < 2\pi \text{ when } \lambda = 1; |z| < |\log\lambda| \text{ when } \lambda \neq 1\bigr) \tag{1.6}$$

with, of course,

$$B_n(x) = \mathcal{B}_n(x;1) \quad \text{and} \quad \mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda), \tag{1.7}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers.

Recently, Luo and Srivastava [5] further extended the Apostol-Bernoulli polynomials as the so-called Apostol-Bernoulli polynomials of order $\alpha$.

**Definition 1.2** (Luo and Srivastava [5]) The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ ($\lambda \in \mathbb{C}$) of order $\alpha$ ($\alpha \in \mathbb{N}$) are defined by means of the following generating function:

$$\left(\frac{z}{\lambda e^z - 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}$$
$$\bigl(|z| < 2\pi \text{ when } \lambda = 1; |z| < |\log\lambda| \text{ when } \lambda \neq 1\bigr) \tag{1.8}$$

with, of course,

$$B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0;\lambda), \tag{1.9}$$

where $\mathcal{B}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Bernoulli numbers of order $\alpha$.

In this sequel, Luo [6] gave an analogous extension of the generalized Euler polynomials which is the so-called Apostol-Euler polynomials of order $\alpha$.

**Definition 1.3** (Luo [6]) The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order $\alpha$ ($\alpha, \lambda \in \mathbb{C}$) are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \quad \bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr) \tag{1.10}$$

with, of course,

$$E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{E}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0;\lambda), \tag{1.11}$$

where $\mathcal{E}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Euler numbers of order $\alpha$.

On the subject of the Genocchi polynomials $G_n(x)$ and their various extensions, a remarkably large number of investigations have appeared in the literature (see, for example, [7–11]). Moreover, Luo (see [12]) introduced and investigated the Apostol-Genocchi polynomials of (real or complex) order $\alpha$, which are defined as follows.

**Definition 1.4** The Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ ($\lambda \in \mathbb{C}$) of order $\alpha$ ($\alpha \in \mathbb{N}$) are defined by means of the following generating function:

$$\left(\frac{2z}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \quad \bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr) \tag{1.12}$$

with, of course,

$$\begin{aligned} &\mathcal{G}_n^{(\alpha)}(x) = \mathcal{G}_n^{(\alpha)}(x;1), \qquad \mathcal{G}_n^{(\alpha)}(\lambda) := \mathcal{G}_n^{(\alpha)}(0;\lambda), \\ &\mathcal{G}_n(x;\lambda) := \mathcal{G}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{G}_n(\lambda) := \mathcal{G}_n^{(1)}(\lambda), \end{aligned} \tag{1.13}$$

where $\mathcal{G}_n(\lambda)$, $\mathcal{G}_n^{(\alpha)}(\lambda)$ and $\mathcal{G}_n(x;\lambda)$ denote the so-called Apostol-Genocchi numbers, the Apostol-Genocchi numbers of order $\alpha$ and the Apostol-Genocchi polynomials, respectively.

Ozden *et al.* [13] introduced and investigated the following unification (and generalization) of the generating functions of the three families of Apostol-type polynomials:

$$\begin{aligned} &\frac{2^{1-\kappa} z^{\kappa}}{\beta^b e^z - a^b} e^{xz} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x;\kappa,a,b)\frac{z^n}{n!} \\ &\quad \bigl(|z| < 2\pi \text{ when } \beta = a; |z| < \bigl|b\log(\beta/a)\bigr| \text{ when } \beta \neq a; \kappa,\beta \in \mathbb{C}; a,b \in \mathbb{C}\backslash\{0\}\bigr). \end{aligned} \tag{1.14}$$

It is found from [14] that Ozden further gave an extension of the above definition (1.14) as follows:

**Definition 1.5**

$$\begin{aligned} &\left(\frac{2^{1-\kappa} z^{\kappa}}{\beta^b e^z - a^b}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b)\frac{z^n}{n!} \\ &\quad \bigl(\alpha \in \mathbb{N}; |z| < 2\pi \text{ when } \beta = a; |z| < \bigl|b\log(\beta/a)\bigr| \text{ when } \beta \neq a; \\ &\qquad \kappa,\beta \in \mathbb{C}; a,b \in \mathbb{C}\backslash\{0\}\bigr). \end{aligned} \tag{1.15}$$

The author [15] obtained a unified relation between the $\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b)$ and the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$, and gave some identities of $\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b)$.

Recently, Luo and Srivastava [16] introduced more general unification (and generalization) of the above-mentioned three families of the generalized Apostol-type polynomials.

**Definition 1.6** (Luo and Srivastava [16]) The generalized Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ ($\alpha \in \mathbb{N}$; $\lambda, \mu, \nu \in \mathbb{C}$) of order $\alpha$ are defined by means of the following generating function:

$$\left(\frac{2^\mu z^\nu}{\lambda e^z + 1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)\frac{z^n}{n!} \quad \left(|z| < \left|\log(-\lambda)\right|\right). \tag{1.16}$$

Clearly, we have

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = (-1)^\alpha \mathcal{F}_n^{(\alpha)}(x;-\lambda;0;1) \quad (\alpha \in \mathbb{N}), \tag{1.17}$$

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;0) \quad (\alpha \in \mathbb{C}), \tag{1.18}$$

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;1) \quad (\alpha \in \mathbb{N}), \tag{1.19}$$

$$\mathcal{Y}_{n,\beta}(x;\kappa,a,b) = -\frac{1}{a^b}\mathcal{F}_n^{(1)}\left(x;-\left(\frac{\beta}{a}\right)^b;1-\kappa;\kappa\right) \tag{1.20}$$

and

$$\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b) = (-1)^\alpha \frac{1}{a^{b\alpha}}\mathcal{F}_n^{(\alpha)}\left(x;-\left(\frac{\beta}{a}\right)^b;1-\kappa;\kappa\right). \tag{1.21}$$

In [5, 6, 17, 18], the authors have researched some elementary properties of the Apostol-type polynomials, and some relationships among the Apostol-type polynomials. More investigations about this subject can be found in [13, 15, 16, 19–30].

The aim of this paper is to study the generalized Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ on the umbral algebra by using the way as the reference [31–33]. We research some fundamental properties of this polynomial family. Some special cases, which include the corresponding results [31–33], are also considered.

## 2 Umbral algebra of Roman

We can use the following notations and definitions, which are given by Roman [34, pp.1-125].

Let $P$ be the algebra of polynomials in a single variable $x$ over the field of complex numbers. Let $P^*$ be the vector space of all linear functionals on $P$. Let $\langle L|p(x)\rangle$ be the action of a linear functional $L$ on a polynomial $p(x)$. Let $\mathcal{F}$ denote the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!}t^k. \tag{2.1}$$

Such algebra is called umbral algebra. Each $f \in \mathcal{F}$ defines a linear functional on $P$ and

$$a_k = \langle f(t)|t^k\rangle \tag{2.2}$$

for all $k \geq 0$.

The order $o(f(t))$ of a power series $f(t)$ is the smallest integer $k$ for which the coefficient of $t^k$ does not vanish. A series $f(t)$ for which $o(f(t)) = 1$ will be called a delta series. When we are considering a delta series $f(t)$ in $\mathcal{F}$ as a linear functional, we will refer to it as a delta functional.

It is well known that $\langle t^k|x^n\rangle = n!\delta_{n,k}$, where $\delta_{n,k}$ denotes the Kronecker symbol. For all $f(t)$ in $\mathcal{F}$,

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k.$$

Let $f(t)$ and $g(t)$ be in $\mathcal{F}$. Then we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle. \tag{2.3}$$

For $y \in \mathbb{C}$, then the evaluation functional is defined to be the power series $e^{yt}$. By (2.2), we have

$$\langle e^{yt}|p(x)\rangle = p(y) \tag{2.4}$$

for all $p(x)$ in $P$. The forward difference functional is the delta functional $e^{yt} - 1$ and

$$\langle e^{yt} - 1|p(x)\rangle = p(y) - p(0). \tag{2.5}$$

The Abel functional is the delta functional $te^{yt}$. We have

$$\langle te^{yt}|p(x)\rangle = p'(y).$$

The Sheffer polynomials are defined by means of the following generating function

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}. \tag{2.6}$$

Roman [34] proved the following theorem which is represented by the Sheffer polynomials (or Sheffer sequences) explicitly.

**Theorem 2.1** *Let $f(t)$ be a delta series and let $g(t)$ be an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality conditions*

$$\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k} \tag{2.7}$$

*for all $k \in \mathbb{N}_0$.*

The sequence $s_n(x)$ in (2.7) is the Sheffer polynomials for pair $(g(t), f(t))$, where $g(t)$ must be invertible and $f(t)$ must be delta series. The Sheffer polynomials for pair $(g(t), t)$ is the Appell polynomials or the Appell sequences for $g(t)$.

The Appell polynomials, the Bernoulli polynomials, the Euler polynomials, the Genocchi polynomials and the Genocchi polynomials of higher order belong to the family of the Sheffer polynomials (*cf.* [31, 34–36]).

The Sheffer polynomials satisfy the following relations:

$$s_n(x) = g(t)^{-1}x^n, \tag{2.8}$$

derivative formula

$$ts_n(x) = s'_n(x) = ns_{n-1}(x), \tag{2.9}$$

recurrence formula

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right)s_n(x), \tag{2.10}$$

expansion theorem

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|s_k(x)\rangle}{k!} g(t)t^k, \tag{2.11}$$

multiplication theorem, for $\alpha \neq 0$,

$$s_n(\alpha x) = \alpha^n \frac{g(t)}{g(\frac{t}{\alpha})} s_n(x), \tag{2.12}$$

and

$$\bigl\langle h(t)|p(ax)\bigr\rangle = \bigl\langle h(at)|p(x)\bigr\rangle. \tag{2.13}$$

## 3 The Apostol-type polynomials on $\mathcal{F}$

We see from Definition 1.6 and (2.6) that the generalized Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ also belong to the Sheffer polynomials where $g(t) = (\frac{\lambda e^t+1}{2^\mu t^\nu})^\alpha$.

In this section, by using the properties of the Sheffer sequences and also the Appell sequences, we prove many fundamental properties of the generalized Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ defined by (1.16).

By using (2.8) and (1.16), we arrive at the following lemma.

**Lemma 3.1**

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = \left(\frac{2^\mu t^\nu}{\lambda e^t + 1}\right)^\alpha x^n. \tag{3.1}$$

**Theorem 3.2**

$$\left\langle \frac{(\lambda e^t+1)^k}{t^{\nu-1}} \middle| \mathcal{F}_n^{(1)}(x;\lambda;\mu;\nu) \right\rangle = 2^\mu \lambda^{k-1} n(k-1)! \sum_{j=0}^{k-1} \left(1+\frac{1}{\lambda}\right)^{k-j-1} \frac{S(n-1,j)}{(k-j-1)!}, \tag{3.2}$$

*where $\mathcal{F}_n^{(1)}(x;\lambda;\mu;\nu)$ and $S(a,b)$ denote the first-order generalized Apostol-type polynomials and the Stirling numbers of the second kind, respectively.*

*Proof* By Lemma 3.1, we obtain

$$\left\langle \frac{(\lambda e^t+1)^k}{t^{\nu-1}} \middle| \mathcal{F}_n^{(1)}(x;\lambda;\mu;\nu) \right\rangle = \left\langle \frac{(\lambda e^t+1)^k}{t^{\nu-1}} \middle| \frac{2^\mu t^\nu}{\lambda e^t+1} x^n \right\rangle.$$

By using (2.3) and (2.9), we get

$$\begin{aligned} &\left\langle \frac{(\lambda e^t+1)^k}{t^{\nu-1}} \middle| \mathcal{F}_n^{(1)}(x;\lambda;\mu;\nu) \right\rangle \\ &\quad = 2^\mu \lambda^{k-1} n \sum_{j=0}^{k-1} \frac{(k-1)!}{(k-j-1)!} \left(1+\frac{1}{\lambda}\right)^{k-j-1} \left\langle \frac{(e^t-1)^j}{j!} \middle| x^{n-1} \right\rangle. \end{aligned} \tag{3.3}$$

Setting

$$S(n-1,j) = \frac{1}{j!} \left\langle (e^t-1)^j | x^{n-1} \right\rangle,$$

where $S(n-1,j)$ denotes the Stirling numbers of second kind (*cf.* [34, p.59]) in (3.3), we arrive at the desired result. □

We deduce the following formulas.

Letting $\lambda \longmapsto -\lambda$, taking $\mu = 0$ and $\nu = 1$ in (3.2) and noting relation (1.17), we deduce the following result.

**Corollary 3.3** (see [32, Remark 19])

$$\left\langle (1-\lambda e^t)^k | \mathcal{B}_n(x;\lambda) \right\rangle = (-1)^k \lambda^{k-1} n(k-1)! \sum_{j=0}^{k-1} \left(1-\frac{1}{\lambda}\right)^{k-j-1} \frac{S(n-1,j)}{(k-j-1)!}, \tag{3.4}$$

*where $\mathcal{B}_n(x;\lambda)$ and $S(a,b)$ denote the Apostol-Bernoulli polynomials and the Stirling numbers of the second kind, respectively.*

Taking $\mu = 1$ and $\nu = 0$ in (3.2) and noting relation (1.18), we deduce the following result.

**Corollary 3.4** (see [32, Remark 21])

$$\left\langle t(\lambda e^t+1)^k | \mathcal{E}_n(x;\lambda) \right\rangle = 2\lambda^{k-1} n(k-1)! \sum_{j=0}^{k-1} \left(1+\frac{1}{\lambda}\right)^{k-j-1} \frac{S(n-1,j)}{(k-j-1)!}, \tag{3.5}$$

*where $\mathcal{E}_n(x;\lambda)$ and $S(a,b)$ denote the Apostol-Euler polynomials and the Stirling numbers of the second kind, respectively.*

Taking $\mu = \nu = 1$ in (3.2) and noting relation (1.19), we deduce the following result.

**Corollary 3.5** (see [32, Remark 20])

$$\left\langle (\lambda e^t+1)^k | \mathcal{G}_n(x;\lambda) \right\rangle = 2\lambda^{k-1} n(k-1)! \sum_{j=0}^{k-1} \left(1+\frac{1}{\lambda}\right)^{k-j-1} \frac{S(n-1,j)}{(k-j-1)!}, \tag{3.6}$$

*where $\mathcal{G}_n(x;\lambda)$ and $S(a,b)$ denote the Apostol-Genocchi polynomials and the Stirling numbers of the second kind, respectively.*

Setting $\lambda = 1$ in (3.6), we deduce Theorem 2 in the work [31, p.758, Theorem 2].

**Corollary 3.6**

$$\left\langle \left(e^t+1\right)^k \middle| G_n(x)\right\rangle = 2n(k-1)!\sum_{j=0}^{k-1} 2^{k-j-1}\frac{S(n-1,j)}{(k-j-1)!}, \tag{3.7}$$

*where $G_n(x)$ and $S(a,b)$ denote the Genocchi polynomials and the Stirling numbers of the second kind, respectively.*

Letting $k \mapsto m$, taking $\lambda = -(\frac{\beta}{a})^b$, $\mu = 1-\kappa$, $\nu = \kappa$ in (3.2) and noting relation (1.20), thus we deduce the following formulas of the polynomials $\mathcal{Y}_{n,\beta}(x;\kappa,a,b)$.

**Corollary 3.7**

$$\begin{aligned}&\left\langle \left[1-\left(\frac{\beta}{a}\right)^b e^t\right]^m t^{1-\kappa} \middle| \mathcal{Y}_{n,\beta}(x;\kappa,a,b)\right\rangle \\ &\quad = (-1)^m 2^{1-\kappa}\beta^{b(m-1)}a^{-bm}n(m-1)!\sum_{j=0}^{m-1}\left[1-\left(\frac{a}{\beta}\right)^b\right]^{m-j-1}\frac{S(n-1,j)}{(m-j-1)!},\end{aligned} \tag{3.8}$$

*where $\mathcal{Y}_{n,\beta}(x;\kappa,a,b)$ and $S(a,b)$ denote the generalization of Apostol type polynomials defined by* (1.14) *and the Stirling numbers of the second kind, respectively.*

By using (2.9), we arrive at the following lemma.

**Lemma 3.8**

$$t\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = n\mathcal{F}_{n-1}^{(\alpha)}(x;\lambda;\mu;\nu). \tag{3.9}$$

**Remark 3.9** An alternative proof of Lemma 3.8 is also obtained from (1.16) by using derivative with respect to $x$. By Lemma 3.8, one can see that

$$\frac{1}{t}\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = \frac{1}{n+1}\mathcal{F}_{n+1}^{(\alpha)}(x;\lambda;\mu;\nu). \tag{3.10}$$

**Theorem 3.10**

$$\left(\frac{t^{\nu-1}}{\lambda e^t+1}\right)\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = \frac{1}{2^\mu(n+1)}\mathcal{F}_{n+1}^{(\alpha+1)}(x;\lambda;\mu;\nu). \tag{3.11}$$

*Proof* By Lemma 3.1, we obtain

$$\left(\frac{t^{\nu-1}}{\lambda e^t+1}\right)\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) = \frac{t^{\nu-1}}{\lambda e^t+1}\left(\frac{2^\mu t^\nu}{\lambda e^t+1}\right)^\alpha x^n. \tag{3.12}$$

After some calculations in the above equation, we have

$$\left(\frac{t^{\nu-1}}{\lambda e^t+1}\right)\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)=\frac{1}{2^\mu t}\left(\frac{2^\mu t^\nu}{\lambda e^t+1}\right)^{\alpha+1}x^n. \tag{3.13}$$

Using (1.16) and (3.10), we obtain the desired result. □

Letting $\lambda\longmapsto-\lambda$, taking $\mu=0$ and $\nu=1$ in (3.11) and noting relation (1.17), we deduce the following result.

**Corollary 3.11** (see [32, Remark 32])

$$\left(\frac{1}{1-\lambda e^t}\right)\mathcal{B}_n^{(\alpha)}(x;\lambda)=\frac{1}{n+1}\mathcal{B}_{n+1}^{(\alpha+1)}(x;\lambda). \tag{3.14}$$

Taking $\mu=1$ and $\nu=0$ in (3.11) and noting relation (1.18), we deduce the following result.

**Corollary 3.12** (see [32, Remark 33])

$$\frac{1}{t(\lambda e^t+1)}\mathcal{E}_n^{(\alpha)}(x;\lambda)=\frac{1}{2(n+1)}\mathcal{E}_{n+1}^{(\alpha+1)}(x;\lambda). \tag{3.15}$$

Taking $\mu=\nu=1$ in (3.11) and noting relation (1.19), we deduce the following result.

**Corollary 3.13** (see [32, Remark 34])

$$\left(\frac{1}{\lambda e^t+1}\right)\mathcal{G}_n^{(\alpha)}(x;\lambda)=\frac{1}{2(n+1)}\mathcal{G}_{n+1}^{(\alpha+1)}(x;\lambda). \tag{3.16}$$

Setting $\lambda=1$ in the above equation, we deduce Lemma 3 in [31, p.758].

**Corollary 3.14**

$$\left(\frac{1}{e^t+1}\right)G_n^{(\alpha)}(x)=\frac{1}{2(n+1)}G_{n+1}^{(\alpha+1)}(x). \tag{3.17}$$

Taking $\lambda=-(\frac{\beta}{a})^b$, $\mu=1-\kappa$, $\nu=\kappa$ in (3.11) and noting relation (1.21), we deduce

**Corollary 3.15**

$$\frac{-t^{\kappa-1}}{a^b[1-(\frac{\beta}{a})^b e^t]}\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b)=\frac{1}{2^{1-\kappa}(n+1)}\mathcal{Y}_{n+1,\beta}^{(\alpha+1)}(x;\kappa,a,b). \tag{3.18}$$

An integral representation of $\langle\frac{e^{ta}-1}{2t}|\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)\rangle$ is given by the following theorem.

**Theorem 3.16**

$$\left\langle\frac{e^{ta}-1}{2t}\middle|\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)\right\rangle=\frac{1}{2}\int_0^a\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)\,dx. \tag{3.19}$$

*Proof* By using Lemma 3.8, we have

$$\left\langle \frac{e^{ta}-1}{2t} \middle| \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) \right\rangle = \left\langle \frac{e^{ta}-1}{2t} \middle| \frac{1}{n+1} t\mathcal{F}_{n+1}^{(\alpha)}(x;\lambda;\mu;\nu) \right\rangle.$$

By (2.3), we obtain

$$\left\langle \frac{e^{ta}-1}{2t} \middle| \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) \right\rangle = \frac{1}{2(n+1)} \left\langle e^{ta}-1 \middle| \mathcal{F}_{n+1}^{(\alpha)}(x;\lambda;\mu;\nu) \right\rangle.$$

Using (2.5), we obtain the desired result. □

Setting $\lambda = \mu = \nu = 1$ in (3.19) and noting relation (1.19), we deduce the Theorem 3 in [31, p.758].

**Corollary 3.17**

$$\left\langle \frac{e^{ta}-1}{2t} \middle| G_n^{(\alpha)}(x) \right\rangle = \frac{1}{2} \int_0^a G_n^{(\alpha)}(x)\,dx. \tag{3.20}$$

A recurrence formula for $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ is given by the next theorem.

**Theorem 3.18** (Recurrence formula)

$$\begin{aligned} &\mathcal{F}_{n+\nu}^{(\alpha+1)}(x;\lambda;\mu;\nu) \\ &\quad = \frac{2^\mu (n+1)(n+1)!}{\alpha(n+\nu)!} \left[ \left(1 - \frac{\alpha\nu}{n+1}\right) \mathcal{F}_{n+1}^{(\alpha)}(x;\lambda;\mu;\nu) + (\alpha - x)\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) \right]. \end{aligned} \tag{3.21}$$

*Proof* Setting

$$g(t) = \left( \frac{\lambda e^t + 1}{2^\mu t^\nu} \right)^\alpha$$

in (2.10), one can obtain

$$\begin{aligned} &\mathcal{F}_{n+1}^{(\alpha)}(x;\lambda;\mu;\nu) \\ &\quad = \left( x - \alpha + \frac{\alpha}{\lambda e^t + 1} + \frac{\alpha\nu}{t} \right) \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) \\ &\quad = (x-\alpha)\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) + \frac{\alpha}{t^{\nu-1}} \cdot \frac{t^{\nu-1}}{\lambda e^t + 1} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) + \alpha\nu \cdot \frac{1}{t} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu). \end{aligned}$$

By using Theorem 3.10 and (3.10), we have

$$\begin{aligned} \mathcal{F}_{n+1}^{(\alpha)}(x;\lambda;\mu;\nu) = {} & (x-\alpha)\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu) + \frac{\alpha(n+\nu)!}{2^\mu (n+1)(n+1)!} \mathcal{F}_{n+\nu}^{(\alpha+1)}(x;\lambda;\mu;\nu) \\ & + \frac{\alpha\nu}{n+1} \mathcal{F}_{n+1}^{(\alpha)}(x;\lambda;\mu;\nu). \end{aligned}$$

After some calculations in the above equation, we get the desired result. □

Letting $\lambda \longmapsto -\lambda$, taking $\mu = 0$ and $\nu = 1$ in (3.21) and noting relation (1.17), we deduce the following known result.

**Corollary 3.19** (see, *e.g.*, [32, Remark 38])

$$\mathcal{B}_{n+1}^{(\alpha+1)}(x;\lambda) = \frac{1}{\alpha}\big[(\alpha - n - 1)\mathcal{B}_{n+1}^{(\alpha)}(x;\lambda) + (n+1)(x-\alpha)\mathcal{B}_{n}^{(\alpha)}(x;\lambda)\big]. \tag{3.22}$$

Taking $\mu = 1$ and $\nu = 0$ in (3.21) and noting relation (1.18), we deduce the following known result.

**Corollary 3.20** (see, *e.g.*, [32, Remark 39])

$$\mathcal{E}_{n}^{(\alpha+1)}(x;\lambda) = \frac{2(n+1)^2}{\alpha}\big[\mathcal{E}_{n+1}^{(\alpha)}(x;\lambda) + (\alpha - x)\mathcal{E}_{n}^{(\alpha)}(x;\lambda)\big]. \tag{3.23}$$

Taking $\mu = \nu = 1$ in (3.21) and noting relation (1.19), we deduce the following known result.

**Corollary 3.21** (see, *e.g.*, [32, Remark 40])

$$\mathcal{G}_{n+1}^{(\alpha+1)}(x;\lambda) = \frac{2}{\alpha}\big[(n-\alpha+1)\mathcal{G}_{n+1}^{(\alpha)}(x;\lambda) + (n+1)(\alpha - x)\mathcal{G}_{n}^{(\alpha)}(x;\lambda)\big]. \tag{3.24}$$

Setting $\lambda = 1$ in the above equation, we have the following.

**Corollary 3.22** (see [31, p.759, Theorem 4])

$$G_{n+1}^{(\alpha+1)}(x) = \frac{2}{\alpha}\big[(n-\alpha+1)G_{n+1}^{(\alpha)}(x) + (n+1)(\alpha - x)G_{n}^{(\alpha)}(x)\big]. \tag{3.25}$$

Taking $\lambda = -(\frac{\beta}{a})^b$, $\mu = 1-\kappa$, $\nu = \kappa$ in (3.21) and noting relation (1.21), thus we deduce the following result.

**Corollary 3.23**

$$\begin{aligned}\mathcal{Y}_{n+\kappa,\beta}^{(\alpha+1)}(x;\kappa,a,b) = {}& \frac{2^{1-\kappa}(n+1)(n+1)!}{\alpha a^b (n+\kappa)!} \\ & \times \left[\left(\frac{\alpha\kappa}{n+1} - 1\right)\mathcal{Y}_{n+1,\beta}^{(\alpha)}(x;k,a,b) + (x-\alpha)\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b)\right].\end{aligned} \tag{3.26}$$

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally in writing this paper, and read and approved the final manuscript.

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