

First Contact Remarks on Umbra Difference Calculus References Streams

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Abstract

The reference links to the modern "classical umbral calculus" (before that properly called Blissard's symbolic method) and to Steffensen-actuarialist.....are numerous. The reference links to the EFOC (Extended Finite Operator Calculus) founded by Rota with numerous outstanding Coworkers, Followers and Others ... and links to Roman-Rota functional formulation of umbra calculiare giant numerous. The reference links to the difference q -calculus - umbra-way treated or without even referring to umbra....are plenty numerous. These reference links now result in counting in thousands the relevant papers and many books . The purpose of the present attempt is to offer one of the keys to enter the world of those thousands of references. This is the first glimpse - not much structured - if at all. The place of entrance was chosen selfish being subordinated to my present interests and workshop purposes. You are welcomed to add your own information.

Motto Herman Weyl April 1939 *The modern evolution has on the whole been marked by a trend of algebraization ...* - from an invited address before AMS in conjunction with the centennial celebration of Duke University.

1 I. First Contact Umbral Remark

" R -calculus" and specifically q -calculus - might be considered as specific cases of umbral calculus as illustrated by a] and b] below :

a) Example 2.2.from [1] quotation:....”The example of ψ -derivative:

$$Q(\partial_\psi) = R(q\hat{Q})\partial_0 \equiv \partial_R$$

i.e.

$$\left\{ [R(q^n)!]^{-1} \right\}_{n \geq 0}$$

one may find in [2] where an advanced theory of general quantum coherent states is being developed. The operator $R(q\hat{Q})$ is not recognized in [2] as an example of ψ -derivative”.

b) specifically q -calculus - might be considered as a particular case of umbral calculus .It is so recognized since tenths of years in numerous references to q -calculus (**Roman, Rota** and Others). From algebraic point of view we deal with specific case of Extended Umbral Calculus (this being particularly evident in the ”Extended Finite Operator Calculus” formulation ; see: [3,4,5,1] (<http://ii.uwb.edu.pl/akk/publ1.htm>) .

The q -case of umbra - has been -with autonomy - q - evolving **in a specification dependent way** evolving -with q -autonomy - despite the Ward 1936 paper [6] ”ultimately” extending the Jackson formulation of q -difference calculi to arbitrary ”reasonable” difference calculi - in the spirit of nowadays Extended Finite Operator Calculus (see: [1] math.CO/0312397 ; see there for ψ -difference calculus and also for **Markowsky** [7] general difference calculus - in Rota-like operator formulation - compare with Roman -Rota, Roman functional formulation).

We share the conviction with more experienced community that this historically established autonomy - adds and still may add more inspiration of the analytical and geometrical character to the modern formulation of umbra idea (see: [1] and references therein) -which is so fruitfully formal algebraic and combinatorial in so many applications (see: [11-21] -via links - **for thousands of references**).

References on modern umbra

The recent modern umbra was born in 19-th century and is known now under the name of **Blissard** calculi. For more and references on this and that see:

[8]Andrew P. Guinand the survey of elementary mnemonic and manipulative uses of the umbral method;

[9] Brian D. Taylor: on difference equations via the classical umbral calculus where you learn what classical umbral calculus is to proceed with next:

[10] G.-C. Rota and B. D. Taylor: just on the Classical Umbral Calculus

[11] **Ira M. Gessel** here also apart of applications of the classical umbral calculus you may learn what classical umbral calculus is in the Algebra Universalis Realm. and do not miss many, **many representative references therein.**

Quite a lot of time , years ago:

[12] **Adams C. R.** in His "Linear q -difference equations" contained around **200** up-that-date (1931) references!

Quite recently (19December 2001)

[13] Thomas Ernst in his **Licentiate** elaborate included **969** references. As natural - the bibliography in this much useful [13] is far from being complete - neither is there q -umbral calculi fully represented.

Compare with the following links :

[14] **Umbral Calculus**

[http : //en2.wikipedia.org/wiki/Umbral_calculusUmbral](http://en2.wikipedia.org/wiki/Umbral_calculusUmbral)

[15] **Blissard symbolic calculus:**

[http : //www.google.pl/search?q = Blissard + symbolic + calculus](http://www.google.pl/search?q=Blissard+symbolic+calculus)

[16] A. Di Bucchianico , D. Loeb A Selected Survey of Umbral Calculus w (**506** references)

[17] **D. Loeb , G. C. Rota** another survey with quite recent contributions to the general calculus of finite differences

see also: *[http : //arxiv.org/list/math.CO/9502](http://arxiv.org/list/math.CO/9502)*

and of course

[18] **S.Roman** in *[http : //www.romanpress.com/me.htm](http://www.romanpress.com/me.htm)*. Here is the splendid , competent source from The Source.

Also other links below are recommended

[19] <http://mathworld.wolfram.com/UmbralCalculus.html>

[20] **R. Stanton** : competent source from The Source

[http : //www.math.umn.edu/ stanton/publist.html](http://www.math.umn.edu/~stanton/publist.html)

altogether with highly appreciate link to NAVIMA - group q -active also in q -research fruitfully

[21] NAVIMA <http://webs.uvigo.es/t10/navima/publications.html>

Though no list is a posteriori complete - this presented here and in statu nascendi listing of references streams may be actualized with **your e-mail help**.

2 II. Second Contact Remark

on $\partial_{q,h}$ -calculus -an umbral difference calculus of course.

$\partial_{q,h}$ -difference calculus of

Hahn [22] may be reduced to q -calculus of **Thomae-Jackson** [23,24,25] due to the following observation . Let

$$h \in F, (E_{q,h}\varphi)(x) = \varphi(qx + h)$$

and let

$$(\partial_{q,h}\varphi)(x) = \frac{\varphi(x) - \varphi(qx + h)}{(1-q)x - h} \quad (1)$$

Then (see Hann [22] and also Appendix A in [26] - fifty years after Hann)

$$\partial_{q,h} = E_{1, \frac{-h}{1-q}} \partial_q E_{1, \frac{h}{1-q}}. \quad (2)$$

Due to (2) it is easy now to derive corresponding formulas including Bernoulli-Taylor $\partial_{q,h}$ -formula obtained in [27] by the Viskov method [28] which for

$$q \rightarrow 1, h \rightarrow 0$$

recovers the content of one of the examples in [28] , while for

$$q \rightarrow 1, h \rightarrow 1$$

one recovers the content of the another example

in [28]. The case $h \rightarrow 0$ is included

in the formulas of q -calculus of Thomae-Jackson easy to be specified from [27] (see also up-date references there). For Bernoulli- Taylor Formula (presented during *PTM - Convention Lodz - 2002*) : contact [27] for its recent version.

Exercise: find the series expansion in of the $\partial_{q,h}$ -delta operator . Note and check that the polynomial sequence $p_n(x) = \left(x - \left[\frac{h}{1-q}\right]\right)^n, n \geq 0$ is the

Appell-Sheffer sequence for the $\partial_{q,h}$ -delta operator . This is consequently not recognized in [29] - see: (5.6) - there.

3 III. Third Contact Remark on τ -calculus and umbral difference calculus

In [29] (page 2) the so called τ derivative was defined and then the exposition of the idea of " τ -calculus" follows with an adaptation of standard methods to obtain solutions of the eigenvalue and eigenvector - this time $-\tau$ -difference equation of second order. (Compare ∂_τ of [29] with divided difference operator from [30])

Our first τ -experience and τ -info is: Inspection of (5.7) formula and its neighborhood from [29] leads to immediate observation:

a) if X_n is standard monomial then ∂_τ is a ∂_ψ derivative where $n_\psi = \frac{c_n}{c_{n-1}}, n > 0$, [3-5] .

b) if $X_n = p_n(x)$ is a polynomial of n-th order then $\{p_n\}_{n \geq 0}$, $\deg p_n = n$ represents ψ -Sheffer sequence and $\partial_\tau = Q(\partial_\psi)$ is the corresponding ∂_ψ -delta operator so that $Q(\partial_\psi)p_n(x) = n_\psi p_{n-1}(x)$ [3-5]. This is not recognized in [29] .

Our next τ -experience and τ -info is: the conclusive illustrative **Example 5.3** in [29] from 8 **December** 2002 comes from the announced (p.18 and p.21 in [29]) A.Dobrogowska, A.Odzijewicz *Second q-difference equations solvable by factorization method* .The calculations in ArXive preprint from **December** 2003 by A.Dobrogowska, A.Odzijewicz *Second q-difference equations solvable by factorization method*

ArXiv : math - ph/031205722Dec2003

rely on calculations from **December** 2002 preprint [29]

ArXiv : math - ph/0208006.

On the 14-th of January 2004 solemnly in public Professor Jan Slawianowski had given the highest appraisal to this investigation. He compared its possible role to be played in the history of physics with the Discovery of Max Planck from **December** 1900. Well. May be this is all because of **December**. Both preprints refer to difference Riccati equation.

One might be then perhaps curious about where q -Riccati equation comes from according to the authors of [29].

Note: before next section see: page 6 in [29] [quotation]” q -Riccati equation of [15]”[end of quotation]. Therefore note also that the q -Riccati equation has been considered in productions of other authors earlier, for example [31-33]

IV. Fourth Contact Remark on ψ -calculus and Viskov-Markowsky and Others -general umbral difference calculus

This section on perspectives is addressed to those who feel Umbra Power as well as to those who ignore it by ... (“umbra dependent reasons” ? .) What might be done to extend the scope of realized application of **EFOC** = **E**xtended **F**inite **O**perator **C**alculus?

Certainly:

ψ -orthogonal OPS : orthogonality in Roman-Rota functional formulation of umbral calculus [34] might be translated up to wish into EFOC - language : see: [5,1] for the beginings of ψ -integration already in EFOC language - to be extended due to inspiration by q, τ, ψ - integrations , taking care of roots coming from Jackson, Carlitz, Hann and so many Others ...

Certainly:

Certainly ψ -difference equations of higher order ... We may first of all get more experienced and activate intuitions with inspirations up to personal choice. Special up to my choice sources of might be further ψ -inspirations are [35-50] .

[35] by Alphonse Magnus

contains : ... first and second order difference equations - orthogonal polynomials satisfying difference equations (Riccati equations, continued fractions etc. Pade, Chebyshev etc.)

contains many up-date relevant q -references and many pages of substantial information - historical - included.

[36] by Ivan A. Dynnikov , Segey V. Smirnov :all six references: principal substantial for the subject

[37] by Mourad E. H. Ismail q -papers - most out of around 200 references ;**contains** : q -orthogonal polynomials, Askey-Wilson divided difference operators, inverses to the Askey-Wilson operators .

[38,39,40,41] q -Riccati equations,special functions, q -series and related topics, q -commuting variables.

[42] by Haret C. Rosu , see also for history of q -difference analogues

[43] by I. Area, E. Godoy, F. Marcelln **contains:** q -Jacobi polynomials; q -Laguerre/Wall polynomials...

[44,45,46,47] q -extension of the Hermite polynomials,coherent pairs and

orthogonal polynomials of a discrete variables, further important q -analogs...

[48] by Natig M. Atakishiyev, Alejandro Frank, and Kurt Bernardo Wolf
contains: Heisenberg q -algebra generators as first-order difference operators, eigenstates of the q -oscillator Hamiltonian in terms of the q -1-Hermite polynomials.... the measure for these q -oscillator states..

[49] by Brian D. Taylor : Difference Equations via the Classical Umbral Calculus

[50] on Umbral calculus and Quantum Mechanics

.....**to be continued**.....

We end this incomplete Reference Survey attempted from finite operator calculus point of view with the link :

<http://www.ms.uky.edu/~jrge/Rota/rota.html>

It is the Gian-Carlo Rota and his grown up Children and Friends -Family Link.

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