Russian Journal of Mathematical Physics 2014, Volume 21, Issue 4, pp. 484-493

UMBRAL CALCULUS ASSOCIATED WITH FROBENIUS-TYPE EULERIAN POLYNOMIALS

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Abstract. In this paper, we study some properties of several polynomials arising from umbral calculus. In particular, we investigate the properties of the orthogonality-type of the Frobenius-type Eulerian polynomials which are derived from umbral calculus. By using our properties, we can derive many interesting identities of special polynomials associated with Frobenius-type Eulerian polynomials. An application to normal ordering is presented.

1. Introduction

It is well known that the Euler numbers have a long history (see [7, 8, 13]). They are of fundamental importance in several parts of mathematics and mathematical physics (see [5–8]). In the last decades, several interesting extensions and modifications were considered along with related combinatorial, probabilistic, and statistical applications (see [9–12, 16]). One of the well known extensions it is the Frobenius-Euler numbers and polynomials [5, 6, 13]). The aim of this paper is to study several properties of the orthogonality-type of the Frobenius-type Eulerian polynomials which are derived from umbral calculus (see [22, 23, 30, 31]). Note that umbral calculus has an application in the physics of gases (see [35]) and in the group theory and quantum mechanics (see [1, 2]). Umbral calculus, in particular Sheffer sequences, has also been applied to the normal ordering of expressions involving bosonic creation and annihilation operators [3, 4].

In this paper, umbral calculus is considered for some special Sheffer polynomials such as *Frobenius-Euler polynomials*, *Changhee polynomials*, *Daehee polynomials* and *Bessel polynomials*. Let Π be the algebra structure of polynomials in a single variable x over $\mathbb{C}$ and let Π^* be the vector space of all linear functionals on Π. The action of a linear functional L on a polynomial $p(x)$ is denoted by $\langle L|p(x)\rangle$. We note that $\langle L|p(x)\rangle$ satisfies $\langle cL+c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle$, for any $c, c' \in \mathbb{C}$ and $L, L' \in \Pi^*$ (see [22, 23, 30, 31]). Let

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}. \tag{1.1}$$

For $f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \in \mathcal{H}$, we define a linear functional on Π by setting

$$\langle f(t)|x^n\rangle = a_n, \text{ for all } n \geq 0, \text{ (see [22, 23, 30, 31]).} \tag{1.2}$$

By (1.1) and (1.2), we have

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \text{ for all } n, k \geq 0, \text{ (see [22, 23, 30, 31]),} \tag{1.3}$$

2000 *Mathematics Subject Classification.* 05A40, 81S05.

Key words and phrases. Normal ordering, Bessel polynomial, Changhee polynomial, Daehee polynomial, Frobenius-Euler polynomial, Sheffer polynomial, Umbral Calculus.

where $\delta_{n,k}$ is the Kronecker's symbol. Let us assume that $f_L(t)=\sum_{k\ge0}\langle L|x^k\rangle\frac{t^k}{k!}$. Then by (1.2), we easily obtain that $\langle f_L(t)|x^n\rangle=\langle L|x^n\rangle$ and $f_L(t)=L$. So, the map $L\mapsto f_L(t)$ is a vector space isomorphic from $\Pi*$ onto $\mathcal{H}$. Henceforth, $\mathcal{H}$ is thought of as both set of formal power series and set of linear functionals. We call $\mathcal{H}$ the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

As is definition, the *order* $O(f(t))$ of a non-zero power series $f(t)$ is the smallest integer k for which the coefficient t^k does not vanish (see [22, 23, 30, 31]). If $O(f(t))=1$ (respectively, $O(f(t))=0$) then $f(t)$ is called a *delta* (respectively, an *invertable*) series. Let us assume that $f(t),g(t)\in\mathcal{H}$ with $O(f(t))=1$ and $O(g(t))=0$, so there exists a unique sequence $S_n(x)$ of polynomials with

$$\langle g(t)(f(t))^k|S_n(x)\rangle=n!\delta_{n,k} \tag{1.4}$$

for all $n,k\ge0$. The sequence $S_n(x)$ is called the *Sheffer* sequence for $(g(t),f(t))$ which is denoted by $S_n(x)\sim(g(t),f(t))$ (see [22, 23, 30, 31]). Let $f(t)\in\mathcal{H}$ and $p(x)\in\Pi$, then we note that

$$\langle e^{yt}|p(x)\rangle=p(y),\quad \langle f(t)g(t)|p(x)\rangle=\langle f(t)|g(t)p(x)\rangle, \tag{1.5}$$

and

$$f(t)=\sum_{k\ge0}\langle f(t)|x^k\rangle\frac{t^k}{k!},\quad p(x)=\sum_{k\ge0}\langle t^k|p(x)\rangle\frac{x^k}{k!}, \tag{1.6}$$

(see [22, 23, 30, 31]). From (1.6), we see that

$$\langle t^k|p(x)\rangle=p^{(k)}(0)\text{ and }\langle 1|p^{(k)}(x)\rangle=p^{(k)}(0), \tag{1.7}$$

where $p^{(k)}(0)$ denotes the k-th derivative of $p(x)$ respect to x at $x=0$. From (1.7) we can derive the following equation $t^kp(x)=p^{(k)}(x)$ (see [22, 23, 30, 31]). For $S_n(x)\sim(g(t),f(t))$, we have

$$\frac{1}{g(\bar f(t))}e^{y\bar f(t)}=\sum_{k\ge0}S_k(y)\frac{t^k}{k!}, \tag{1.8}$$

for all $y\in\mathbb{C}$, where $\bar f(t)$ is the compositional inverse of $f(t)$ (see [22,23,30,31]). For $S_n(x)\sim(g(t),f(t))$ and $R_n(x)\sim(h(t),\ell(t))$, let us assume that $S_n(x)=\sum_{k=0}^{n}C_{n,k}R_k(x)$. Then we see that

$$C_{n,k}=\frac{1}{k!}\left\langle\frac{h(\bar f(t))}{g(\bar f(t))}(\ell(\bar f(t)))^k|x^n\right\rangle, \tag{1.9}$$

For all $n,k\ge0$ (see [22, 23, 30, 31]).

Throughout this paper, we assume that $\lambda\in\mathbb{C}$ with $\lambda\ne1$. The *Frobenius-type Eulerian polynomials of order* r are also given by

$$\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^r e^{xt}=\sum_{n\ge0}A_n^{(r)}(x|\lambda)\frac{t^n}{n!}\text{ (see [7, 8, 14, 15, 19, 20])}, \tag{1.10}$$

where r is a positive integer. In particular case, $x=0$, $A_n^{(r)}(0|\lambda)=A_n^{(r)}(\lambda)$ are called the *n-th Frobenius-Euler numbers of order r*. As is well known, the Frobenius-Euler polynomials of order r are defined by the generating function to be

$$\left(\frac{1-\lambda}{e^{t}-\lambda}\right)^r e^{xt}=\sum_{n\ge0}F_n^{(r)}(x|\lambda)\frac{t^n}{n!}\text{ (see [23–25])}. \tag{1.11}$$

In the special case, $x = 0$, $F_n^{(r)}(0|\lambda) = F_n^{(r)}(\lambda)$ are called the *n-th Frobenius-Euler numbers of order r*. The *Hermite polynomials* are defined by the generating function to be

$$e^{2xt-t^2} = \sum_{n\geq 0} H_n(x)\frac{t^n}{n!} \text{ (see [21, 22, 30, 31]).} \tag{1.12}$$

In the special case, $x = 0$, $H_n(0) = H_n$ are called the *n-th Hermite numbers*. From (1.12), we note that $H_n(x) = \sum_{j=0}^{n} 2^j \binom{n}{j} H_{n-j} x^j$. It is well known that the *Poisson-Charlier sequence* is given by

$$C_n(x;a) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a^{-k} (x)_k \sim \left(e^{a(e^t-1)}, a(e^t-1)\right) \text{ and } \sum_{k\geq 0} C_n(k;a)\frac{t^k}{k!} = \left(\frac{t-a}{a}\right)^n e^t,$$

(see [22, 23, 30, 31]), where $a \neq 0$, $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and $(x)_k = x(x-1)\cdots(x-k+1)$. The solution of the *Bessel differential equation* $x^2y'' + 2(x+1)y' + n(n+1)y = 0$ is given by

$$y_n(x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!}\left(\frac{x}{2}\right)^k. \tag{1.13}$$

The *Stirling numbers of the second kind* is defined by the generating function to be

$$(e^t - 1)^n = n! \sum_{j\geq n} S_n(j,n)\frac{t^j}{j!} \text{(see [22, 30]).} \tag{1.14}$$

In this paper, we present some properties of several polynomials arising from umbral calculus. In particular, we investigate the properties of the orthogonality-type of the Frobenius-type Eulerian polynomials which are derived from umbral calculus. Finally, in the last section, we establish a connection between our results and the problem of normal ordering.

2. Umbral calculus associated with Frobenius-type Eulerian polynomials

From (1.8), (1.12) and (1.10) we note that

$$A_n^{(r)}(x|\lambda) \sim \left(\left(\frac{e^{t(1-\lambda)} - \lambda}{1-\lambda}\right)^r, t\right) \text{ and } H_n(x) \sim (e^{t^2/4}, t/2). \tag{2.1}$$

Let us assume that

$$A_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} C_{n,k} H_k(x). \tag{2.2}$$

By (1.9) and (2.2), we get that

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle \frac{e^{t^2/4}}{\left(\frac{e^{t(\lambda-1)}-\lambda}{1-\lambda}\right)^r}\frac{t^k}{2^k}|x^n\right\rangle = \frac{1}{2^k}\binom{n}{k}\left\langle \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^r |e^{t^2/4}x^{n-k}\right\rangle\\
&= \frac{1}{2^k}\binom{n}{k}\sum_{j=0}^{(n-k)/2}\frac{(n-k)_{2j}}{4^j j!}\left\langle \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^r |x^{n-k-2j}\right\rangle\\
&= \frac{1}{2^k}\binom{n}{k}\sum_{j=0}^{(n-k)/2}\frac{(2j)!}{2^{2j}j!}\binom{n-k}{2j}\left\langle 1|A^{(r)}_{n-k-2j}(x|\lambda)\right\rangle\\
&= \frac{1}{2^k}\binom{n}{k}\sum_{j=0}^{(n-k)/2}\frac{(2j)!}{2^{2j}j!}\binom{n-k}{2j}A^{(r)}_{n-k-2j}(\lambda)\\
&= n!\sum_{j=0,\ j\text{ even}}^{n-k}\frac{A^{(r)}_{n-k-j}(\lambda)}{2^{j+k}k!(j/2)!(n-k-j)!}.
\end{aligned}$$

Therefore, by (2.2), we obtain the following result.

Theorem 2.1. *Let $r \in \mathbb{Z}_+$. For $n \geq 0$,*

$$A^{(r)}_n(x|\lambda) = n!\sum_{k=0}^{n}\left(\sum_{j=0,\ j\ \mathit{even}}^{n-k}\frac{A^{(r)}_{n-k-j}(\lambda)}{2^{j+k}k!(j/2)!(n-k-j)!}\right)H_k(x).$$

Now, let us assume that

$$H_n(x) = \sum_{k=0}^{n} C_{n,k}A^{(r)}_k(x|\lambda). \tag{2.3}$$

Then, by (1.9) and (2.3), we get

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!}\left\langle \left(\frac{e^{2t(\lambda-1)}-\lambda}{1-\lambda}\right)^r 2^k t^k e^{-t^2}|x^n\right\rangle\\
&= 2^k\binom{n}{k}\sum_{j=0}^{(n-k)/2}\frac{(-1)^j(n-k)_{2j}}{j!}\left\langle \left(\frac{e^{2t(\lambda-1)}-\lambda}{1-\lambda}\right)^r |x^{n-k-2j}\right\rangle\\
&= \frac{2^k}{(1-\lambda)^r}\binom{n}{k}\sum_{j=0}^{(n-k)/2}\frac{(-1)^j(n-k)_{2j}}{j!}\left\langle \left(e^{2t(\lambda-1)}-\lambda\right)^r |x^{n-k-2j}\right\rangle. \qquad (2.4)
\end{aligned}$$

From (1.14), we note that

$$\begin{aligned}
\left(e^{2t(1-\lambda)}-\lambda\right)^r &= \left(e^{2t(1-\lambda)}-1+1-\lambda\right)^r = \sum_{d=0}^{r}\binom{r}{d}(1-\lambda)^{r-d}(\left(e^{2t(\lambda-1)}-1\right)^d\\
&= \sum_{d=0}^{r}\sum_{m\geq 0}\frac{(-1)^{r-d}j!2^{m+d}\binom{r}{d}(\lambda-1)^{r+m}}{(m+d)!}S_2(m+d,d)t^{m+d},
\end{aligned}$$

which implies

$$\begin{aligned}&\left(e^{2t(1-\lambda)}-\lambda\right)^r x^{n-k-2j}\\ &=\sum_{d=0}^{r}\sum_{m\geq 0}\frac{(-1)^{r-d}d!2^{m+d}\binom{r}{d}(\lambda-1)^{r+m}}{(m+d)!}S_2(m+d,d)(n-k-2j)_{m+d}x^{n-k-2j-m-d}.\end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we have

$$C_{n,k}=r!\binom{n}{k}\sum_{j=0}^{(n-k)/2}\sum_{d=0}^{r}\frac{(-1)^{j+d}(2j)!d!}{j!}\binom{n-k}{2j}\binom{r}{d}(\lambda-1)^{n-k-2j-d}2^{n-2j}S_2(n-k-2j,d).$$

Therefore, by (2.3), we can state the following result.

Theorem 2.2. *Let $r\in\mathbb{Z}_+$. For all $n\geq 0$,*

$$\begin{aligned}&H_n(x)\\ &=r!\sum_{k=0}^{n}\left(\sum_{j=0}^{(n-k)/2}\sum_{d=0}^{r}\frac{(-1)^{j+d}(2j)!d!}{j!}\binom{n-k}{2j}\binom{r}{d}\binom{n}{k}(\lambda-1)^{n-k-2j-d}2^{n-2j}S_2(n-k-2j,d)\right)A_k^{(r)}(x|\lambda).\end{aligned}$$

From (1.13), Carlitz defined the *Bessel function* as follows:

$$J_n(x)=x^ny_n(1/x)\sim(1,t-t^2/2),\ \text{for all } n\geq 1,\ (\text{see [22,30,31]}). \tag{2.6}$$

By (2.6), we can derive the generating function of $J_n(x)$ as follows: $\sum_{n\geq 0}J_n(x)\frac{t^n}{n!}=e^{x(1+\sqrt{1-2t})}$. By (1.13), we easily obtain

$$J_n(x)=\sum_{k=0}^{n-1}\frac{(n-1+k)!}{(n-1-k)!k!2^k}x^{n-k}\sim(1,t-t^2/2). \tag{2.7}$$

Let us assume that

$$A_n^{(r)}(x|\lambda)=\sum_{k=0}^{n}C_{n,k}J_k(x). \tag{2.8}$$

Then, by (1.9) and (2.8), we have

$$\begin{aligned}C_{n,k}&=\binom{n}{k}\left\langle\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^r\left(\frac{2-t}{t}\right)^k\Big|x^{n-k}\right\rangle\\ &=(-1)^k\binom{n}{k}\sum_{j=0}^{n-k}\frac{C_k(j;2)}{j!}\left\langle\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^r e^{-t}\Big|t^jx^{n-k}\right\rangle\\ &=(-1)^k\binom{n}{k}\sum_{j=0}^{n-k}C_k(j;2)\binom{n-k}{j}\left\langle\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^r\Big|(x-1)^{n-k-j}\right\rangle\\ &=(-1)^k\binom{n}{k}\sum_{j=0}^{n-k}C_k(j;2)\binom{n-k}{j}A_{n-k-j}^{(r)}(-1|\lambda).\end{aligned}$$

Therefore, by (2.8) we have the following result.

Theorem 2.3. *Let $r \in \mathbb{Z}_+$. For $n \geq 0$, we have*

$$A_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} \left(\sum_{j=0}^{n-k} (-1)^k \binom{n}{k} \binom{n-k}{j} C_k(j;2) A_{n-k-j}^{(r)}(-1|\lambda) \right) J_k(x).$$

It is well known that *Eulerian-type Chaughee polynomials* are defined by the generating function to be

$$\sum_{n\geq 0} Ch_n(x|\lambda) = \frac{(1+t)^{\lambda-1}-\lambda}{1-\lambda}(1+t)^x \text{ (see [19, 22, 23, 30, 31]).} \tag{2.9}$$

From (1.8) and (2.9), we can derive $Ch_n(x|\lambda) \sim (\frac{1-\lambda}{e^{t(1-\lambda)}-\lambda}, e^t-1)$. Let us assume that

$$A_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} C_{n,k} Ch_k(x|\lambda). \tag{2.10}$$

Then, by (1.9), we obtain

$$\begin{aligned}
C_{n,k} &= \frac{1}{k!} \left\langle \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda} \right)^{r+1} \left(e^t-1\right)^k |x^n \right\rangle \\
&= \sum_{j=0}^{n-k} \frac{S_2(j+k,k)}{(j+k)!} \left\langle \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda} \right)^{r+1} |t^{k+j} x^n \right\rangle \\
&= \sum_{j=0}^{n-k} S_2(j+k,k) \binom{n}{j+k} \left\langle 1| \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda} \right)^{r+1} x^{n-k-j} \right\rangle \\
&= \sum_{j=0}^{n-k} S_2(j+k,k) \binom{n}{j+k} A_{n-k-j}^{(r+1)}(\lambda).
\end{aligned}$$

Hence, by (2.10) we can sate the following theorem.

Theorem 2.4. *Let $r \in \mathbb{Z}_+$. For $n \geq 0$, we have*

$$A_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} \left(\sum_{j=0}^{n-k} S_2(j+k,k) \binom{n}{j+k} A_{n-k-j}^{(r+1)}(\lambda) \right) Ch_k(x|\lambda).$$

Let us consider the Eulerian-type Daehee polynomials of the second kind as follows:

$$D_n^*(x|\lambda) \sim (\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}, \frac{e^t-1}{e^t-\lambda}). \tag{2.11}$$

From (1.8) and (2.11), we can derive the generating function of (2.11) as follows:

$$\sum_{n\geq 0} D_n^*(x|\lambda) \frac{t^n}{n!} = \frac{(1-\lambda t)^{x+\lambda-1} - \lambda(1-t)^{\lambda-1}(1-\lambda t)^x}{(1-\lambda)(1-t)^{x+\lambda-1}},$$

where $t \neq 1$. Let us assume that

$$A_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} C_{n,k} D_k^*(x|\lambda). \tag{2.12}$$

Then, by (1.9) and (2.12), we have

$$C_{n,k} = \frac{1}{k!}\left\langle \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{r+1}\left(\frac{e^t-1}{e^t-\lambda}\right)^k \Big| x^n \right\rangle$$
$$= \frac{1}{(1-\lambda)^k}\sum_{j=0}^{n-k}\binom{n}{j+k}S_2(j+k,k)\left\langle \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{r+1} \Big| F_{n-k-j}^{(k)}(x|\lambda)\right\rangle$$
$$= \frac{1}{(1-\lambda)^k}\sum_{j=0}^{n-k}\sum_{m=0}^{n-k-j}\binom{n}{j+k}S_2(j+k,k)F_{n-k-j-m}^{(k)}(x|\lambda)A_m^{(r+1)}(\lambda).$$

Therefore, by (2.12), we obtain the following result.

Theorem 2.5. *Let $r \in \mathbb{Z}_+$. For $n \geq 0$, we have*

$$A_n^{(r)}(x|\lambda) = \sum_{k=0}^{n}\left(\sum_{j=0}^{n-k}\sum_{m=0}^{n-k-j}\frac{\binom{n}{j+k}S_2(j+k,k)}{(1-\lambda)^k}F_{n-k-j-m}^{(k)}(x|\lambda)A_m^{(r+1)}(\lambda)\right)Ch_k(x|\lambda).$$

Let $p_n(x) \sim (1, f(t))$ and $q_n(x) \sim (1, g(t))$ $(n \geq 0)$. Then we note that

$$q_n(x) = x\left(\frac{f(t)}{g(t)}\right)^n x^{-1}p_n(x). \tag{2.13}$$

Let us consider the following Sheffer sequences

$$S_n(x|\lambda) \sim \left(1, \frac{e^{t(\lambda-1)}}{1-\lambda}t\right) \text{ and } x^n \sim (1,t). \tag{2.14}$$

From (2.13) and (2.14), we can derive

$$S_n(x|\lambda) = x\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^n x^{n-1} \tag{2.15}$$

and

$$S_n(x|\lambda) = (1-\lambda)^n x(e^{t(\lambda-1)}-1+1-\lambda)^{-n}x^{n-1}$$
$$= x\sum_{j\geq 0}\binom{n+j-1}{j}\left(\frac{e^{t(\lambda-1)}-1}{1-\lambda}\right)^j x^{n-1}$$
$$= x\sum_{j\geq 0}\binom{n+1-j}{j}(1-\lambda)^{-j}(e^{t(\lambda-1)}-1)^j x^{n-1}. \tag{2.16}$$

By (1.14), we easily get

$$(e^{t(\lambda-1)}-1)^j = \sum_{m\geq 0}\frac{j!}{(m+j)!}S_2(m+j,j)(\lambda-1)^{j+m}t^{j+m}. \tag{2.17}$$

Thus, from (2.16) and (2.17), we have

$$S_n(x|\lambda) = x\sum_{j=0}^{n-1}\sum_{m=0}^{n-1-j}j!\binom{n+1-j}{j}\binom{n-1}{m+j}(\lambda-1)^m S_2(m+j,j)x^{n-1-m-j}. \tag{2.18}$$

Hence, by (2.15) and (2.18), we obtain the following result.

Theorem 2.6. *Let $r \in \mathbb{Z}_+$. For $n \geq 1$, we have*

$$xA_{n-1}^{(r)}(x|\lambda) = \sum_{j=0}^{n-1}\sum_{m=0}^{n-1-j} j!\binom{n+1-j}{j}\binom{n-1}{m+j}(\lambda-1)^m S_2(m+j,j)x^{n-1-m-j}.$$

If we consider the following Sheffer sequences

$$p_n(x) \sim \left(1, \left(\frac{e^{t(\lambda-1)}}{1-\lambda}\right)^r t\right) \text{ and } x^n \sim (1,t), \tag{2.19}$$

then, by (2.13) and (2.19), we get

$$p_n(x) = x\left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^{rn} x^{n-1} = xA_{n-1}^{rn}(x|\lambda). \tag{2.20}$$

Therefore, we can state the following result.

Theorem 2.7. *Let $r \in \mathbb{Z}_+$. For $n \geq 1$, we have*

$$xA_{n-1}^{(rn)}(x|\lambda) \sim \left(1, \left(\frac{1-\lambda}{e^{t(\lambda-1)}-\lambda}\right)^r t\right).$$

3. Orthogonality-type

Let $\Pi_n = \{p(x) \in \mathbb{C}[x] | \deg p(x) \leq n\}$. Then we note that Π_n is the $(n+1)$-dimensional vector space over $\mathbb{C}$. It is not difficult to see that $\{A_0^{(r)}(x|\lambda), A_1^{(r)}(x|\lambda), \ldots, A_n^{(r)}(x|\lambda)\}$ is a basis for Π_n. For $p(x) \in \Pi_n$, let us assume that

$$p(x) = \sum_{k=0}^{n} a_k A_k^{(r)}(x|\lambda), (n \geq 0). \tag{3.1}$$

From (1.4), (2.1) and (3.1), we can derive

$$\left\langle \left(\frac{e^{t(\lambda-1)}-\lambda}{1-\lambda}\right)^r t^k \Big| p(x)\right\rangle = \sum_{j=0}^{n} a_j \left\langle \left(\frac{e^{t(\lambda-1)}-\lambda}{1-\lambda}\right)^r t^k \Big| A_j^{(r)}(x|\lambda)\right\rangle = \sum_{j=0}^{n} j! a_j \delta_{j,k} = k! a_k.$$

So,

$$\begin{aligned} a_k &= \frac{1}{k!}\left\langle \left(\frac{e^{t(\lambda-1)}-\lambda}{1-\lambda}\right)^r t^k \Big| p(x)\right\rangle = \frac{1}{k!}\left\langle \left(\frac{e^{t(\lambda-1)}-\lambda}{1-\lambda}\right)^r \Big| D^k p(x)\right\rangle \\ &= \frac{1}{k!(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j}\langle 0|D^k p(x+j(\lambda-1))\rangle. \end{aligned}$$

Therefore, by (3.1), we obtain the following theorem.

Theorem 3.1. *For $r \in \mathbb{Z}_+$ and $p(x) \in \Pi_n$, let $p(x) = \sum_{k=0}^{n} a_k A_k^{(r)}(x|\lambda)$. Then*

$$a_k = \frac{1}{k!(1-\lambda)^r}\sum_{j=0}^{r}\binom{r}{j}(-\lambda)^{r-j} D^k p(j(\lambda-1)).$$

Now, we present several applications for the above theorem. At first, let us take

$$p(x) = L_n(-x) = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{n!}{m!} x^m \sim (1, t/(1-t)), \tag{3.2}$$

where $L_n(x)$ is the *n-th Laguerre polynomial*. By Theorem 3.1, we obtain

$$\begin{aligned} a_k &= \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j(\lambda-1)) \\ &= \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \sum_{m=1}^{n} \binom{r}{j} (-\lambda)^{r-j} \binom{n-1}{m-1} \binom{m}{k} \frac{n!k!}{m!} (j(\lambda-1))^{m-k} \\ &= \sum_{j=0}^{r} \sum_{m=1}^{n} \binom{r}{j} \binom{n-1}{m-1} \binom{m}{k} (-\lambda)^{r-j} (\lambda-1)^{m-k-r} (-1)^r j^{m-k} \frac{n!}{m!}. \end{aligned}$$

Hence, by Theorem 3.1 we can state the following theorem.

Theorem 3.2. *For $r, n \in \mathbb{Z}_+$, we have*

$$L_n(-x) = \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \sum_{m=1}^{n} \binom{r}{j} \binom{n-1}{m-1} \binom{m}{k} (-\lambda)^{r-j} (\lambda-1)^{m-k-r} (-1)^r j^{m-k} \frac{n!}{m!} \right\} A_k^{(r)}(x|\lambda).$$

Let us take $p(x) = J_n(x) = \sum_{m=0}^{n-1} \frac{(n-1+m)!}{m!(n-1-m)!2^m} x^{n-m} \sim (1, t - t^2/2)$, where $J_n(x)$ is the n-th Bessel function. By Theorem 3.1, we obtain

$$\begin{aligned} a_k &= \frac{1}{k!(1-\lambda)^r} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j(\lambda-1)) \\ &= \sum_{j=0}^{r} \sum_{m=0}^{n-1} \binom{r}{j} (-\lambda)^{r-j} \frac{(n-1+m)!}{m!(n-1-m)!2^m} \binom{n-m}{k} (-1)^r (\lambda-1)^{n-k-m-r} j^{n-k-m} \end{aligned}$$

Hence, by Theorem 3.1 we can state the following theorem.

Theorem 3.3. *For $r, n \in \mathbb{Z}_+$, we have*

$$J_n(x) = \sum_{k=0}^{n} \left\{ \sum_{j=0}^{r} \sum_{m=0}^{n-1} \binom{r}{j} \binom{n-m}{k} \frac{(-1)^j j^{n-k-m} (n-1+m)! \lambda^{r-j} (\lambda-1)^{n-k-m-r}}{m!(n-1-m)!2^m} \right\} A_k^{(r)}(x|\lambda).$$

4. Application to normal ordering

Since the seminal work of Katriel [17] the combinatorial aspects of normal ordering arbitrary words in the creation and annihilation operators $a^\dagger$ and a of a single-mode boson having the usual commutation relations $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$, $[a, a] = 0$ and $[a^\dagger, a^\dagger] = 0$ have been studied intensively since the seventies, see [3, 4, 17, 18, 26–29, 32–34] and references therein. From a more mathematical point of view the consequences of the noncommutative calculus of operators has been considered, in particular by Maslov [29]. Recall that *normal ordering* $\mathcal{N}(F(a, a^\dagger))$ is a functional representation of an operator function $F(a,a^\dagger)$ in which all the creation operators stand to the left of the annihilation operators. For example, Katriel [17] showed that

$$\mathcal{N}[(a^\dagger a)^n]=\sum_{k=0}^n S_2(n,k)(a^\dagger)^k a^k.$$

By the properties of coherent states (for instance, see [3]), the last identity can be written as

$$\langle z|e^{ta^\dagger a}|z\rangle=\sum_{n\geq 0}\langle z|(a^\dagger a)^n|z\rangle\frac{t^n}{n!}=e^{|z|^2(e^t-1)}.$$

Now, we can state the relation between normal ordering and Sheffer sequences as follows. Let $S_n(x)\sim(g(t),f(t))$ and $R_n(x)\sim(h(t),\ell(t))$ be any two Sheffer sequences. Then one has

$$\langle z|[M_{g,f}(a,a^\dagger)]^n|z'\rangle=\sum_{k=0}^n\frac{1}{k!}\left\langle\frac{h(\bar{f}(t))}{g(\bar{f}(t))}(\ell(\bar{f}(t)))^k|x^n\right\rangle\langle z|[M_{h,\ell}(a,a^\dagger)]^k|z'\rangle, \tag{4.1}$$

where $\bar{f}$ denotes the compositional inverse of f and $M_{g,f}(x,y)=\left(y-\frac{g'(x)}{g(x)}\right)\frac{1}{f'(x)}$. By (4.1) and the results in the previous sections, we can obtain several nice normal ordering identities. In the following, we present several examples.

Example 4.1. *Let* $(g,f)=(1,t-t^2/2)$ *and* $(h,\ell)=(\left(\left(\frac{e^{t(1-\lambda)}-\lambda}{1-\lambda}\right)^r,t\right)$, *so* $M_{g,f}(a,a^\dagger)=\frac{a^\dagger}{1-a}=a^\dagger\sum_{j\geq0}a^j$ *and* $M_{h,\ell}(a,a^\dagger)=a^\dagger-\frac{r(1-\lambda)e^{a(1-\lambda)}}{e^{a(1-\lambda)}-\lambda}$. *Then, by the proof of Theorem 2.1 and (4.1), we obtain*

$$n!\sum_{k=0}^n\left(\sum_{j=0,\ j\ even}^{n-k}\frac{A^{(r)}_{n-k-j}(\lambda)}{2^{j+k}k!(j/2)!(n-k-j)!}\right)\langle z|\left(2a^\dagger-a\right)^k|0\rangle=\langle z|(a^\dagger-\frac{r(1-\lambda)e^{a(1-\lambda)}}{e^{a(1-\lambda)}-\lambda})^n|0\rangle.$$

Example 4.2. *Let* $(g,f)=\left(\left(\frac{e^{t(1-\lambda)}-\lambda}{1-\lambda}\right)^r,t\right)$ *and* $(h,\ell)=(1,t-t^2/2)$, *so* $M_{g,f}(a,a^\dagger)=a^\dagger-\frac{r(1-\lambda)e^{a(1-\lambda)}}{e^{a(1-\lambda)}-\lambda}$ *and* $M_{h,\ell}(a,a^\dagger)=a^\dagger(1-a)^{-1}=a^\dagger\sum_{j\geq0}a^j$. *Then, by the proof of Theorem 2.3 and (4.1), we obtain*

$$\sum_{k=0}^n\left(\sum_{j=0}^{n-k}(-1)^k\binom{n}{k}\binom{n-k}{j}C_k(j;2)A^{(r)}_{n-k-j}(-1|\lambda)\right)\langle z|\left(a^\dagger(1-a)^{-1}\right)^k|0\rangle$$
$$=\langle z|(a^\dagger-\frac{r(1-\lambda)e^{a(1-\lambda)}}{e^{a(1-\lambda)}-\lambda})^n|0\rangle.$$

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