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Some identities of Bernoulli, Euler and Abel polynomials arising from umbral calculus

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Abstract
In this paper, we derive some identities of Bernoulli, Euler, and Abel polynomials arising from umbral calculus.
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1 Introduction

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F}=\left\{f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k \,\middle|\, a_k\in\mathbb{C}\right\}. \tag{1.1}$$

Let us assume that $\mathbb{P}$ is the algebra of polynomials in the variable x over $\mathbb{C}$ and $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on a polynomial $p(x)$, and we remind that the vector space structure on $\mathbb{P}^*$ is defined by

$$\langle L+M|p(x)\rangle=\langle L|p(x)\rangle+\langle M|p(x)\rangle,$$
$$\langle cL|p(x)\rangle=c\langle L|p(x)\rangle,$$

where c is a complex constant (see [1–4]).

The formal power series

$$f(t)=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\in\mathcal{F} \tag{1.2}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle=a_n,\quad \text{for all } n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}. \tag{1.3}$$

Thus, by (1.2) and (1.3), we get

$$\langle t^k|x^n\rangle=n!\delta_{n,k}\quad (n,k\geq 0), \tag{1.4}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [3]).

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, from (1.4), we have

$$\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle, \quad n \geq 0. \tag{1.5}$$

By (1.5), we get $L = f_L(t)$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. So, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ is thought of as both a formal power series and a linear functional (see [1–3]). We call $\mathcal{F}$ the *umbral algebra*, and the study of umbral algebra is called *umbral calculus* (see [1–3]).

The order $o(f(t))$ of the nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a *delta series*. If $o(f(t)) = 0$, then $f(t)$ is called an *invertible series* (see [3]).

Let $S_n(x)$ be polynomials in the variable x with degree n, and let $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $S_n(x)$ such that $\langle g(t)f(t)^k|S_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$. The sequence $S_n(x)$ is called the *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$ (see [3]).

For $f(t), g(t) \in \mathcal{F}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k, \tag{1.6}$$

and

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \quad \text{(see [3])}. \tag{1.7}$$

By (1.6), we get

$$\left.\frac{d^k p(x)}{dx^k}\right|_{x=0} = p^{(k)}(0) = \langle t^k|p(x)\rangle \quad \text{and} \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0). \tag{1.8}$$

Thus, from (1.8), we have

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{(see [1–3])}. \tag{1.9}$$

For $S_n(x) \sim (g(t), f(t))$, the following equations from (1.10) to (1.14) are well known in [3]:

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t)|S_k(x)\rangle}{k!} g(t)f(t)^k, \quad h(t) \in \mathcal{F}, \tag{1.10}$$

$$p(x) = \sum_{k=0}^{\infty} \frac{\langle g(t)f(t)^k|p(x)\rangle}{k!} S_k(x), \quad p(x) \in \mathbb{P}, \tag{1.11}$$

$$f(t)S_n(x) = nS_{n-1}(x), \qquad \langle h(t)|p(\alpha x)\rangle = \langle h(\alpha t)|p(x)\rangle, \tag{1.12}$$

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!}, \quad \text{for all } y \in \mathbb{C}, \tag{1.13}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$, and

$$S_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(y) S_{n-k}(x) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) S_{n-k}(y), \tag{1.14}$$

where $p_k(y) = g(t)S_k(y) \sim (1, f(t))$.

The *Euler polynomials* of order r are defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = e^{E^{(r)}(x)t} = \sum_{n=0}^{\infty} \frac{E_n^{(r)}(x)}{n!} t^n \quad \text{(see [1–3, 5–16])} \tag{1.15}$$

with the usual convention about replacing $(E^{(r)}(x))^n$ by $E_n^{(r)}(x)$. In the special case, $x = 0$, $E_n^{(r)}(0) = E_n^{(r)}$ are called the *Euler numbers* of order r.

As is well known, the higher-order Bernoulli polynomials are also defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = e^{B^{(r)}(x)t} = \sum_{n=0}^{\infty} \frac{B_n^{(r)}(x)}{n!} t^n \quad \text{(see [1–3, 5–16])} \tag{1.16}$$

with the usual convention about replacing $(B^{(r)}(x))^n$ by $B_n^{(r)}(x)$. In the special case, $x = 0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the *Bernoulli numbers* of order r.

Recently, several researchers have studied the umbral calculus related to special polynomials. In this paper, we derive some interesting identities related to Bernoulli, Euler, and Abel polynomials arising from umbral calculus.

2 Some identities of special polynomials

It is known [3] that

$$xB_{n-1}^{(na)}(x) \sim \left(1, \left(\frac{e^t-1}{t}\right)^a t\right), \quad x^n \sim (1, t), \tag{2.1}$$

where $n \in \mathbb{N}$ and $a \neq 0$. From (2.1), we have

$$\begin{aligned} x^n &= x\left(\frac{e^t-1}{t}\right)^{an} x^{-1} x B_{n-1}^{(na)}(x) = x\left(\frac{e^t-1}{t}\right)^{an} B_{n-1}^{(na)}(x) \\ &= x \sum_{l=0}^{\infty} \frac{(an)!}{(l+an)!} S_2(l+an, an) t^l B_{n-1}^{(na)}(x) \\ &= x \sum_{l=0}^{n-1} \frac{(an)!}{(l+an)!} S_2(l+an, an)(n-1)_l B_{n-1-l}^{(na)}(x), \end{aligned} \tag{2.2}$$

where $S_2(n, l)$ is the Stirling number of the second kind. Therefore, by (2.2), we obtain the following theorem.

Theorem 2.1 *For $n \in \mathbb{N}$ and $a \neq 0$, we have*

$$x^{n-1} = \sum_{l=0}^{n-1} \frac{(an)!}{(l+an)!} S_2(l+an, an)(n-1)_l B_{n-1-l}^{(na)}(x),$$

where $(a)_n = a(a-1)\cdots(a-n+1)$.

In [3], we note that

$$S_n(x)=\sum_{k=1}^{n}\binom{-an}{n-k}(n-1)_{n-k}x^k\sim\bigl(1,t(1+t)^a\bigr),\tag{2.3}$$

and

$$\phi_n(x)=\sum_{k=0}^{n}S_2(n,k)x^k\sim\bigl(1,\log(1+t)\bigr),\tag{2.4}$$

where $a\neq 0$.

For $n\geq 1$, we have

$$\begin{aligned}\phi_n(x)&=x\left(\frac{t(1+t)^a}{\log(1+t)}\right)^n x^{-1}S_n(x)\\&=x\left(\frac{t(1+t)^a}{\log(1+t)}\right)^n\sum_{l=1}^{n}\binom{-an}{n-l}(n-1)_{n-l}x^{l-1}.\end{aligned}\tag{2.5}$$

The Bernoulli polynomials $b_n(x)$ of the second kind are defined by the generating function to be

$$\frac{t}{\log(1+t)}(1+t)^x=\sum_{k=0}^{\infty}b_k(x)\frac{t^k}{k!}\quad\text{(see [3])}.\tag{2.6}$$

By (2.5) and (2.6), we get

$$\begin{aligned}\phi_n(x)&=\sum_{l=1}^{n}\binom{-an}{n-l}(n-1)_{n-l}x\left(\frac{t(1+t)^a}{\log(1+t)}\right)^n x^{l-1}\\&=\sum_{l=1}^{n}\binom{-an}{n-l}(n-1)_{n-l}x\left(\sum_{k=0}^{\infty}\frac{b_k(a)}{k!}t^k\right)^n x^{l-1}\\&=\sum_{l=1}^{\infty}\binom{-an}{n-l}(n-1)_{n-l}x\sum_{k=0}^{\infty}\left(\sum_{l_1+\cdots+l_n=k}\binom{k}{l_1,\ldots,l_n}b_{l_1}(a)\cdots b_{l_n}(a)\right)\frac{t^k}{k!}x^{l-1}\\&=\sum_{l=1}^{n}\binom{-an}{n-l}(n-1)_{n-l}x\sum_{k=0}^{l-1}\left(\sum_{l_1+\cdots+l_n=k}\binom{k}{l_1,\ldots,l_n}b_{l_1}(a)\cdots b_{l_n}(a)\right)\frac{(l-1)_k}{k!}x^{l-1-k}\\&=\sum_{l=1}^{n}\sum_{k=0}^{l-1}\sum_{l_1+\cdots+l_n=k}\binom{-an}{n-l}(n-1)_{n-l}\binom{l-1}{k}\binom{k}{l_1,\ldots,l_n}b_{l_1}(a)\cdots b_{l_n}(a)x^{l-k}\\&=\sum_{l=1}^{n}\sum_{m=1}^{l}\sum_{l_1+\cdots+l_n=l-m}\binom{-an}{n-l}(n-1)_{n-l}\binom{l-1}{m-1}\binom{l-m}{l_1,\ldots,l_n}b_{l_1}(a)\cdots b_{l_n}(a)x^m\\&=\sum_{m=1}^{n}\left\{\sum_{l=m}^{n}\sum_{l_1+\cdots+l_n=l-m}\binom{-an}{n-l}(n-1)_{n-l}\binom{l-1}{m-1}\binom{l-m}{l_1,\ldots,l_n}\right.\\&\qquad\left.\times b_{l_1}(a)\cdots b_{l_n}(a)\right\}x^m.\end{aligned}\tag{2.7}$$

Therefore, by (2.4) and (2.7), we obtain the following theorem.

Theorem 2.2 *For $a \neq 0$, $n \geq 1$ with $1 \leq m \leq n$, we have*

$$S_2(n,m) = \sum_{l=m}^{n} \sum_{l_1+\cdots+l_n=l-m} \binom{-an}{n-l}(n-1)_{n-l}\binom{l-1}{m-1}\binom{l-m}{l_1,\ldots,l_n} b_{l_1}(a)\cdots b_{l_n}(a).$$

It is well known (see [3]) that

$$\left(\frac{t}{\log(1+t)}\right)^n (1+t)^{x-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(x)\frac{t^k}{k!}. \tag{2.8}$$

Thus, by (2.8), we get

$$\left(\frac{t(1+t)^a}{\log(1+t)}\right)^n = \sum_{k=0}^{\infty} B_k^{(k-n+1)}(an+1)\frac{t^k}{k!}, \tag{2.9}$$

and

$$\left(\frac{t(1+t)^a}{\log(1+t)}\right)^n = \sum_{k=0}^{\infty}\left(\sum_{l_1+\cdots+l_n=k}\binom{k}{l_1,\ldots,l_n} b_{l_1}(a)\cdots b_{l_n}(a)\right)\frac{t^k}{k!}. \tag{2.10}$$

Therefore, by (2.9) and (2.10), we obtain the following lemma.

Lemma 2.3 *For $n, k \in \mathbb{Z}_+$, we have*

$$\sum_{l_1+\cdots+l_n=k}\binom{k}{l_1,\ldots,l_n} b_{l_1}(a)\cdots b_{l_n}(a) = B_k^{(k-n+1)}(an+1).$$

Let us consider the following sequences:

$$\begin{aligned} &S_n(x) \sim \left(1, \left(\frac{e^t+1}{2}\right)^a t\right) \quad (a \in \mathbb{R}), \\ &x^n \sim (1,t) \quad (n \geq 0). \end{aligned} \tag{2.11}$$

Then from (2.11), we have

$$\begin{aligned} S_n(x) &= x\left(\frac{2}{e^t+1}\right)^{an} x^{-1}x^n = x\left(\frac{2}{e^t+1}\right)^{an} x^{n-1} \\ &= xE_{n-1}^{(an)}(x). \end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following proposition.

Proposition 2.4 *For $a \in \mathbb{R}$, $n \in \mathbb{N}$, we have*

$$xE_{n-1}^{(an)}(x) \sim \left(1, \left(\frac{e^t+1}{2}\right)^a t\right).$$

The Abel sequence is given by

$$A_n(x;b) = x(x-bn)^{n-1} \sim \left(1, te^{bt}\right) \quad (b \neq 0). \tag{2.13}$$

By Proposition 2.4 and (2.13), we get

$$\begin{aligned} xE_{n-1}^{(na)}(x) &= x\left(\frac{te^{bt}}{(\frac{e^t+1}{2})^a t}\right)^n x^{-1}A_n(x;b) \\ &= x\left(\frac{2}{e^t+1}\right)^{an} e^{bnt}x^{-1}A_n(x;b) \\ &= x\left(\sum_{k=0}^{\infty}\frac{E^{(an)}(bn)}{k!}t^k\right)(x-bn)^{n-1} \\ &= x\sum_{k=0}^{n-1}\binom{n-1}{k}E_k^{(an)}(bn)(x-bn)^{n-1-k}. \end{aligned} \tag{2.14}$$

Therefore, by (2.14), we obtain the following theorem.

Theorem 2.5 *For $n \in \mathbb{N}$ and $a \in \mathbb{R}$, we have*

$$\begin{aligned} E_{n-1}^{(an)}(x) &= \sum_{k=0}^{n-1}\binom{n-1}{k}E_k^{(an)}(bn)(x-bn)^{n-1-k} \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}E_{n-1-k}^{(an)}(bn)(x-bn)^{k}. \end{aligned}$$

Let us consider the following Sheffer sequences:

$$\begin{aligned} &G_n(x;a,b) \sim \left(1, e^{at}\left(e^{bt}-1\right)\right) \quad (b \neq 0), \\ &A_n(x;c+a) \sim \left(1, te^{(c+a)t}\right) \quad (c+a \neq 0). \end{aligned} \tag{2.15}$$

By (2.15), we note that

$$G_n(x;a,b) = \frac{x}{b}\left(\frac{x-an}{b}-1\right)_{n-1}. \tag{2.16}$$

For $n \geq 1$, from (2.15), we have

$$\begin{aligned} A_n(x;c+a) &= x\left(\frac{e^{at}(e^{bt}-1)}{te^{(c+a)t}}\right)^n x^{-1}G_n(x;a,b) \\ &= x\left(\frac{e^{bt}-1}{te^{ct}}\right)^n x^{-1}G_n(x;a,b), \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \frac{(e^t-1)^n}{e^{tx}t^n} &= \frac{1}{t^n}\left(n!\sum_{j=n}^{\infty}S_2(j,n)\frac{t^j}{j!}\right)\left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}x^l t^l\right) \\ &= \left(n!\sum_{j=0}^{\infty}S_2(j+n,n)\frac{t^j}{(j+n)!}\right)\left(\sum_{l=0}^{\infty}\frac{(-1)^l x^l}{l!}t^l\right) \\ &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}S_2(j+n,n)\frac{(-1)^{k-j}\binom{k}{j}}{\binom{j+n}{j}}x^{k-j}\right)\frac{t^k}{k!}. \end{aligned} \tag{2.18}$$

From (2.18), we can derive the following equation (2.19):

$$\frac{(e^{bt}-1)^n}{e^{bt(\frac{c}{b}n)}(bt)^n} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^{k-j}\frac{\binom{k}{j}S_2(j+n,n)}{\binom{j+n}{j}}\left(\frac{c}{b}n\right)^{k-j}\right)\frac{(bt)^k}{k!}. \tag{2.19}$$

Thus, by (2.19), we get

$$\left(\frac{e^{bt}-1}{te^{ct}}\right)^n = b^n\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-cn)^{k-j}\frac{\binom{k}{j}S_2(j+n,n)}{\binom{j+n}{j}}b^j\right)\frac{t^k}{k!}. \tag{2.20}$$

From (2.16), (2.17), and (2.20), we can derive the following equation (2.21):

$$\begin{aligned} &A_n(x;c+a) \\ &\quad = b^{n-1}\sum_{k=0}^{n-1}\left(\sum_{j=0}^{k}(-cn)^{k-j}\frac{\binom{k}{j}S_2(j+n,n)b^j}{\binom{j+n}{j}}\right)x\frac{t^k}{k!}\left(\frac{x-an}{b}-1\right)_{n-1}, \end{aligned} \tag{2.21}$$

and

$$\left(\frac{x-an}{b}-1\right)_{n-1} = \sum_{l=0}^{n-1}S_1(n-1,l)\left(\frac{x-an}{b}-1\right)^l, \tag{2.22}$$

where $S_1(n,l)$ is the Stirling number of the first kind. By (2.22), we get

$$\frac{t^k}{k!}\left(\frac{x-an}{b}-1\right)_{n-1} = \sum_{l=k}^{n-1}S_1(n-1,l)\binom{l}{k}\left(\frac{x-an}{b}-1\right)^{l-k}b^{-k}. \tag{2.23}$$

Thus, by (2.21) and (2.23), we get

$$\begin{aligned} &A_n(x;c+a) \\ &\quad = b^{n-1}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{l=k}^{n-1}\left(-\frac{cn}{b}\right)^{k-j}\frac{\binom{k}{j}\binom{l}{k}S_2(j+n,n)S_1(n-1,l)}{\binom{j+n}{j}}x\left(\frac{x-an}{b}-1\right)^{l-k}. \end{aligned} \tag{2.24}$$

From (1.14), we have

$$A_n(x;c+a) = x\bigl(x-(c+a)n\bigr)^{n-1}. \tag{2.25}$$

Therefore, by (2.24) and (2.25), we obtain the following theorem.

Theorem 2.6 *For $n \geq 1$, $b \neq 0$, $c+a \neq 0$, we have*

$$\begin{aligned} &\bigl(x-(c+a)n\bigr)^{n-1} \\ &\quad = b^{n-1}\sum_{k=0}^{n-1}\sum_{j=0}^{k}\sum_{l=k}^{n-1}\left(-\frac{cn}{b}\right)^{k-j}\frac{\binom{k}{j}\binom{l}{k}S_2(j+n,n)S_1(n-1,l)}{\binom{j+n}{j}}\left(\frac{x-an}{b}-1\right)^{l-k}. \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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