Journal of Number Theory 147 (2015) 871–882

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Umbral calculus associated with Bernoulli polynomials

Dae San Kim [a], Taekyun Kim [b,*]

[a] Department of Mathematics, Sogang University, Seoul 121-741, Republic of Korea
[b] Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea

ARTICLE INFO

Article history:
Received 6 November 2012
Received in revised form 10 September 2013
Accepted 17 September 2013
Available online 9 December 2013
Communicated by David Goss

Keywords:
Bernoulli polynomials
Umbral calculus
Linear functional

ABSTRACT

Text. Recently, R. Dere and Y. Simsek have studied applications of umbral algebra to generating functions for the Hermite type Genocchi polynomials and numbers [6]. In this paper, we investigate some interesting properties arising from umbral calculus. These properties are useful in deriving some identities of Bernoulli polynomials.

Video. For a video summary of this paper, please click here or visit http://youtu.be/r7_Bbf0BDbs.

1. Introduction

As is well known, the Bernoulli polynomials are defined by the generating function to be

$$\frac{te^{xt}}{e^t - 1} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1}$$

* Corresponding author.
E-mail addresses: dskim@sogang.ac.kr (D.S. Kim), tkkim@kw.ac.kr (T. Kim).

http://dx.doi.org/10.1016/j.jnt.2013.09.013

with the usual convention about replacing $B^n(x)$ by $B_n(x)$. In the special case, $x = 0$, $B_n(0) = B_n$ are called the n-th Bernoulli numbers. From (1), we note that

$$B_0 = 1, \qquad (B+1)^n - B^n = B_n(1) - B_n = \delta_{1,n} \quad \bigl(\text{see [2–4]}\bigr),$$

where $\delta_{m,k}$ is the Kronecker symbol.

In particular, by (1), we set

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l \quad \bigl(\text{see [1–4,9,14]}\bigr). \tag{2}$$

By (2), we see that $B_n(x)$ is a monic polynomial of degree n. We recall the Euler polynomials are defined by the generating function to be

$$\frac{2e^{xt}}{e^t+1} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad \bigl(\text{see [1–4,6,9–24]}\bigr), \tag{3}$$

with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers. From (3), we can derive the following equation:

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l} x^l \quad \bigl(\text{see [11,14]}\bigr). \tag{4}$$

Thus by (4), we see that $E_n(x)$ is also a monic polynomial of degree n. By (4), we get

$$E_0 = 1, \qquad (E+1)^n + E^n = E_n(1) + E_n = 2\delta_{0,n}. \tag{5}$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \colon\ a_k \in \mathbb{C} \right\}.$$

We use the notation $\mathbb{P} = \mathbb{C}[x]$ and $\mathbb{P}^*$ denotes the vector space of all linear functional on $\mathbb{P}$.

Let $\langle L \mid p(x) \rangle$ be the action of a linear functional L on a polynomial $p(x)$, and we remind that the vector space operations on $\mathbb{P}^*$ are defined by

$$\bigl\langle L + M \bigm| p(x) \bigr\rangle = \bigl\langle L \bigm| p(x) \bigr\rangle + \bigl\langle M \bigm| p(x) \bigr\rangle,$$
$$\bigl\langle cL \bigm| p(x) \bigr\rangle = c\bigl\langle L \bigm| p(x) \bigr\rangle \quad \bigl(\text{see [6,18]}\bigr),$$

where c is any constant in $\mathbb{C}$.

The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F} \tag{6}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t) \mid x^n \rangle = a_n \quad \text{for all } n \geqslant 0. \tag{7}$$

Thus, by (6) and (7), we have

$$\langle t^k \mid x^n \rangle = n!\delta_{n,k} \quad (\text{see [6,18]}). \tag{8}$$

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k \rangle}{k!} t^k$. Then we see that $\langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle$ and so as linear functionals $L = f_L(t)$ (see [6,18]). As is known in [18], the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will denote both the algebra of formal poser series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We shall call $\mathcal{F}$ the umbral algebra. The umbral calculus is the study of umbral algebra and modern classical umbral calculus can be described as a systematic study of the class of Sheffer sequences (see [18]).

The order $ord(f(t))$ of a nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. If a series $f(t)$ is with $ord(f(t)) = 1$, then $f(t)$ is called a delta series. If a series $f(t)$ is with $ord(f(t)) = 0$, then $f(t)$ is called an invertible series (see [6,18]). For $f(t), g(t) \in \mathcal{F}$, we have

$$\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle \quad (\text{see [6,18]}). \tag{9}$$

Let us assume that $S_n(x)$ denotes a polynomial of degree n. If $f(t)$ is a delta series and $g(t)$ is an invertible series, then there exists a unique sequence $S_n(x)$ such that $\langle g(t)f(t)^k \mid S_n(x) \rangle = n!\delta_{n.k}$, $n, k \geqslant 0$ (see [6]). The sequence $S_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$, denoted by $S_n(x) \sim (g(t), f(t))$. If $S_n(x) \sim (1, f(t))$, then $S_n(x)$ is called the associated sequence for $f(t)$ or $S_n(x)$ is associated to $f(t)$. If $S_n(x) \sim (g(t), t)$, then $S_n(x)$ is called the Appell sequence for $g(t)$ or $S_n(x)$ is Appell for $g(t)$ (see [6,18]). For $p(x) \in \mathbb{P}$, it is known (see [6,18]) that

$$\left\langle \frac{e^{yt} - 1}{t} \,\middle|\, p(x) \right\rangle = \int_0^y p(u)\, du, \tag{10}$$

$$\langle f(t) \mid xp(x) \rangle = \langle \partial_t f(t) \mid p(x) \rangle = \langle f'(t) \mid p(x) \rangle, \tag{11}$$

and

$$\langle e^{yt} - 1 \mid p(x) \rangle = p(y) - p(0) \quad (\text{see [6,18]}). \tag{12}$$

Let us assume that $S_n(x) \sim (g(t), f(t))$. Then we have the following equations (13)–(16):

$$h(t) = \sum_{k=0}^{\infty} \frac{\langle h(t) \mid S_k(x) \rangle}{k!} g(t) f(t)^k, \quad h(t) \in \mathcal{F}, \tag{13}$$

$$p(t) = \sum_{k=0}^{\infty} \frac{\langle g(t) f(t)^k \mid p(x) \rangle}{k!} S_k(x), \quad p(t) \in \mathbb{P}, \tag{14}$$

$$f(t) S_n(x) = n S_{n-1}(x) \quad \big(n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}\big), \tag{15}$$

and

$$\frac{1}{g(\bar{f}(t))} e^{y \bar{f}(t)} = \sum_{k=0}^{\infty} \frac{S_k(y)}{k!} t^k, \quad \text{for all } y \in \mathbb{C}, \tag{16}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [18]).

Let $f_1(t), \dots, f_m(t) \in \mathcal{F}$. Then as is well known, we have

$$\big\langle f_1(t) f_2(t) \cdots f_m(t) \mid x^n \big\rangle = \sum \binom{n}{i_1, \dots, i_m} \big\langle f_1(t) \mid x^{i_1} \big\rangle \cdots \big\langle f_m(t) \mid x^{i_m} \big\rangle, \tag{17}$$

where the sum is over all nonnegative integers $i_1, \dots, i_m$ such that $i_1 + \cdots + i_m = n$ (see [6,18]).

In [6], R. Dere and Y. Simsek have studied applications of umbral algebra to generating functions for the Hermite type Genocchi polynomials and numbers. In this paper, we derive some interesting properties of Bernoulli polynomials arising from umbral calculus. These properties will be used in studying identities on the Bernoulli polynomials

2. Umbral calculus and Bernoulli polynomials

Let $\mathbb{P}_n = \{p(x) \in \mathbb{C}[x]\colon \deg p(x) \leqslant n\}$ and let $S_n(x) \sim (g(t), t)$. From (16), we have

$$\frac{1}{g(t)} x^n = S_n(x) \quad \Leftrightarrow \quad x^n = g(t) S_n(x) \quad (n \geqslant 0). \tag{18}$$

Let us take $g(t) = \frac{1}{t}(e^t - 1) \in \mathcal{F}$. Then $g(t)$ is invertible series. By (1), we get

$$\sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}. \tag{19}$$

Thus by (19), we have

$$\frac{1}{g(t)} x^n = B_n(x) \quad (n \geqslant 0), \tag{20}$$

and

$$tB_n(x) = B_n'(x) = nB_{n-1}(x). \tag{21}$$

From (20) and (21), we note that $B_n(x)$ is an Appell sequence for $\frac{1}{t}(e^t - 1)$. By (2), we get

$$\begin{aligned}\int_x^{x+y} B_n(u)\,du &= \frac{1}{n+1}\{B_{n+1}(x+y) - B_{n+1}(x)\} \\ &= \sum_{k=1}^{\infty} \frac{y^k}{k!} t^{k-1} B_n(x) = \frac{e^{yt}-1}{t} B_n(x).\end{aligned} \tag{22}$$

In particular, for $y = 1$, we have

$$B_n(x) = \frac{t}{e^t - 1} \int_x^{x+1} B_n(u)\,du = \frac{t}{e^t - 1} x^n. \tag{23}$$

By (15), we easily get

$$B_n(x) = t\left\{\frac{1}{n+1} B_{n+1}(x)\right\}. \tag{24}$$

From (24), we can derive the following equation:

$$\left\langle \frac{e^{yt}-1}{t} \,\middle|\, B_n(x) \right\rangle = \left\langle e^{yt} - 1 \,\middle|\, \frac{1}{n+1} B_{n+1}(x) \right\rangle = \int_0^y B_n(u)\,du. \tag{25}$$

For $r \in \mathbb{N}$, the n-th Bernoulli polynomials of order r are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!} \quad \big(\text{see [2,3]}\big). \tag{26}$$

In the special case, $x = 0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the n-th Bernoulli numbers of order r. By (26), we get

$$B_n^{(r)}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l}^{(r)} x^r \quad \big(\text{see [2,3]}\big). \tag{27}$$

Note that

$$B_n^{(r)} = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1,\ldots,l_r} B_{l_1}\cdots B_{l_r}. \tag{28}$$

From (27) and (28), we note that $B_n^{(r)}(x)$ is a monic polynomial with coefficients in $\mathbb{Q}$. By (27), we get

$$\begin{aligned} \int_x^{x+y} B_n^{(r)}(u)\,du &= \frac{1}{n+1}\left\{B_{n+1}^{(r)}(x+y) - B_{n+1}^{(r)}(x)\right\} \\ &= \sum_{k=1}^{\infty} \frac{y^k}{k!} t^{k-1} B_n(x) = \frac{e^{yt}-1}{t} B_n^{(r)}(x), \end{aligned} \tag{29}$$

and

$$B_n^{(r)}(x+1) - B_n^{(r)}(x) = nB_{n-1}^{(r-1)}(x). \tag{30}$$

From (29) and (30), we note that

$$\frac{e^t-1}{t} B_n^{(r)}(x) = \int_x^{x+1} B_n^{(r)}(u)\,du = B_n^{(r-1)}(x). \tag{31}$$

By (31), we get

$$B_n^{(r)}(x) = \left(\frac{t}{e^t-1}\right) B_n^{(r-1)}(x) = \left(\frac{t}{e^t-1}\right)^{r-1} B_n(x) = \left(\frac{t}{e^t-1}\right)^{r} x^n, \tag{32}$$

and

$$tB_n^{(r)}(x) = n\left(\frac{t}{e^t-1}\right)^{r} x^{n-1} = nB_{n-1}^{(r)}(x). \tag{33}$$

It is easy to show that $(\frac{e^t-1}{t})^r$ is an invertible series in $\mathcal{F}$. Therefore, by (32) and (33), we obtain the following lemma.

Lemma 1. *$B_n^{(r)}(x)$ is the Appell sequence for $(\frac{e^t-1}{t})^r$.*

By (33), we get

$$B_n^{(r)}(x) = t\left\{\frac{1}{n+1} B_{n+1}^{(r)}(x)\right\} \quad (n \geqslant 0). \tag{34}$$

Thus, from (34), we have

$$\left\langle \frac{e^{yt}-1}{t} \,\middle|\, B_n^{(r)}(x) \right\rangle = \left\langle e^{yt}-1 \,\middle|\, \frac{1}{n+1} B_{n+1}^{(r)}(x) \right\rangle = \int_0^y B_n^{(r)}(u)\,du. \tag{35}$$

In the special case, $y = 1$, we have

$$\left\langle \frac{e^{t}-1}{t} \,\middle|\, B_n^{(r)}(x) \right\rangle = \int_0^1 B_n^{(r)}(u)\,du = B_n^{(r-1)}.$$

By (17), we get

$$\left\langle \left(\frac{t}{e^t-1}\right)^r \,\middle|\, x^n \right\rangle = \sum_{n=i_1+\cdots+i_r} \binom{n}{i_1,\ldots,i_r} \left\langle \frac{t}{e^t-1} \,\middle|\, x^{i_1} \right\rangle \cdots \left\langle \frac{t}{e^t-1} \,\middle|\, x^{i_r} \right\rangle \tag{36}$$

and

$$\left\langle \frac{t}{e^t-1} \,\middle|\, x^n \right\rangle = B_n, \qquad \left\langle \left(\frac{t}{e^t-1}\right)^r \,\middle|\, x^n \right\rangle = B_n^{(r)}. \tag{37}$$

Thus, from (36) and (37), we have

$$\sum_{n=i_1+\cdots+i_r} \binom{n}{i_!,\ldots,i_r} B_{i_1}\cdots B_{i_r} = B_n^{(r)}.$$

Let us take $p(x) \in \mathbb{P}_n$ with

$$p(x) = \sum_{k=0}^{n} b_k B_k(x). \tag{38}$$

From (20) and (21), we note that $B_n(x) \sim (\frac{e^t-1}{t}, t)$. By the definition of Appell sequences, we get

$$\left\langle \frac{e^t-1}{t} t^k \,\middle|\, B_n(x) \right\rangle = n!\delta_{n,k} \quad (n,k \geqslant 0), \tag{39}$$

and, from (38), we have

$$\left\langle \frac{e^t-1}{t} t^k \,\middle|\, p(x) \right\rangle = \sum_{l=0}^{n} b_l \left\langle \frac{e^t-1}{t} t^k \,\middle|\, B_l(x) \right\rangle = \sum_{l=0}^{n} b_l l!\delta_{l,k} = k!b_k. \tag{40}$$

Thus, by (25) and (40), we get

$$b_k = \frac{1}{k!}\left\langle \frac{e^t-1}{t} t^k \,\middle|\, p(x) \right\rangle = \frac{1}{k!}\left\langle \frac{e^t-1}{t} \,\middle|\, p^{(k)}(x) \right\rangle = \frac{1}{k!}\int_0^1 p^{(k)}(u)\,du, \tag{41}$$

where $p^{(k)}(u) = \frac{d^k}{du^k}p(u)$. Therefore, by (38) and (41), we obtain the following theorem.

Theorem 2. *Let $p(x) \in \mathbb{P}_n$ with $p(x) = \sum_{k=0}^n b_k B_k(x)$. Then we have*

$$b_k = \frac{1}{k!}\left\langle \frac{e^t-1}{t} \,\middle|\, p^{(k)}(x) \right\rangle = \frac{1}{k!}\int_0^1 p^{(k)}(u)\,du,$$

where $p^{(k)}(u) = \frac{d^k}{du^k}p(u)$.

Let $p(x) = B_n^{(r)}(x) \in \mathbb{P}_n$ with $p(x) = \sum_{k=0}^n b_k B_k(x)$. Then we have

$$p^{(k)}(x) = k!\binom{n}{k} B_{n-k}^{(r)}(x), \tag{42}$$

and

$$b_k = \frac{1}{k!}\left\langle \frac{e^t-1}{t} \,\middle|\, p^{(k)}(x) \right\rangle = \binom{n}{k}\left\langle \frac{e^t-1}{t} \,\middle|\, B_{n-k}^{(r)}(x) \right\rangle. \tag{43}$$

Therefore, by Theorem 2 and (43), we obtain the following corollary.

Corollary 3. *For $n \geqslant 0$, we have*

$$B_n^{(r)}(x) = \sum_{k=0}^n \binom{n}{k}\left\langle \frac{e^t-1}{t} \,\middle|\, B_{n-k}^{(r)}(x) \right\rangle B_k(x).$$

In other words,

$$B_n^{(r)}(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k}^{(r-1)} B_k(x), \quad \textit{where } n-k \geqslant 0.$$

From the definition of Appell sequences, we note that

$$\left\langle \left(\frac{e^t-1}{t}\right)^r t^k \,\middle|\, B_n^{(r)}(x) \right\rangle = n!\delta_{n,k} \quad (n,k \geqslant 0). \tag{44}$$

Let $p(x) \in \mathbb{P}_n$ with $p(x) = \sum_{k=0}^{n} b_k^{(r)} B_k^{(r)}(x)$. By (44), we get

$$\left\langle \left(\frac{e^t-1}{t}\right)^r t^k \,\middle|\, p(x) \right\rangle = \sum_{l=0}^{n} b_l^{(r)} \left\langle \left(\frac{e^t-1}{t}\right)^r t^k \,\middle|\, B_l^{(r)}(x) \right\rangle$$
$$= \sum_{l=0}^{n} b_l^{(r)} l! \delta_{l,k} = k! b_k^{(r)}. \tag{45}$$

Thus, by (45), we have

$$b_k^{(r)} = \frac{1}{k!} \left\langle \left(\frac{e^t-1}{t}\right)^r t^k \,\middle|\, p(x) \right\rangle. \tag{46}$$

Therefore, by (46), we obtain the following theorem.

Theorem 4. *Let $p(x) \in \mathbb{P}_n$ with $p(x) = \sum_{k=0}^{n} b_k^{(r)} B_k^{(r)}(x)$. Then we have*

$$b_k^{(r)} = \frac{1}{k!} \left\langle \left(\frac{e^t-1}{t}\right)^r t^k \,\middle|\, p(x) \right\rangle.$$

Let us consider $p(x) = B_n(x)$ with

$$B_n(x) = p(x) = \sum_{k=0}^{n} b_k^{(r)} B_k^{(r)}(x). \tag{47}$$

By Theorem 4 and (47), we get

$$b_k^{(r)} = \frac{1}{k!} \left\langle \left(\frac{e^t-1}{t}\right)^r t^k \,\middle|\, p(x) \right\rangle = \frac{1}{k!} \left\langle \left(\frac{e^t-1}{t}\right)^r t^k \,\middle|\, B_n(x) \right\rangle. \tag{48}$$

For $k < r$, we have

$$b_k^{(r)} = \frac{1}{k!} \left\langle \left(\frac{e^t-1}{t}\right)^r t^k \,\middle|\, B_n(x) \right\rangle$$
$$= \frac{1}{k!} \left\langle \left(e^t-1\right)^r \,\middle|\, \frac{B_{n+r-k}(x)}{(n+1)\cdots(n+r-k)} \right\rangle$$
$$= \frac{1}{k!(r-k)!\binom{n+r-k}{r-k}} \left\langle \left(e^t-1\right)^r \mid B_{n+r-k}(x) \right\rangle$$
$$= \frac{\binom{r}{k}}{r!\binom{n+r-k}{r-k}} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} \left\langle e^{jt} \mid B_{n+r-k}(x) \right\rangle$$
$$= \frac{\binom{r}{k}}{r!\binom{n+r-k}{r-k}} \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j} B_{n+r-k}(j). \tag{49}$$

Let $k \geqslant r$. Then by (48), we get

$$b_k^{(r)} = \frac{1}{k!}\langle (e^t-1)^r t^{k-r} \mid B_n(x)\rangle = \frac{1}{k!}\langle (e^t-1)^r \mid t^{k-r}B_n(x)\rangle$$
$$= \frac{1}{k!}\binom{n}{k-r}(k-r)!\langle (e^t-1)^r \mid B_{n+r-k}(x)\rangle$$
$$= \frac{\binom{n}{k-r}}{r!\binom{k}{r}}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\langle e^{jt} \mid B_{n+r-k}(x)\rangle$$
$$= \frac{\binom{n}{k-r}}{r!\binom{k}{r}}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}B_{n+r-k}(j). \tag{50}$$

Therefore, by (47), (49) and (50), we obtain the following theorem.

Theorem 5. *For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have*

$$B_n(x) = \sum_{k=0}^{r-1}\frac{\binom{r}{k}}{r!\binom{n+r-k}{r-k}}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}B_{n+r-k}(j)B_k^{(r)}(x)$$
$$+\sum_{k=r}^{n}\frac{\binom{n}{k-r}}{r!\binom{k}{r}}\sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}B_{n+r-k}(j)B_k^{(r)}(x).$$

3. Further remarks

For $n, m \in \mathbb{Z}_+$ with $n-m \geqslant 0$, we have

$$B_n^{(r)}(x)B_{n-m}^{(r)}(x) = \left(\sum_{l=0}^{n}\binom{n}{l}B_{n-l}^{(r-1)}B_l(x)\right)\left(\sum_{p=0}^{n-m}\binom{n-m}{p}B_{n-m-p}^{(r-1)}B_p(x)\right)$$
$$= \sum_{k=0}^{2n-m}\sum_{p=0}^{k}\binom{n-m}{p}\binom{n}{k-p}B_{n-m-p}^{(r-1)}B_{n-k+p}^{(r-1)}B_p(x)B_{k-p}(x), \tag{51}$$

where $n-m-p \geqslant 0$.

Let us consider $p(x) = E_n(x)$ with

$$E_n(x) = p(x) = \sum_{k=0}^{n} b_k B_k(x). \tag{52}$$

Then we have

$$p^{(k)}(x) = k!\binom{n}{k}E_{n-k}(x), \tag{53}$$

and

$$b_k = \frac{1}{k!}\left\langle \frac{e^t-1}{t} \,\middle|\, p^{(k)}(x) \right\rangle = \binom{n}{k}\left\langle \frac{e^t-1}{t} \,\middle|\, E_{n-k}(x) \right\rangle$$
$$= \binom{n}{k}\frac{E_{n-k+1}(1)-E_{n-k+1}}{n-k+1} = -2\binom{n}{k}\frac{E_{n-k+1}}{n-k+1}. \tag{54}$$

By (52) and (54), we get

$$E_n(x) = -2\sum_{k=0}^{n}\binom{n}{k}\frac{E_{n-k+1}}{n-k+1}B_k(x). \tag{55}$$

From (55), we can derive the following equation.

$$\begin{aligned} &E_n(x)E_{n-m}(x) \\ &= 4\left(\sum_{l=0}^{n}\binom{n}{l}\frac{E_{n-l+1}}{n-l+1}B_l(x)\right)\left(\sum_{p=0}^{n-m}\binom{n-m}{p}\frac{E_{n-m-p+1}}{n-m-p+1}B_p(x)\right) \\ &= 4\sum_{k=0}^{2n-m}\sum_{l=0}^{k}\binom{n-m}{l}\binom{n}{k-l}\frac{E_{n-m-l+1}E_{n-k+l+1}}{(n-m-l+1)(n-k+l+1)}B_l(x)B_{k-l}(x), \end{aligned}$$

where $n, m \in \mathbb{Z}_+$ with $n-m \geqslant 0$.

Remark. Recently, R. Dere, Y. Simsek and H.M. Srivastava have studied special polynomials with viewpoint of umbral calculus (see [5,7,8]).

Acknowledgments

The authors would like to express their gratitude for the valuable comments and suggestions of referees. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MOE) (No. 2012R1A1A2003786).

Appendix A. Supplementary material

The online version of this article contains additional supplementary material.

Please visit http://dx.doi.org/10.1016/j.jnt.2013.09.013.

References

[1] S. Araci, M. Acikgoz, A note on the Frobenius–Euler numbers and polynomials associated with Bernstein polynomials, Adv. Stud. Contemp. Math. 22 (3) (2012) 399–406.
[2] L. Carlitz, A note on Bernoulli numbers and polynomials of higher order, Proc. Amer. Math. Soc. 3 (1952) 608–613.
[3] L. Carlitz, Note on the integral of the product of several Bernoulli polynomials, J. Lond. Math. Soc. 34 (1959) 361–363.
[4] J. Choi, D.S. Kim, T. Kim, Y.H. Kim, Some arithmetic identities on Bernoulli and Euler numbers arising from the p-adic integrals on Z_p, Adv. Stud. Contemp. Math. 22 (2) (2012) 239–247.

[5] R. Dere, Y. Simsek, Genocchi polynomials associated with the umbral algebra, Appl. Math. Comput. 218 (2011) 756–761.

[6] R. Dere, Y. Simsek, Applications of umbral algebra to some special polynomials, Adv. Stud. Contemp. Math. 22 (3) (2012) 433–438.

[7] R. Dere, Y. Simsek, Remarks on the Frobenius–Euler polynomials on the umbral algebra, in: Numerical Analysis and Applied Mathematics, in: AIP Conf. Proc., vol. 1479, 2012, pp. 348–351.

[8] R. Dere, Y. Simsek, H.M. Srivastava, A unified presentation of three families of generalized Apostol type polynomials based upon the theory of the umbral calculus and the umbral algebra, J. Number Theory 133 (10) (2013) 3245–3263.

[9] D. Ding, J. Yang, Some identities related to the Apostol–Euler and Apostol–Bernoulli polynomials, Adv. Stud. Contemp. Math. 20 (1) (2010) 7–21.

[10] T. Kim, q-Volkenborn integration, Russ. J. Math. Phys. 9 (2002) 288–299.

[11] T. Kim, Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials, J. Difference Equ. Appl. 14 (2008) 1267–1277.

[12] D. Kim, T. Kim, Some identities of Bernoulli and Euler polynomials arising from umbral calculus, Adv. Stud. Contemp. Math. 23 (1) (2013) 159–171.

[13] G. Kim, B. Kim, J. Choi, The DC algorithm for computing sums of powers of consecutive integers and Bernoulli numbers, Adv. Stud. Contemp. Math. 17 (2008) 137–145.

[14] D.S. Kim, N. Lee, J. Na, H.K. Park, Identities of symmetry for higher-order Euler polynomials in three variables (I), Adv. Stud. Contemp. Math. 22 (1) (2012) 51–74.

[15] H. Ozden, I.N. Cangul, Y. Simsek, Remarks on q-Bernoulli numbers associated with Daehee numbers, Adv. Stud. Contemp. Math. 18 (1) (2009) 41–48.

[16] S.-H. Rim, J. Jeong, On the modified q-Euler numbers of higher order with weight, Adv. Stud. Contemp. Math. 22 (1) (2012) 93–98.

[17] S.-H. Rim, J. Lee, Some identities on the twisted (h, q)-Genocchi numbers and polynomials associated with q-Bernstein polynomials, Int. J. Math. Math. Sci. 2011 (2011), Art. ID 482840, 8 pp.

[18] S. Roman, The Umbral Calculus, Dover Publ. Inc., New York, 2005.

[19] C.S. Ryoo, On the generalized Barnes type multiple q-Euler polynomials twisted by ramified roots of unity, Proc. Jangjeon Math. Soc. 13 (2) (2010) 255–263.

[20] C.S. Ryoo, T. Kim, A new identities on the q-Bernoulli numbers and polynomials, Adv. Stud. Contemp. Math. 21 (2) (2011) 161–169.

[21] K. Shiratani, S. Yokoyama, An application of p-adic convolutions, Mem. Fac. Sci. Kyushu Univ. Ser. A 36 (1) (1982) 73–83.

[22] Y. Simsek, Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation function, Adv. Stud. Contemp. Math. 16 (2008) 251–278.

[23] Y. Simsek, Special functions related to Dedekind type DC-sums and their applications, Russ. J. Math. Phys. 17 (2010) 495–508.

[24] Z. Zhang, H. Yang, Some closed formulas for generalized Bernoulli–Euler numbers and polynomials, Proc. Jangjeon Math. Soc. 11 (2) (2008) 191–198.