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A symbolic handling of Sheffer polynomials

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Abstract We revisit the theory of Sheffer sequences by means of the formalism introduced in Rota and Taylor (SIAM J Math Anal 25(2):694–711, 1994) and developed in Di Nardo and Senato (Umbral nature of the Poisson random variables. Algebraic combinatorics and computer science, pp 245–256, Springer Italia, Milan, 2001, European J Combin 27(3):394–413, 2006). The advantage of this approach is twofold. First, this new syntax allows us noteworthy computational simplification and conceptual clarification in several topics involving Sheffer sequences, most of the open questions proposed in Taylor (Comput Math Appl 41:1085–1098, 2001) finds answer. Second, most of the results presented can be easily implemented in a symbolic language. To get a general idea of the effectiveness of this symbolic approach, we provide a formula linking connection constants and Riordan arrays via generalized Bell polynomials, here defined. Moreover, this link allows us to smooth out many results involving Bell Polynomials and Lagrange inversion formula.

Keywords Umbral calculus · Sheffer sequences · Connection constants · Riordan arrays · Lagrange inversion formula

Mathematics Subject Classification (2000) 05A40 · 05A15 · 11B83

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1 Introduction

Sheffer polynomials are a large class of polynomial sequences that include Laguerre polynomials, first and second kind Meixner polynomials, Bernoulli polynomials, Poisson–Charlier polynomials, Stirling polynomials and many others. The subject traces back to the end of thirties [24], but it is still a relevant topic because of its applications in dealing with a variety of subjects. Sheffer sequences of integral type [3] have applications in signal processing [2,16]. More recently, a connection between Sheffer polynomials and Lévy processes has been investigated in Ref. [17], see references therein. Applications in financial mathematics can be found in Ref. [23], and applications in statistics can be found in Ref. [15]. Symbolic computations of classical, boolean and free cumulants using generalized Abel polynomials (a subclass of Sheffer polynomials) have been given in Ref. [11]. New applications of Sheffer polynomials have also been found in mathematical physics, as reported in Ref. [29].

One of the most successful language commonly used for handling Sheffer sequences is the so-called umbral calculus. As well known, umbral calculus was formalized in the language of linear operators by Gian-Carlo Rota in a series of papers that have produced a plenty of applications (see references in Ref. [7]). In 1994, Rota and Taylor [20] came back to the foundation of umbral calculus with the aim to restore, in a light formal setting, the computational power of the original tools, heuristically applied by founders Blissard, Cayley and Sylvester. In this new setting, to which we refer as *the classical umbral calculus*, there are two basic devices. The first one is to represent a unital sequence of numbers by a symbol α, called an umbra, that is, to represent the sequence $1, a_1, a_2, \ldots$ by means of the sequence $1, \alpha, \alpha^2, \ldots$ of powers of α via an operator E, resembling the expectation operator of random variables. The second device is to represent the same sequence $1, a_1, a_2, \ldots$ by means of distinct umbrae, as it happens also in probability theory with independent and identically distributed random variables.

At first glance, the classical umbral calculus seems just a notation for dealing with exponential generating functions. Nevertheless, this new syntax has given rise noteworthy computational simplifications and conceptual clarifications in different contexts. Applications to bilinear generating functions for polynomial sequences are given by Gessel [13]. Connections with wavelet theory have been investigated in Refs. [22,25]. In Ref. [10], the theory of k-statistics and polykays has been completely rewritten, carrying out a unifying framework for these estimators, in both the univariate and multivariate cases. Moreover, very fast algorithms for computing these estimators have been carried out.

Apart from the preliminary paper of Taylor [28], Sheffer sequences have not been described in terms of umbrae. Indeed, most of the recent papers involving Sheffer sequences still use the language of linear operators (see for instance [31]). We believe that a symbolic theory of Sheffer sequences in terms of umbrae has a number of advantages. Here, by keeping the length of the present paper within bounds, we have chosen two fundamental topics, presented in Sect. 5, to which this new theory of Sheffer sequences can be fruitfully applied. The first application shows the connection between Abel and binomial polynomials. This connection allows us to simplify many results involving Bell polynomials and the Lagrange inversion formula. Moreover, this relation has smoothed the way to a symbolic treatment of free cumulant theory [12]. The latter application gives a closed form formula expressing the coefficients of a polynomial sequence $\{s_n(x)\}$ in terms of a different sequence of polynomials $\{p_n(x)\}$, which is particularly suited to be implemented in symbolic software. It is proved that this closed form formula gives also the expression of the elements of Riordan arrays.

The rest of the paper is structured as follows. In Sect. 2, we introduce terminology, notation and some basic definitions of the umbral calculus, adding the notion of the adjoint of an umbra. Using umbral polynomials, we stress a feature of the classical umbral calculus, that is the construction of new umbrae by suitable symbolic substitutions. Section 3 is devoted to Sheffer umbrae and their characterizations. In particular, Theorem 3.1 gives the umbral version of the well-known Sheffer identity with respect to the associated sequence. Theorem 3.3 gives a second characterization, that is $\{s_n(x)\}$ is said to be a Sheffer sequence with respect to a delta operator Q, when $Qs_n(x) = ns_{n-1}(x)$ for all $n \geq 0$. Finally, in Sect. 4, we characterize two special Sheffer umbrae whose moments are binomial and Appell sequences, respectively.

2 The classical umbral calculus

In the following, we recall terminology, notation and some basic definitions of the classical umbral calculus, as it has been introduced by Rota and Taylor in Ref. [20] and further developed in Refs. [8,9]. An umbral calculus consists of the following data:

a) a set $A = \{\alpha, \beta, \ldots\}$, called the *alphabet*, whose elements are named *umbrae*;
b) a commutative integral domain R whose quotient field is of characteristic zero;
c) a linear functional E, called *evaluation*, defined on the polynomial ring $R[A]$ and taking values in R, such that $E[1] = 1$ and

$$E\left[\alpha^i \beta^j \cdots \gamma^k\right] = E\left[\alpha^i\right] E\left[\beta^j\right] \cdots E\left[\gamma^k\right], \quad \text{(uncorrelation property)}$$

for any set of distinct umbrae in A and for $i, j, \ldots, k$ nonnegative integers.
d) an element $\epsilon \in A$, called *augmentation* [19], such that $E[\epsilon^n] = \delta_{0,n}$, for any nonnegative integer n, and an element $u \in A$, called *unity* umbra [8], such that $E[u^n] = 1$, for any nonnegative integer n.

A sequence $a_0 = 1, a_1, a_2, \ldots$ in R is umbrally represented by an umbra α when $E[\alpha^n] = a_n$ for all nonnegative integers n. The elements $\{a_n\}$ are called *moments* of the umbra α.

Example 2.1 Singleton umbra. The singleton umbra χ is such that $E[\chi^1] = 1$ and $E[\chi^n] = 0$ for $n = 2, 3, \ldots$.

The *factorial moments* of an umbra α are the elements $a_{(0)} = 1, a_{(n)} = E[(\alpha)_n]$ for all nonnegative integers n, where $(\alpha)_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1)$ is the lower factorial.

Example 2.2 Bell umbra. The Bell umbra β is such that $E[(\beta)_n] = 1$ for all nonnegative integers n. The moments of the Bell umbra are the Bell numbers $\{B_n\}$, $E[\beta^n] = B_n$, i.e. the number of partitions of a finite nonempty set with n elements, or the n-th coefficient in the Taylor series expansion of the function $\exp(e^t - 1)$.

An umbral polynomial is a polynomial $p \in R[A]$. The support of p is the set of all umbrae occurring in p. If p and q are two umbral polynomials, then p and q are *uncorrelated* if and only if their supports are disjoint. Moreover, p and q are *umbrally equivalent* if and only if $E[p] = E[q]$, in symbols $p \simeq q$.

Similar umbrae and dot-product Two umbrae α and γ are *similar* when α^n is umbrally equivalent to γ^n, for all $n = 0, 1, 2, \ldots$ in symbols $\alpha \equiv \gamma \Leftrightarrow \alpha^n \simeq \gamma^n$.

Example 2.3 Bernoulli umbra. The Bernoulli umbra ι is characterized by the following umbral relations $\iota + u \equiv -\iota$ and $(\iota + u)^n \simeq \iota^n$ for $n > 1$. Its moments are the Bernoulli numbers [20].

Thanks to the notion of similar umbrae, the alphabet A has been extended with the so-called *auxiliary* umbrae, resulting from operations among similar umbrae. This leads to the construction of a saturated umbral calculus, in which auxiliary umbrae are handled as elements of the alphabet. The symbol $n\boldsymbol{.}\alpha$ denotes the *dot-product* of n and α, an auxiliary umbra [20] similar to the sum $\alpha' + \alpha'' + \cdots + \alpha'''$, where $\{\alpha', \alpha'', \ldots, \alpha'''\}$ is a set of n distinct umbrae, each one similar to the umbra α. We assume $0\boldsymbol{.}\alpha \equiv \epsilon$. Properties of the dot-product have been investigated with full details in Ref. [8].

Two umbrae α and γ are *inverse* to each other when $\alpha + \gamma \equiv \epsilon$. The inverse of the umbra α is denoted by $-1\boldsymbol{.}\alpha$. Note that α and $-1\boldsymbol{.}\alpha$ are uncorrelated. In dealing with a saturated umbral calculus, the inverse of an umbra is not unique, but any two umbrae inverse to any given umbra are similar.

The symbol $\alpha^{\boldsymbol{.} n}$ denotes the *dot-power* of α, an auxiliary umbra [8] similar to the product $\alpha'\alpha'' \cdots \alpha'''$, where $\{\alpha', \alpha'', \ldots, \alpha'''\}$ is a set of n distinct umbrae, each one similar to the umbra α. We assume $\alpha^{\boldsymbol{.} 0} \equiv u$. Properties of the dot-power have been investigated with full details in Ref. [8]. In particular, the moments of the umbra $\alpha^{\boldsymbol{.} n}$ are the n-th power of the moments of the umbra α, that is $E[(\alpha^{\boldsymbol{.} n})^k] = a_k^n$ for any nonnegative integers k and n.

Moments of $n\boldsymbol{.}\alpha$ can be expressed using the notions of integer partition and dot-power [10]. An expression in terms of exponential Bell polynomials will be given in (2).

The generating function of an umbra The formal power series $u + \sum_{n \geq 1} \alpha^n \frac{t^n}{n!}$ is the *generating function* (g.f.) of the umbra α, and it is denoted by $e^{\alpha t}$. The notion of umbral equivalence and similarity can be extended coefficientwise to formal power series, that is $\alpha \equiv \beta \Leftrightarrow e^{\alpha t} \simeq e^{\beta t}$ (see [27] for a formal construction). Note that any exponential formal power series $f(t) = 1 + \sum_{n \geq 1} a_n t^n / n!$ can be umbrally represented by a formal power series. In fact, if the sequence $1, a_1, a_2, \ldots$ is umbrally represented by α, then $f(t) = E\left[e^{\alpha t}\right]$, that is $f(t) \simeq e^{\alpha t}$, assuming that we extend E by linearity. In this case, instead of $f(t)$, we write $f(\alpha, t)$ and we say that $f(\alpha, t)$ is umbrally represented by α. Henceforth, when no confusion occurs, we just say that $f(\alpha, t)$ is the g.f. of α. For example, the g.f. of the augmentation umbra ϵ is $f(\epsilon, t) = 1$, while the g.f. of the unity umbra u is $f(u, t) = e^t$. The g.f. of the singleton umbra χ is $f(\chi, t) = 1 + t$, the g.f. of the Bell umbra is $f(\beta, t) = \exp\left(e^t - 1\right)$, and the g.f. of the Bernoulli umbra is $f(\iota, t) = t / \left(e^t - 1\right)$.

The advantage of an umbral notation for g.f.'s is the representation of operations among g.f.'s through symbolic operations among umbrae. For example, the product of exponential g.f.'s is umbrally represented by a sum of the corresponding umbrae:

$$f(\alpha, t)\, f(\gamma, t) \simeq e^{(\alpha+\gamma)t} \quad \text{with} \quad f(\alpha, t) \simeq e^{\alpha t} \text{ and } f(\gamma, t) \simeq e^{\gamma t}. \tag{1}$$

If α is an umbra with g.f. $f(\alpha, t)$, then $f(-1\boldsymbol{.}\alpha, t) = 1/f(\alpha, t)$. Via (1), the g.f. of $n\boldsymbol{.}\alpha$ is $f(n\boldsymbol{.}\alpha, t) = f(\alpha, t)^n$. Via g.f., the moments of $n\boldsymbol{.}\alpha$ can be expressed as [8]

$$E\left[(n\boldsymbol{.}\alpha)^i\right] = \sum_{j=1}^{i} (n)_j \, B_{i,j}(a_1, a_2, \ldots, a_{i-j+1}), \quad i = 1, 2, \ldots \tag{2}$$

where $B_{i,j}$ are the (partial) Bell exponential polynomials and a_i are the moments of the umbra α.

The introduction of the g.f.device leads to the definition of new auxiliary umbrae, improving the computational power of the umbral syntax. For this purpose, we could replace R by a suitable polynomial ring having coefficients in R and any desired number of indeterminates. Then, an umbra is said to be *scalar* if the moments are elements of R while it is said to be *polynomial* if the moments are polynomials. In this paper, we deal with $R[x, y]$.

In particular, we define the dot-product of x and α via g.f., i.e. $x\boldsymbol{.}\alpha$ is the auxiliary umbra having g.f. $f(x\boldsymbol{.}\alpha, t) = f(\alpha, t)^x$. The polynomial umbra $x\boldsymbol{.}\alpha$ has been introduced in Ref. [28], and its properties have been investigated with full details in Ref. [8].

Example 2.4 Bell polynomial umbra. The umbra $x\boldsymbol{.}\beta$ is the Bell polynomial umbra [8]. Its factorial moments are powers of x, that is $(x\boldsymbol{.}\beta)_n \simeq x^n$, and its moments are the exponential polynomials, $(x\boldsymbol{.}\beta)^n \simeq \sum_{k=0}^n S(n,k)\, x^k$. The g.f.is $f(x\boldsymbol{.}\beta, t) = \exp[x(e^t - 1)]$.

Umbral polynomials Let $\{q_n(x)\}$ be a polynomial sequence of $R[x]$ such that $q_n(x) = \sum_{k=0}^n q_{n,k}\, x^k$, with $q_{n,n} \neq 0$ for all nonnegative integers n. The sequence $\{q_n(\alpha)\}$ consists of umbral polynomials with support α such that $E[q_n(\alpha)] = \sum_{k=0}^n q_{n,k}\, a_k$ for any nonnegative integer n. Now suppose $q_0(x) = 1$ and consider an auxiliary umbra η such that $E[\eta^n] = E[q_n(\alpha)]$, for any nonnegative integer n. In order to underline that the moments of η depend on those of α, we add the subscript α to the umbra η so that we shall write $\eta_\alpha^n \simeq q_n(\alpha)$ for any nonnegative integer n. If $\alpha \equiv x.u$, then $\eta_{x.u}$ is a polynomial umbra with moments $q_n(x)$, so we shall simply denote it by η_x. Let us consider some simple consequences of the notations here introduced.

Proposition 2.1 *If η_x is a polynomial umbra and α and γ are scalar or polynomial umbrae, then $\eta_\alpha \equiv \eta_\gamma \Leftrightarrow \alpha \equiv \gamma$.*

Proof For any nonnegative integer n, there exist constants $c_{n,k}, k = 0, 1, \ldots, n$ such that $x^n = \sum_{k=0}^n c_{n,k}\, q_k(x)$. Since $\eta_\alpha \equiv \eta_\gamma$, then $q_k(\alpha) \simeq q_k(\gamma)$ for all nonnegative integers k and so for all nonnegative integers n we have $\alpha^n \simeq \sum_{k=0}^n c_{n,k}\, q_k(\alpha) \simeq \sum_{k=0}^n c_{n,k}\, q_k(\gamma) \simeq \gamma^n$. The other direction of the proof is straightforward. □

Proposition 2.2 *If η_x and ζ_x are polynomial umbrae and α is a scalar or polynomial umbra, then $\eta_\alpha \equiv \zeta_\alpha \Leftrightarrow \eta_x \equiv \zeta_x$.*

Proof Suppose $\eta_\alpha \equiv \zeta_\alpha$. Let $\{q_n(x)\}$ be the moments of η_x and let $\{z_n(x)\}$ be the moments of ζ_x. For all nonnegative integers n, there exist constants $c_{n,k}, k = 0, 1, \ldots, n$ such that $q_n(x) = \sum_{k=0}^n c_{n,k}\, z_k(x)$. Because $q_n(\alpha) \simeq z_n(\alpha)$ for all nonnegative integers n, we have $c_{n,k} = \delta_{n,k}$ for $k = 0, 1, \ldots, n$ by which we have $q_n(x) = z_n(x)$. The other direction of the proof is straightforward. □

Special auxiliary umbrae A feature of the classical umbral calculus is the construction of new auxiliary umbrae by suitable symbolic substitutions. This symbolic substitution relies on the simple observation that $E[(n\boldsymbol{.}\alpha)^k]$ is a polynomial in n, as first remarked by Ray [18]. In Ref. [8], the explicit expression (2) of these polynomials has been carried out. So in $n\boldsymbol{.}\alpha$, we can replace the integer n by an umbra γ. From (2), the new auxiliary umbra $\gamma\boldsymbol{.}\alpha$ has moments

$$E\left[(\gamma\boldsymbol{.}\alpha)^i\right] = \sum_{j=1}^{i} g_{(j)} B_{i,j}(a_1, a_2, \ldots, a_{i-j+1}), \quad i = 1, 2, \ldots \tag{3}$$

where $g_{(j)}$ are the factorial moments of the umbra γ. The auxiliary umbra $\gamma\boldsymbol{.}\alpha$ is the *dot-product* of α and γ with g.f. $f(\gamma\boldsymbol{.}\alpha, t) = f(\alpha, \log f(\gamma, t))$. Observe that $E[\gamma\boldsymbol{.}\alpha] = g_1\, a_1 = E[\gamma]\, E[\alpha.]$ In Ref. [8], we have proved that

$$\alpha\boldsymbol{.}x \equiv \alpha\boldsymbol{.}(xu) \equiv x(\alpha\boldsymbol{.}u) \equiv x\alpha. \tag{4}$$

Here, we recall some properties that will be often used in the following. If $\alpha, \gamma, \eta \in A$ then $(\alpha + \eta)\boldsymbol{.}\gamma \equiv \alpha\boldsymbol{.}\gamma + \eta\boldsymbol{.}\gamma$ and $\eta\boldsymbol{.}(\gamma\boldsymbol{.}\alpha) \equiv (\eta\boldsymbol{.}\gamma)\boldsymbol{.}\alpha$. If $\eta\boldsymbol{.}\alpha \equiv \eta\boldsymbol{.}\gamma$ for some umbra η, then

$\alpha \equiv \gamma$. If $c \in R$, then $\eta{\boldsymbol .}(c\alpha) \equiv c(\eta{\boldsymbol .}\alpha)$ for any two distinct umbrae α and η. The proofs and some discussion on these properties can be found in Ref. [8].

In the following, we recall some useful dot-products of umbrae, cf. [8,9] for further details.

Example 2.5 Exponential umbral polynomials. Suppose we replace x with a generic umbra α in the Bell polynomial umbra $x{\boldsymbol .}\beta$, [8]. We get the auxiliary umbra $\alpha{\boldsymbol .}\beta$, whose factorial moments are $(\alpha{\boldsymbol .}\beta)_n \simeq \alpha^n$, for all nonnegative integers. The g.f. is $f(\alpha{\boldsymbol .}\beta, t) = f(\alpha, e^t - 1)$, and the moments are given by the exponential polynomials $\{\Phi_n(x)\}$ with x replaced by α :

$$(\alpha{\boldsymbol .}\beta)^n \simeq \Phi_n(\alpha) \simeq \sum_{i=0}^{n} S(n,i)\alpha^i \quad n = 0, 1, 2, \dots. \tag{5}$$

Example 2.6 α-partition umbra. The α-partition umbra is the umbra $\beta{\boldsymbol .}\alpha$, where β is the Bell umbra. Since the factorial moments of β are all equal to 1, Eq. (3) gives

$$E\left[(\beta{\boldsymbol .}\alpha)^i\right] = \sum_{j=1}^{i} B_{i,j}(a_1, a_2, \dots, a_{i-j+1}) = Y_i(a_1, a_2, \dots, a_i), \tag{6}$$

for $i = 1, 2, \dots,$ where Y_i are the complete exponential polynomials. The umbra $x{\boldsymbol .}\beta{\boldsymbol .}\alpha$ is the polynomial α-partition umbra. Since the factorial moments of $x{\boldsymbol .}\beta$ are powers of x, Eq. (3) gives

$$E\left[(x{\boldsymbol .}\beta{\boldsymbol .}\alpha)^i\right] = \sum_{j=1}^{i} x^j\, B_{i,j}(a_1, a_2, \dots, a_{i-j+1}), \tag{7}$$

so the g.f. is $f(x{\boldsymbol .}\beta{\boldsymbol .}\alpha, t) = \exp[x(f(\alpha, t) - 1)]$. One of the most relevant properties of the polynomial α-partition umbra is $(x + y){\boldsymbol .}\beta{\boldsymbol .}\alpha \equiv x{\boldsymbol .}\beta{\boldsymbol .}\alpha + y{\boldsymbol .}\beta{\boldsymbol .}\alpha$.

The α-partition umbra $\beta{\boldsymbol .}\alpha$ plays a crucial role in the umbral representation of the composition of exponential g.f.'s. Indeed, the *composition umbra* of α and γ is $\gamma{\boldsymbol .}\beta{\boldsymbol .}\alpha$, with g.f. $f(\gamma{\boldsymbol .}\beta{\boldsymbol .}\alpha, t) = f[\gamma, f(\alpha, t) - 1]$. From Eq. (7), the moments are $E[(\gamma{\boldsymbol .}\beta{\boldsymbol .}\alpha)^i] = \sum_{j=1}^{i} g_j\, B_{i,j}(a_1, a_2, \dots, a_{i-j+1})$, where g_j and a_i are moments of the umbra γ and α, respectively. The umbra $\alpha^{<-1>}$ is the compositional inverse of α, with g.f. $f(\alpha^{<-1>}, t) = f^{<-1>}(\alpha, t)$, where $f^{<-1>}(\alpha, t)$ is the compositional inverse of $f(\alpha, t)$ in the sense that $f[\alpha^{<-1>}, f(\alpha, t) - 1] = f[\alpha, f(\alpha^{<-1>}, t) - 1] = 1 + t$. So we have $\alpha{\boldsymbol .}\beta{\boldsymbol .}\alpha^{<-1>} \equiv \alpha^{<-1>}{\boldsymbol .}\beta{\boldsymbol .}\alpha \equiv \chi$. In particular, for the unity umbra, we have

$$\beta{\boldsymbol .}u^{<-1>} \equiv u^{<-1>}{\boldsymbol .}\beta \equiv \chi, \quad \text{and} \quad \beta{\boldsymbol .}\chi \equiv u \equiv \chi{\boldsymbol .}\beta. \tag{8}$$

We have also $f(u^{<-1>}, t) = 1 + \log(1 + t)$, see [9] for more details.

Example 2.7 α-cumulant umbra. The umbra $\chi{\boldsymbol .}\alpha$ is the α-cumulant umbra, having g.f. $f(\chi{\boldsymbol .}\alpha, t) = 1 + \log[f(\alpha, t)]$. The α-cumulant umbra gives rise to an umbral representation theorem for formal power series of logarithmic type (cf. Open problems in Ref. [28]). Properties of cumulant umbrae are investigated in details in Ref. [9].

Example 2.8 α-factorial umbra. The umbra $\alpha{\boldsymbol .}\chi$ is the α-factorial umbra, having g.f. $f(\alpha{\boldsymbol .}\chi, t) = f[\alpha, \log(1+t)]$. In particular, we have $(\alpha{\boldsymbol .}\chi)^n \simeq (\alpha)_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$ for all integers $n \geq 1$. Properties of factorial umbrae are investigated in details in Ref. [9].

3 Sheffer sequences

Up to now, the umbral calculus we have summarized has been set up in the papers [8,20] and [9]. Now we add one more notion that is a key to construct a symbolic theory of Sheffer polynomials. For the rest of the section, assume γ be an umbra with $E[\gamma] = g_1 \neq 0$ so that its g.f. $f(\gamma, t)$ admits compositional inverse.

Definition 3.1 The adjoint umbra of γ is $\gamma^* = \beta{\bf .}\gamma^{<-1>}$, the $\gamma^{<-1>}$-partition umbra.

The adjoint umbra has g.f. $f(\gamma^*, t) = \exp[f^{<-1>}(\gamma, t) - 1]$. The name parallels the adjoint of an umbral operator [19] since $\gamma{\bf .}\alpha^*$ gives the umbral composition of γ and $\alpha^{<-1>}$.

Example 3.1 Adjoint of the singleton umbra χ. The compositional inverse of χ is the umbra χ itself. So we have $\chi^* \equiv \beta{\bf .}\chi^{<-1>} \equiv \beta{\bf .}\chi \equiv u$.

Example 3.2 Adjoint of the unity umbra u. By virtue of equivalence (8), the adjoint of the unity umbra u is $u^* \equiv \beta{\bf .}u^{<-1>} \equiv \chi$.

Example 3.3 Adjoint of the Bell umbra β. We have $\beta^* \equiv u^{<-1>}$. Indeed, $\beta{\bf .}\beta{\bf .}\beta^{<-1>} \equiv \chi$ and taking the left-hand side dot-product by χ, we have $\chi{\bf .}\beta{\bf .}\beta{\bf .}\beta^{<-1>} \equiv \chi{\bf .}\chi \equiv u^{<-1>}$. The last equivalence follows observing that $f(\chi{\bf .}\chi, t) = 1 + \log(1+t)$ from Example 2.7 and $f(u^{<-1>}, t) = 1 + \log(1+t)$, see [9] for more details. The result follows recalling the second equivalence in (8) and $\beta{\bf .}\beta^{<-1>} \equiv \beta^*$.

Example 3.4 Adjoint of $u^{<-1>}$. From Definition 3.1, we have $(\gamma^{<-1>})^* \equiv \beta{\bf .}\gamma$, so that $(u^{<-1>})^* \equiv \beta{\bf .}u \equiv \beta$.

Note that $\chi{\bf .}\gamma^* \equiv \gamma^{<-1>}$ and $\chi{\bf .}(\gamma^{<-1>})^* \equiv \gamma$. From Definition 3.1, we have

$$\gamma{\bf .}\gamma^* \equiv \chi \Rightarrow \beta{\bf .}\gamma{\bf .}\gamma^* \equiv \beta{\bf .}\chi \equiv u. \tag{9}$$

Proposition 3.1 *If γ is an umbra with compositional inverse, then $(\gamma^{<-1>})^*{\bf .}\gamma^* \equiv u$ and $\gamma^*{\bf .}(\gamma^{<-1>})^* \equiv u$.*

Proof Because $(\gamma^{<-1>})^* \equiv \beta{\bf .}\gamma$, the first equivalence follows because $(\gamma^{<-1>})^*{\bf .}\beta{\bf .}\gamma^{<-1>} \equiv \beta{\bf .}\gamma{\bf .}\beta{\bf .}\gamma^{<-1>} \equiv \beta{\bf .}\chi \equiv u$. The second equivalence follows from the first replacing $\gamma^{<-1>}$ with γ. □

Proposition 3.2 *For $\alpha, \gamma \in A$ provided with compositional inverses, we have $(\alpha{\bf .}\beta{\bf .}\gamma)^* \equiv \gamma^*{\bf .}\alpha^*$.*

Proof By Definition 3.1, we have $(\alpha{\bf .}\beta{\bf .}\gamma)^* \equiv \beta{\bf .}(\alpha{\bf .}\beta{\bf .}\gamma)^{<-1>}$. Since

$$\left(\gamma^{<-1>}{\bf .}\beta{\bf .}\alpha^{<-1>}\right){\bf .}\beta{\bf .}\alpha{\bf .}\beta{\bf .}\gamma \equiv \chi$$

and also $(\alpha{\bf .}\beta{\bf .}\gamma)^{<-1>}{\bf .}\beta{\bf .}\alpha{\bf .}\beta{\bf .}\gamma \equiv \chi$, we have $(\alpha{\bf .}\beta{\bf .}\gamma)^{<-1>} \equiv \gamma^{<-1>}{\bf .}\beta{\bf .}\alpha^{<-1>}$, so that $(\alpha{\bf .}\beta{\bf .}\gamma)^* \equiv (\beta{\bf .}\gamma^{<-1>}){\bf .}(\beta{\bf .}\alpha^{<-1>}) \equiv \gamma^*{\bf .}\alpha^*$. □

Definition 3.2 A polynomial umbra σ_x is said to be a Sheffer umbra for (α, γ) if

$$\sigma_x \equiv \alpha + x{\bf .}\gamma^*. \tag{10}$$

We denote a Sheffer umbra by $\sigma_x^{(\alpha,\gamma)}$ in order to make explicit the dependence on α and γ. The g.f. of $\sigma_x^{(\alpha,\gamma)}$ is

$$f\left[\sigma_x^{(\alpha,\gamma)}, t\right] = f(\alpha, t)\, e^{x\,[f^{<-1>}(\gamma,t)-1]}. \tag{11}$$

Given the umbra γ, any Sheffer umbra is uniquely determined by its moments evaluated at 0, since via equivalence (10) we have $\sigma_0^{(\alpha,\gamma)} \equiv \alpha$.

Example 3.5 Poisson–Charlier polynomials. The polynomials $\{c_n(x;a)\}$ such that $c_n(x;a) \simeq \{(x{\boldsymbol .}\chi - a)/a\}^n$ are the Poisson–Charlier polynomials. Indeed, since $(x{\boldsymbol .}\chi)^k \simeq (x)_k$ it is straightforward to prove $c_n(x;a) = a^{-n} \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k} (x)_k$. Denoting by $\omega_{x,a}$ the polynomial umbra whose moments are $c_n(x;a)$, we have $\omega_{x,a} \equiv (x{\boldsymbol .}\chi - a)/a$. The umbra $\omega_{x,a}$ is called the Poisson–Charlier polynomial umbra. We show that $\omega_{x,a}$ is a Sheffer umbra for $(-u, \chi{\boldsymbol .}a{\boldsymbol .}\beta)$. Indeed, we have $\omega_{x,a} \equiv (x{\boldsymbol .}\chi - au)/a \equiv -u + x{\boldsymbol .}(\chi/a)$. Observe that $(\chi/a) \equiv (\chi{\boldsymbol .}a{\boldsymbol .}\beta)^*$. Indeed, $(\chi{\boldsymbol .}a{\boldsymbol .}\beta)^* \equiv (\chi{\boldsymbol .}a{\boldsymbol .}\beta{\boldsymbol .}u)^* \equiv u^*{\boldsymbol .}(a\chi)^* \equiv \chi{\boldsymbol .}(a\chi)^*$. Since $\chi^* \equiv u$ then $(a\chi)^* \equiv u/a$ and $(\chi{\boldsymbol .}a{\boldsymbol .}\beta)^* \equiv \chi/a$. Thus, $\omega_{x,a} \equiv -u + x{\boldsymbol .}(\chi{\boldsymbol .}a{\boldsymbol .}\beta)^*$.

Theorem 3.1 (Generalized Sheffer identity) *A polynomial umbra σ_x is a Sheffer umbra if and only if there exists an umbra γ, provided with a compositional inverse, such that*

$$\sigma_{\eta+\zeta} \equiv \sigma_\eta + \zeta{\boldsymbol .}\gamma^*, \quad \textit{for any } \eta, \zeta \in A. \tag{12}$$

Proof Let σ_x be a Sheffer umbra for (α, γ). From equivalence (10), we have $\sigma_{\eta+\zeta}^{(\alpha,\gamma)} \equiv \alpha + (\eta + \zeta){\boldsymbol .}\gamma^* \equiv \sigma_\eta^{(\alpha,\gamma)} + \zeta{\boldsymbol .}\gamma^*$, because $(\eta + \zeta){\boldsymbol .}\gamma^* \equiv \eta{\boldsymbol .}\gamma^* + \zeta{\boldsymbol .}\gamma^*$. Vice versa, in equivalence (12) choose the umbra $x{\boldsymbol .}u$ as umbra ζ and the augmentation umbra as umbra η. Observe that σ_ϵ is a scalar umbra. Thus, equivalence (12) becomes $\sigma_x \equiv \sigma_\epsilon + x{\boldsymbol .}\gamma^*$. By Definition 3.2, the polynomial umbra σ_x is a Sheffer umbra for $(\sigma_\epsilon, \gamma)$. □

Corollary 3.1 (The Sheffer identity) *A polynomial umbra σ_x is a Sheffer umbra if and only if there exists an umbra γ, provided with a compositional inverse, such that*

$$\sigma_{x+y} \equiv \sigma_x + y{\boldsymbol .}\gamma^*. \tag{13}$$

Proof If σ_x is a Sheffer umbra for (α, γ), equivalence (13) follows from equivalence (12) choosing the umbra $x{\boldsymbol .}u$ as umbra η and the umbra $y{\boldsymbol .}u$ as umbra ζ. Vice versa, if the polynomial umbra σ_x satisfies equivalence (13), it satisfies also equivalence (12), due to Proposition 2.2. □

Equivalence (13) gives the well-known Sheffer identity. Indeed, setting $s_n(x + y) = E[\sigma_{x+y}^n]$, $s_k(x) = E[\sigma_x^k]$ and $p_{n-k}(y) = E[(y{\boldsymbol .}\gamma^*)^{n-k}]$ and using the binomial expansion, we have $s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)\, p_{n-k}(y)$. In the next section, we will prove that the moments of umbrae such $y{\boldsymbol .}\gamma^*$ are binomial sequences.

Theorem 3.2 (The Expansion Theorem) *If $\sigma_x^{(\alpha,\gamma)}$ is a Sheffer umbra for (α, γ), then $\eta \equiv (-1{\boldsymbol .}\alpha + \sigma_\eta^{(\alpha,\gamma)}){\boldsymbol .}\beta{\boldsymbol .}\gamma$ for any umbra η.*

Proof Replacing x with η in equivalence (10), we obtain $\sigma_\eta^{(\alpha,\gamma)} \equiv \alpha + \eta{\boldsymbol .}\gamma^*$. Take the right dot-product with $\beta{\boldsymbol .}\gamma$ of both sides, then $\sigma_\eta^{(\alpha,\gamma)}{\boldsymbol .}\beta{\boldsymbol .}\gamma \equiv \alpha{\boldsymbol .}\beta{\boldsymbol .}\gamma + \eta$. The result follows adding $-1{\boldsymbol .}\alpha{\boldsymbol .}\beta{\boldsymbol .}\gamma$ to both sides of the previous equivalence. □

Theorem 3.3 *A polynomial umbra σ_x is a Sheffer umbra if and only if there exists an umbra γ, provided with compositional inverse, such that*

$$\sigma_{\gamma+x.u} \equiv \chi + \sigma_x. \tag{14}$$

Proof If σ_x is a Sheffer umbra for (α, γ), equivalence (14) follows from equivalence (12) choosing $x.u$ as umbra η and the umbra γ as umbra ζ, because $\gamma.\gamma^* \equiv \chi$. Vice versa, let σ_x be a polynomial umbra such that equivalence (14) holds for some umbra γ, provided with compositional inverse. Set $x = 0$ in equivalence (14). We have $\sigma_\gamma \equiv \chi + \sigma_0 \Rightarrow \sigma_\gamma.\beta.\gamma \equiv \chi.\beta.\gamma + \sigma_0.\beta.\gamma \equiv \gamma + \sigma_0.\beta.\gamma$. Due to Theorem 3.2, the Sheffer umbra for (σ_0, γ) satisfies $\sigma_\gamma^{(\sigma_0,\gamma)}.\beta.\gamma \equiv \gamma + \sigma_0.\beta.\gamma$, therefore $\sigma_\gamma \equiv \sigma_\gamma^{(\sigma_0,\gamma)}$ and $\sigma_x \equiv \sigma_x^{(\sigma_0,\gamma)}$ by Proposition 2.2. □

Corollary 3.2 *A polynomial umbra σ_x is a Sheffer umbra if and only if there exists an umbra γ, provided with compositional inverse, such that $\sigma^k_{\gamma+x.u} \simeq \sigma^k_x + k\,\sigma^{k-1}_x$ for $k = 1, 2, \ldots$.*

Proof Take the k-th moment of both sides in equivalence (14). □

Theorem 3.4 (Orthogonality) *Let $\{s_n(x)\}$ be the moments of a Sheffer umbra for (α, γ). The polynomial sequence $\{s_n(x)\}$ is the unique polynomial sequence such that $s_n(-1.\alpha.\beta.\gamma + k.\gamma) \simeq (k.\chi)^n$ for all nonnegative integers $n, k \geq 0$.*

Proof The uniqueness follows from Proposition 2.2. The existence follows from equivalence (10), since when x is replaced by $-1.\alpha.\beta.\gamma + k.\gamma$, we have $-1.\alpha.\beta.\gamma.\gamma^* + \alpha + k.\gamma.\gamma^* \equiv -1.\alpha.\beta.\chi + \alpha + k.\chi \equiv -1.\alpha + \alpha + k.\chi \equiv k.\chi$ for all nonnegative k. □

In general, if $q_n(x) = \sum_{k=0}^n q_{n,k}\, x^k$ and $p_n(x)$ are sequence of polynomials, the *umbral composition* of $q_n(x)$ and $p_n(x)$ is the sequence [19]

$$q_n(\mathbf{p}(x)) = \sum_{k=0}^{n} q_{n,k}\, p_k(x). \tag{15}$$

If $q_n(x)$ and $p_n(x)$ are Sheffer sequences, the following theorem holds.

Theorem 3.5 (Umbral composition and Sheffer umbrae) *If $\{q_n(x)\}$ are the moments of a Sheffer umbra for (α, γ) and $\{p_n(x)\}$ are the moments of a Sheffer umbra for (η, ζ), then the polynomials $\{q_n(\mathbf{p}(x))\}$ given in (15) are moments of the Sheffer umbra for $(\alpha+\eta.\gamma^*, \gamma.\beta.\zeta)$.*

Proof We have $E\{(\alpha + x.\gamma^*)^n\} = q_n(x) = \sum_{k=0}^n q_{n,k}\, x^k$. By symbolic substitution of x with the umbra $\eta + x.\zeta^*$, we have

$$[\alpha + (\eta + x.\zeta^*).\gamma^*]^n \simeq q_n(\eta + x.\zeta^*) = \sum_{k=0}^{n} q_{n,k}\,(\eta + x.\zeta^*)^k.$$

Then we have $q_n(\eta + x.\zeta^*) \simeq q_n(\mathbf{p}(x))$ given in (15). Recalling Proposition 3.2, we have $\zeta^*.\gamma^* \simeq (\gamma.\beta.\zeta)^*$ and

$$q_n(\mathbf{p}(x)) \simeq \left\{(\alpha + \eta.\gamma^*) + x.(\gamma.\beta.\zeta)^*\right\}^n \tag{16}$$

by which the result follows. □

Two Sheffer sequences are said to be inverse of each other if and only if their umbral composition (15) gives the sequence $\{x^n\}$.

Corollary 3.3 (Inverse of Sheffer sequences) *The sequence of moments corresponding to the Sheffer umbra for* $(-1.\alpha.\beta.\gamma, \gamma^{<-1>})$ *are inverses of the sequence of moments corresponding to the Sheffer umbra for* (α, γ).

Proof From equivalence (16), we have $q_n(\mathbf{p}(x)) = x^n$ for all $n \geq 1$ if and only if $\zeta^*.\gamma^* \equiv u$ and $\alpha + \eta.\gamma^* \equiv \epsilon$, up to similarity. From the first equivalence, taking into account Proposition 3.1, we have $\zeta \equiv \gamma^{<-1>}$. From the second equivalence, we have $\eta.\gamma^* \equiv -1.\alpha$ and $\eta.\gamma^*.(\gamma^{<-1>})^* \equiv \eta \equiv -1.\alpha.(\gamma^{<-1>})^*$. By recalling that $(\gamma^{<-1>})^* \equiv \beta.\gamma$, the result follows. □

4 Two special Sheffer umbrae

Among Sheffer umbrae, a special role is played by the associated umbra and the Appell umbra. The associated umbrae are polynomial umbrae whose moments $\{p_n(x)\}$ satisfies the well-known binomial identity

$$p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y) \tag{17}$$

for all $n = 0, 1, 2, \ldots$. Polynomial sequences satisfying property (17) are said to be of binomial type. Every sequence of binomial type is a Sheffer sequence but most Sheffer sequences are not of binomial type. The Appell umbrae are polynomial umbrae whose moments $\{p_n(x)\}$ satisfies the identity

$$\frac{d}{dx} p_n(x) = n p_{n-1}(x) \quad n = 1, 2, \ldots. \tag{18}$$

Associated umbrae Let us consider a Sheffer umbra for the umbrae (ϵ, γ), where γ has compositional inverse and ϵ is the augmentation umbra.

Definition 4.1 The polynomial umbra $\sigma_x \equiv x.\gamma^*$ is the associated umbra of γ.

The g.f. of $x.\gamma^*$ is $f(x.\gamma^*, t) = e^{x[f^{<-1>}(\gamma,t)-1]}$, because in Eq. (11) we have $f(\alpha, t) = f(\epsilon, t) = 1$.

Theorem 4.1 *An umbra* $\sigma_x^{(\alpha,\gamma)}$ *is a Sheffer umbra for* (α, γ) *if and only if* $\sigma^{(\alpha,\gamma)}_{-1.\alpha.\beta.\gamma+x.u}$ *is the umbra associated with* γ.

Proof If $\sigma_x^{(\alpha,\gamma)}$ is a Sheffer umbra for (α, γ), then $\sigma^{(\alpha,\gamma)}_{-1.\alpha.\beta.\gamma+x.u} \equiv -1.\alpha.\beta.\gamma.\gamma^* + \alpha + x.\gamma^*$, by which $\sigma^{(\alpha,\gamma)}_{-1.\alpha.\beta.\gamma+x.u} \equiv x.\gamma^*$, because $\beta.\gamma.\gamma^* \equiv u$ and $-1.\alpha + \alpha \equiv \epsilon$. Thus, the result follows from Definition 4.1. Vice versa, let η_x be a polynomial umbra such that $\eta_{-1.\alpha.\beta.\gamma+x.u} \equiv x.\gamma^*$. Replacing x with $k.\gamma$, we have $\eta_{-1.\alpha.\beta.\gamma+k.\gamma} \equiv k.\gamma.\gamma^* \equiv k.\chi$. The result follows from Theorem 3.4. □

We will say that a polynomial sequence $\{p_n(x)\}$ is *associated* with an umbra γ if and only if $p_n(x) \simeq (x.\gamma^*)^n$, for $n = 0, 1, 2, \ldots$ or

$$p_n(k.\gamma) \simeq (k.\chi)^n \quad n, k = 0, 1, 2, \ldots. \tag{19}$$

Theorem 4.2 (Umbral characterization of associated sequences) *The sequence* $\{p_n(x)\}$ *is associated with the umbra* γ *if and only if*:

$$p_n(\epsilon) \simeq \epsilon^n \quad \textit{for } n = 0, 1, 2, \ldots \tag{20}$$

$$p_n(\gamma + x.u) \simeq p_n(x) + n\, p_{n-1}(x) \quad \textit{for } n = 1, 2, \ldots. \tag{21}$$

Proof If the sequence $\{p_n(x)\}$ is associated with γ, then $p_n(\epsilon) \simeq (\epsilon\boldsymbol{.}\gamma^*)^n \simeq \epsilon^n$ for all $n = 0, 1, 2, \ldots,$ since $\epsilon\boldsymbol{.}\alpha \equiv \epsilon$ for any umbra α. Equivalence (21) follows from Corollary 3.2 choosing a Sheffer umbra for (ϵ, γ). Vice versa, if equivalences (20) and (21) hold, we prove by induction that the sequence $\{p_n(x)\}$ satisfies (19). Indeed, by equivalence (20), we have $p_n(0\boldsymbol{.}\gamma) \simeq p_n(\epsilon) \simeq 1$ if $n = 0$ otherwise being 0 for $n = 1, 2, \ldots, j$. Suppose that equivalence (19) holds for $k = m$

$$p_n(m\boldsymbol{.}\gamma) \simeq (m\boldsymbol{.}\chi)^n \quad n = 0, 1, 2, \ldots. \tag{22}$$

By equivalence (21), we have $p_n[(m+1)\boldsymbol{.}\gamma)] \simeq p_n(\gamma + m\boldsymbol{.}\gamma) \simeq p_n(m\boldsymbol{.}\gamma) + np_{n-1}(m\boldsymbol{.}\gamma)$ for $n = 1, 2, \ldots$. Due to induction hypothesis (22), we have $p_n[(m+1)\boldsymbol{.}\gamma] \simeq (m\boldsymbol{.}\chi)^n + n(m\boldsymbol{.}\chi)^{n-1} \simeq (\chi + m\boldsymbol{.}\chi)^n \simeq [(m+1)\boldsymbol{.}\chi]^n$ for $n = 1, 2, \ldots$. Since the sequence $\{p_n(x)\}$ verifies (19), it is associated with γ. □

Theorem 4.3 (The binomial identity) *The sequence* $\{p_n(x)\}$ *is associated with the umbra* γ *if and only if it satisfies the binomial identity* (17) *for all* $n = 0, 1, 2, \ldots$.

Proof If the sequence $\{p_n(x)\}$ is associated with the umbra γ, then identity (17) follows from the property $(x+y)\boldsymbol{.}\gamma^* \equiv x\boldsymbol{.}\gamma^* + y\boldsymbol{.}\gamma^*$. Vice versa, suppose the sequence $\{p_n(x)\}$ satisfies identity (17). Let η_x be a polynomial umbra such that $E[\eta_x^n] = p_n(x)$. By identity (17), we have

$$\eta_{x+y} \equiv \eta_x + \eta'_y \tag{23}$$

with η_x and η'_y uncorrelated polynomial umbrae and $E[(\eta'_y)^n] = p_n(y)$ for all nonnegative integers n. In particular, if we replace y with ϵ in equivalence (23), then $\eta_x \equiv \eta_x + \eta'_\epsilon$ and hence $\eta'_\epsilon \equiv \epsilon$. So the polynomials $\{p_n(x)\}$ are such that $p_n(\epsilon) \simeq \epsilon^n$, i.e. they satisfy equivalence (20). By induction on equivalence (23), we have

$$\eta_{\underbrace{x+\cdots+x}_{k}} \equiv \underbrace{\eta_x + \cdots + \eta'_x}_{k},$$

where the polynomial umbrae on the right-hand side are uncorrelated and similar to η_x. If the x's are replaced by uncorrelated umbrae similar to any umbra γ, provided with a compositional inverse, then $\eta_{k\boldsymbol{.}\gamma} \equiv k\boldsymbol{.}\eta_\gamma$. Since $E[\gamma] \neq 0$, we can choose an umbra γ such that $\eta_\gamma \equiv \chi$ thus $\eta_{k\boldsymbol{.}\gamma} \equiv k\boldsymbol{.}\chi$, and the result follows from this last equivalence and equivalence (19). □

Remark 4.1 Since any polynomial sequence of binomial type can be represented by a polynomial umbra $x\boldsymbol{.}\gamma$, a polynomial sequence of integral type [3] can be represented by replacing x with n. So the theory of these sequences can be recovered using the tools introduced in Ref. [8] (see Open problems in Ref. [28]).

Example 4.1 The umbra $x\boldsymbol{.}u$ is associated with the umbra χ. Indeed, we have $\chi^* \equiv \beta\boldsymbol{.}\chi^{<-1>} \equiv u$ and so $x\boldsymbol{.}\chi^* \equiv x\boldsymbol{.}u$. Therefore the polynomial sequence $\{x^n\}$ is associated with the umbra χ. The g.f. is $f(x\boldsymbol{.}u, t) = e^{xt}$ and the binomial identity becomes the well-known $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k\, y^{n-k}$.

Example 4.2 The umbra $x\boldsymbol{.}u^* \equiv x\boldsymbol{.}\chi$ is associated with the umbra u. The associated polynomial sequence is given by $\{(x\boldsymbol{.}\chi)^n\} \simeq \{(x)_n\}$, see Example 2.8. The g.f. is $f(x\boldsymbol{.}u^*, t) = (1+t)^x$, and the binomial identity becomes $(x+y)_n = \sum_{k=0}^n \binom{n}{k} (x)_k\, (y)_{n-k}$.

Example 4.3 The umbra $x.(u^{<-1>})^* \equiv x.\beta$ is associated with the umbra $u^{<-1>}$. The associated polynomial sequence is given by $\{(x.\beta)^n\} \simeq \{\Phi_n(x)\}$, where $\{\Phi_n(x)\}$ are the exponential polynomials (5).

Appell umbrae Let us consider a Sheffer umbra for the umbrae (α, χ).

Definition 4.2 The polynomial umbra $\sigma_x \equiv \alpha + x.u$ is the Appell umbra of α.

This definition is an updated version in terms of Sheffer sequences of the one given in Ref. [20]. By identity (11), the g.f. of the Appell umbra of α is $f(\alpha + x.u, t) = f(\alpha, t)\, e^{xt}$. We will say that a polynomial sequence $\{p_n(x)\}$ is an Appell sequence if and only if $p_n(x) \simeq (\alpha + x.u)^n$, for $n = 0, 1, 2, \ldots$.

Theorem 4.4 (The Appell identity) *The polynomial umbra σ_x is an Appell umbra for some umbra α if and only if $\sigma_{x+y} \equiv \sigma_x + y.u$.*

Proof The result follows immediately, choosing as umbra γ the singleton umbra χ in the Sheffer identity. □

Corollary 4.1 *A polynomial umbra σ_x is an Appell umbra for some umbra α if and only if $\sigma^n_{\chi+x.u} \simeq \sigma^n_x + n\sigma^{n-1}_x$.*

Proof The result follows from Corollary 3.2, choosing as umbra γ the singleton umbra χ □

Roughly speaking, Corollary 4.1 says that when in the Appell umbra we replace x by $\chi + x.u$, the umbra χ acts as a derivative operator. Indeed, by the binomial expansion, we have

$$\begin{aligned}
(\alpha + \chi + x.u)^n &\simeq \sum_{k \geq 0} \binom{n}{k} \alpha^{n-k} \, (\chi + x.u)^k \\
&\simeq \sum_{k \geq 0} \binom{n}{k} \alpha^{n-k} \left[(x.u)^k + k(x.u)^{k-1} \right] \\
&\simeq (\alpha + x.u)^n + \sum_{k \geq 0} \binom{n}{k} \alpha^{n-k} \, D_x \left[(x.u)^k \right] \\
&\simeq (\alpha + x.u)^n + D_x \left[(\alpha + x.u)^n \right].
\end{aligned}$$

Therefore, we have

$$D_x \left[(\alpha + x.u)^n \right] \simeq (\alpha + \chi + x.u)^n - (\alpha + x.u)^n$$

and this is why Corollary 4.1 umbrally expresses Eq. (18).

Example 4.4 Power polynomials. The umbra $x.u$ is the Appell umbra of the augmentation. So the power polynomials $\{x^n\}$ are Appell polynomials.

Example 4.5 Bernoulli polynomials. The Appell umbra for the Bernoulli umbra is $\iota + x.u$. From the binomial expansion, its moments are the Bernoulli polynomials $E[(\iota + x.u)^n] = \sum_{k \geq 0} \binom{n}{k} \mathcal{B}_{n-k}\, x^k$, where $\mathcal{B}_n$ are the Bernoulli numbers.

5 Applications

Abel polynomials and Lagrange inversion formula Abel polynomials play a leading role in the theory of associated sequences of polynomials. The main result of this section is the proof that any sequence of binomial type can be represented as Abel polynomials, heart of the paper [21]. Here, we give a very simple proof by introducing the notion of the derivative of an umbra. The connection between polynomial sequences of binomial type and Abel polynomials allows a *one line* proof of the Lagrange version formula and a series of results, some of which we are going to use in the next paragraph, and others useful in the construction of an umbral theory of free cumulants [12].

Definition 5.1 The derivative umbra α_D of an umbra α is the umbra whose powers are such that $(\alpha_D)^n \simeq \partial_\alpha \alpha^n \simeq n\alpha^{n-1}$ for $n = 1, 2, \ldots$.

We have $f(\alpha_D, t) = 1 + t\, f(\alpha, t)$, because $e^{\alpha_D t} \simeq u + \sum_{n\geq 1} n\alpha^{n-1} t^n/n!$. Note that $E[\alpha_D] = 1$. In particular, we have

$$\left(e^{\alpha_D t} - u\right)^{\boldsymbol{.}k} \simeq t^k \left(e^{\alpha t}\right)^{\boldsymbol{.}k} \simeq t^k e^{(k\boldsymbol{.}\alpha)t}. \tag{24}$$

Example 5.1 Singleton umbra. The singleton umbra χ is the derivative umbra of the augmentation umbra ϵ, that is $\epsilon_D \equiv \chi$.

Example 5.2 Bernoulli umbra. We have $u \equiv (-1\boldsymbol{.}\iota)_D$. Indeed, we have $f(-1\boldsymbol{.}\iota, t) = (e^t - 1)/t$ so that $f[(-1\boldsymbol{.}\iota)_D, t] = 1 + t(e^t - 1)/t = e^t = f(u, t)$.

Example 5.3 Bernoulli-factorial umbra. We have $u^{<-1>} \equiv (\iota\boldsymbol{.}\chi)_D$. Indeed, $f(\iota\boldsymbol{.}\chi, t) = \log(1+t)/(e^{\log(1+t)} - 1) = \log(1+t)/t$ so that $f[(\iota\boldsymbol{.}\chi)_D, t] = 1 + t\log(1+t)/t = f(u^{<-1>}, t)$.

Theorem 5.1 (Abel representation of binomial sequences) *If γ is an umbra provided with a compositional inverse, then for all $x \in R$*

$$\left(x\boldsymbol{.}\gamma_D^*\right)^n \simeq x(x - n\boldsymbol{.}\gamma)^{n-1}, \quad n = 1, 2, \ldots. \tag{25}$$

In the following, we refer to polynomials $x(x - n\boldsymbol{.}\gamma)^{n-1}$ as *umbral Abel polynomials*.

Proof On the basis of Theorem 4.2, the result follows showing that umbral Abel polynomials are associated with the umbra γ_D, i.e. showing that such polynomials satisfy equivalences (20) and (21). Since $\epsilon(\epsilon - n\boldsymbol{.}\gamma)^{n-1} \simeq \epsilon^n$, equivalences (20) are satisfied. Moreover, it is easy to check by simple calculations that $(x\boldsymbol{.}u + \gamma_D)^n - x^n \simeq n(x\boldsymbol{.}u + \gamma)^{n-1}$, for $n = 1, 2, \ldots$ and more in general $p(x\boldsymbol{.}u + \gamma_D) - p(x) \simeq p'(x\boldsymbol{.}u + \gamma)$ for any polynomial $p(x) \in R[A][x]$, where $p'(x)$ denotes the derivative with respect to x of $p(x)$. In particular for $p_n(x) = x(x - n\boldsymbol{.}\gamma)^{n-1}$, we have $p_n(x\boldsymbol{.}u + \gamma_D) - p_n(x) \simeq p_n'(x\boldsymbol{.}u + \gamma)$. Since $p_n'(x) \simeq n\,(x - n\boldsymbol{.}\gamma)^{n-2}\,(x - 1\boldsymbol{.}\gamma)$ then $p_n'(x\boldsymbol{.}u + \gamma) \simeq n\,x\,(x - (n-1)\boldsymbol{.}\gamma)^{n-2} \simeq n\,p_{n-1}(x)$, and so equivalences (21) are satisfied. □

Theorem 5.1 includes the well-known Transfer Formula [19]. The following corollary gives the Lagrange inversion formula.

Corollary 5.1 *For any umbra γ, we have*

$$\left(\gamma_D^{<-1>}\right)^n \simeq (-n\boldsymbol{.}\gamma)^{n-1}, \quad n = 1, 2, \ldots. \tag{26}$$

Proof Since $\chi.\beta \equiv u$ then $\gamma_D^{<-1>} \equiv \chi.\beta.\gamma_D^{<-1>}$. From equivalence (25), with x replaced by χ, we have $(\gamma_D^{<-1>})^n \simeq (\chi.\beta.\gamma_D^{<-1>})^n \simeq \chi(\chi - n.\gamma)^{n-1}$, for $n = 1, 2, \ldots$. Because $\chi^{k+1} \simeq 0$ for $k = 1, 2, \ldots, n-1$, we have $\chi(\chi - n.\gamma)^{n-1} \simeq \sum_{k=0}^{n-1} \binom{n-1}{k} \chi^{k+1}(-n.\gamma)^{n-1-k} \simeq (-n.\gamma)^{n-1}$ by which equivalence (26) follows. □

The generalization of Theorem 14 required in Ref. [28] (see Open problems) is the following corollary.

Corollary 5.2 *If γ is an umbra provided with a compositional inverse, then for all $\alpha \in A$ we have $(\alpha.\gamma_D^*)^n \simeq \alpha(\alpha - n.\gamma)^{n-1}$, for $n = 1, 2, \ldots$.*

One more application of Theorem 5.1 is the proof of the following theorem, giving a property of Abel polynomials, known as *Abel identity*.

Theorem 5.2 (Abel identity) *If $\gamma \in A$, then*

$$(x + y)^n \simeq \sum_{k \geq 0} \binom{n}{k} y(y - k.\gamma)^{k-1}(x + k.\gamma)^{n-k}. \tag{27}$$

Proof We have $e^{(y.\beta.\gamma_D)t} \simeq \sum_{k\geq 0} y^k (e^{\gamma_D t} - u)^{.k}/k!$. Replace y by $y.\gamma_D^*$. Since $\gamma_D^*.\beta.\gamma_D \equiv u$ we have $e^{yt} \simeq \sum_{k\geq 0}(y.\gamma_D^*)^k(e^{\gamma_D t} - u)^{.k}/k! \simeq \sum_{k\geq 0}(y.\gamma_D^*)^k(t^k e^{(k.\gamma)t})/k!$ due to (24), and $e^{yt} \simeq 1 + \sum_{k\geq 1} y(y - k.\gamma)^{k-1}(t^k e^{(k.\gamma)t})/k!$ from Theorem 5.1. Multiplying both sides by e^{xt}, we have $e^{(x+y)t} \simeq 1 + \sum_{k\geq 1} y(y - k.\gamma)^{k-1}[t^k e^{(x+k.\gamma)t}]/k!$. By expanding both sides of the last equivalence, the result follows immediately because the coefficients of the same powers of t are similar. □

The umbral polynomials in (27)

$$\binom{n}{k} (x + k.\gamma)^{n-k} \tag{28}$$

play a leading role. Indeed, let us underline that using Theorem 5.1, equivalence (27) can be rewritten as $(x + y)^n \simeq \sum_{k\geq 0} \binom{n}{k} (y.\gamma_D^*)^k(x + k.\gamma)^{n-k}$. By replacing y with $y.\beta.\gamma_D$ in this last equivalence and recalling equivalence (9), we recover the following equivalence

$$(x + y.\beta.\gamma_D)^n \simeq \sum_{k\geq 0} \binom{n}{k} y^k(x + k.\gamma)^{n-k}. \tag{29}$$

By replacing y with u in (29), we have

$$(x + \beta.\gamma_D)^n \simeq \sum_{k\geq 0} \binom{n}{k} (x + k.\gamma)^{n-k}. \tag{30}$$

From equivalence (29), we recover the following corollary that gives the umbral expression of Bell exponential polynomials in (2).

Corollary 5.3 (Umbral representation of Bell exponential polynomials) *For all nonnegative n, we have*

$$(x.\beta.\gamma_D)^n \simeq \sum_{k\geq 0} \binom{n}{k} (k.\gamma)^{n-k} x^k. \tag{31}$$

So we call *generalized Bell polynomials* the polynomials given in (28). These polynomials provide a link between Riordan arrays and connection constants, when x is replaced by an umbra α, see next section.

Corollary 5.4 (Closed formulae for Stirling numbers) *If ι is the Bernoulli umbra, then the Stirling numbers of first and second kind can be umbrally represented as*

$$s(n,k) \simeq \binom{n}{k} (k.\iota.\chi)^{n-k} \quad \textit{and} \quad S(n,k) \simeq \binom{n}{k} (-k.\iota)^{n-k} \quad \textit{for} \quad k = 0, 1, \ldots, n.$$

Proof From Example 5.3, we have $(x)_n \simeq (x.\chi)^n \simeq (x.\beta.u^{<-1>})^n \simeq [x.\beta.(\iota.\chi)_D]^n$. The umbral representation of Stirling numbers of first kind follows from comparing the n-th moment of $x.\beta.(\iota.\chi)_D$, computed with equivalence (31), with the well-known equality $(x)_n = \sum_{k\geq 0} s(n,k)\, x^k$. For the Stirling numbers of second kind, choose $-1.\iota$ as umbra γ in equivalence (31). From Example 5.2, we have $x.\beta.(-1.\iota)_D \equiv x.\beta$. Again, the umbral representation of Stirling numbers of second kind follows from comparing the n-th moment of $x.\beta.(-1.\iota)_D$, computed with equivalence (31), with the umbral equivalence $(x.\beta)^n \simeq \sum_{k\geq 0} S(n,k) x^k$, given in Example 2.4. □

Connection constants and Riordan arrays Theorem 5.1 and equivalence (31) refer to normalized binomial polynomials, i.e. sequences $\{p_n(x)\}$ such that $p_1(x)$ is monic, since they involve umbrae having first moment equal to 1.

More in general, if one would recover umbral expressions of Sheffer sequences coefficients, an additional step is necessary in order to consider umbrae having first moment different from zero and not equal to one.

Proposition 5.1 (Generalized Lagrange inversion formula) *For all umbrae γ provided with compositional inverse and moments $\{g_n\}$, we have*

$$\gamma^{.n} \left(\gamma^{<-1>}\right)^n \simeq (-n.\tilde{\gamma})^{n-1}, \tag{32}$$

where $\tilde{\gamma}$ is the umbra with moments $E\left[\tilde{\gamma}^{\,n-1}\right] = g_n/(n g_1)$ for $n = 1, 2, \ldots$.

Proof For any umbra γ provided with compositional inverse (that is $g_1 \neq 0$), there exists [1] an umbra α such that $\gamma/g_1 \equiv \alpha_D$. Such an umbra α has moments $\alpha^{n-1} \simeq \gamma^n/(n\, g_1^n)$ for $n = 1, 2, \ldots$ and g.f. $f(\alpha, t) = [f(\gamma, t/g_1) - 1]/t$. In particular, we have $g_1\alpha \equiv \tilde{\gamma}$, so multiplying by g_1^{n-1} both sides of equivalence (26), written for the umbra α, we have $g_1^{n-1}(\alpha_D^{<-1>})^n \simeq g_1^{n-1}(-n.\alpha)^{n-1} \simeq [-n.(g_1\alpha)]^{n-1} \simeq (-n.\tilde{\gamma})^{n-1}$. Equivalence (32) follows since we have $(\gamma/g_1)^{<-1>} \equiv \chi.g_1.\beta.\gamma^{<-1>}$ from equivalence (4) and $(\alpha_D^{<-1>})^n \simeq [(\gamma/g_1)^{<-1>}]^n \simeq g_1(\gamma^{<-1>})^n$. □

The generalization of equivalence (29) to umbrae γ with first moment $g_1 \neq 1$ can be stated using the same arguments. Indeed, since $(x.\beta.\gamma)^n \simeq \gamma^{.n}(x.\beta.\alpha_D)^n$, we have

$$(x + y.\beta.\gamma)^n \simeq \sum_{k=0}^{n} \binom{n}{k} y^k \gamma^{.k} (x + k.\tilde{\gamma})^{n-k}. \tag{33}$$

By setting $x = 0$ in (33), we have

$$(y.\beta.\gamma)^n \simeq \sum_{k\geq 0} \binom{n}{k} y^k \gamma^{.k} (k.\tilde{\gamma})^{n-k}, \tag{34}$$

[1] In this case, in the setting of the umbral calculus R must be a field.

and by comparing (34) to (7), we have $B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) \simeq \binom{n}{k} \gamma^{\boldsymbol{.}k}[k\boldsymbol{.}\tilde{\gamma}]^{n-k}$. Using equivalence (33), it is straightforward to prove the following proposition.

Proposition 5.2 (Umbral representation of Sheffer polynomials) *If* $\{s_n(x)\}$ *are moments of a Sheffer umbra for* (α, γ)*, then for all nonnegative integers* n

$$s_n(x) \simeq \sum_{k=0}^{n} \binom{n}{k} x^k \delta^{\boldsymbol{.}k}(\alpha + k\boldsymbol{.}\tilde{\delta})^{n-k} \quad \textit{with} \quad \delta \equiv \gamma^{<-1>}.$$

An (exponential) Riordan array is a pair $(g(t), f(t))$ of (exponential) formal power series, where $g(t)$ is an invertible series and $f(0) = 0$. The pair defines an infinite lower triangular array $(d_{n,k})_{0 \leq k,n < \infty}$ according to the rule $d_{n,k} = n$-th coefficient of $g(t)[f(t)]^k/k!$. Riordan arrays are known also as *recursive matrices* [4]. The reader may consult [26] for more results on the theory of Riordan arrays. In recent years, this subject has aroused some interest. We mention some results of recent literature that would benefit of the umbral approach, see [1,5,6,14,30]. If $s_n(x) = \sum_{k=0}^{n} s_{n,k} x^k$ is a polynomial of Sheffer type, then from (11) we have $s_{n,k} = n$-th coefficient of $f(\alpha, t)[f^{<-1>}(\gamma, t) - 1]^k/k!$. From Proposition 5.2, we have $s_{n,k} \simeq \binom{n}{k} \delta^{\boldsymbol{.}k}(\alpha + k\boldsymbol{.}\tilde{\delta})^{n-k}$. Hence, the following corollary is proved.

Corollary 5.5 (Umbral representation of exponential Riordan arrays) *The elements of the exponential Riordan array* $(f(\alpha, t), f^{<-1>}(\gamma, t) - 1)$ *are umbrally represented by*

$$d_{n,k} \simeq \binom{n}{k} \delta^{\boldsymbol{.}k}(\alpha + k\boldsymbol{.}\tilde{\delta})^{n-k} \quad \textit{with} \quad \delta \equiv \gamma^{<-1>}. \tag{35}$$

Also the connection constants have an umbral representation similar to the elements of a Riordan array (35). Recall that when $s_n(x)$ and $r_n(x)$ are Sheffer sequences, the constants $c_{n,k}$ in the expression

$$r_n(x) = \sum_{k=0}^{n} c_{n,k} s_k(x) \tag{36}$$

are known as *connection constants*. Via umbral syntax, we provide an easy implementable closed form formula for $\{c_{n,k}\}$.

Theorem 5.3 (Umbral representation of connection constants) *Let* $\{s_n(x)\}$ *be the moments of a Sheffer umbra for* (α, γ) *and* $\{r_n(x)\}$ *be the moments of a Sheffer umbra for* (δ, ζ)*. The connection constants* $\{c_{n,k}\}$ *in (36) are such that*

$$c_{n,k} \simeq \binom{n}{k} \xi^{\boldsymbol{.}k}(\varsigma + k\boldsymbol{.}\tilde{\xi})^{n-k}, \quad \textit{with} \quad \varsigma \equiv (-1\boldsymbol{.}\alpha\boldsymbol{.}\beta\boldsymbol{.}\gamma + \delta\boldsymbol{.}\beta\boldsymbol{.}\zeta)\boldsymbol{.}\zeta^* \quad \textit{and} \quad \xi \equiv \zeta\boldsymbol{.}\gamma^*. \tag{37}$$

Proof Suppose η_x be a polynomial umbra such that $E[\eta_x^n] = q_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$. From (36), we have

$$\delta + x\boldsymbol{.}\zeta^* \equiv \eta_{\alpha + x\boldsymbol{.}\gamma^*}. \tag{38}$$

In equivalence (38), replace x with $-1\boldsymbol{.}\alpha\boldsymbol{.}\beta\boldsymbol{.}\gamma + x\boldsymbol{.}\beta\boldsymbol{.}\gamma$. Due to equivalence (9), the right-hand side of equivalence (38) becomes $\eta_{\alpha+(-1\boldsymbol{.}\alpha\boldsymbol{.}\beta\boldsymbol{.}\gamma+x\boldsymbol{.}\beta\boldsymbol{.}\gamma)\boldsymbol{.}\gamma^*} \equiv \eta_{-1\boldsymbol{.}\alpha+\alpha+x\boldsymbol{.}\beta\boldsymbol{.}\gamma\boldsymbol{.}\gamma^*} \equiv \eta_{x\boldsymbol{.}u}$. The left-hand side can be simply rewritten as $\delta + (-1\boldsymbol{.}\alpha\boldsymbol{.}\beta\boldsymbol{.}\gamma + x\boldsymbol{.}\beta\boldsymbol{.}\gamma)\boldsymbol{.}\zeta^* \equiv (-1\boldsymbol{.}\alpha\boldsymbol{.}\beta\boldsymbol{.}\gamma +$

$\delta.\beta.\zeta).\zeta^* + x.\beta.\gamma.\zeta^*$. Now we have $x.\beta.\gamma.\zeta^* \equiv x.(\gamma^{<-1>})^*.\zeta^* \equiv x.(\zeta.\beta.\gamma^{<-1>})^*$ due to Proposition 3.2. Therefore, equivalence (38) can be simply rewritten as

$$\eta_x \equiv (-1.\alpha.\beta.\gamma + \delta.\beta.\zeta).\zeta^* + x.(\zeta.\beta.\gamma^{<-1>})^*. \tag{39}$$

To get the explicit expression of $\{c_{n,k}\}$ in (36), it is necessary to expand the n-th moment of η_x in (39) and to get the coefficient of x^k. The result follows taking into account Proposition 5.2. □

So Riordan arrays and connection constants have a similar umbral representation via generalized Bell polynomials (28) with x replaced by an umbra α.

Corollary 5.6 *The Riordan array for the pair* $(f(\alpha,t), f(\gamma,t)-1)$ *is the inverse of the Riordan array for the pair* $(1/f[\alpha, f^{<-1>}(\gamma,t)-1], f^{<-1>}(\gamma,t)-1)$.

Proof From Corollary 5.5, the elements of the Riordan array for the pair $(f(\alpha,t), f(\gamma,t)-1)$ have the form (35) with $\delta \equiv \gamma$. They correspond to the coefficients of $\{s_n(x)\}$, moments of a Sheffer umbra for $(\alpha, \gamma^{<-1>})$. The compositional inverses of $\{s_n(x)\}$ are the polynomials $\sum_{k=0}^n c_{n,k} x^k$ such that $x^n = \sum_{k=0}^n c_{n,k} s_k(x)$. The coefficients $\{c_{n,k}\}$ are the elements of the inverses of the Riordan arrays for the pair $(f(\alpha,t), f(\gamma,t)-1)$. Since x^n are moments of a Sheffer umbra for (ϵ, χ), then $\{c_{n,k}\}$ are connection constants (37) with $\varsigma \equiv -1.\alpha.\beta.\gamma^{<-1>}$ and $\xi \equiv \gamma$. So from Corollary 5.5, the result follows. □

References

1. Barry, P.: On a family of generalized pascal triangles defined by exponential riordan arrays. J. Integer Seq. **10**(3), Article 07.3.5, 21pp (2007)
2. Barnabei, M., Guerrini, C., Montefusco, L.B.: Some algebraic aspects of signal processing. Linear Alg. Appl. **284**, 3–17 (1998)
3. Barnabei, M., Brini, A., Nicoletti, G.: Polynomial sequences of integral type. J. Math. Anal. Appl. **78**, 598–617 (1980)
4. Barnabei, M., Brini, A., Nicoletti, G.: Recursive matrices and umbral calculus. J. Algebra **75**, 546–573 (1982)
5. Della Riccia, G.: Riordan arrays, Sheffer sequences and orthogonal polynomials. J. Integer Seq. (11), Article 08.5.3. (2008)
6. Deutsch, E., Ferrari, L., Rinaldi, S.: Production matrices and Riordan arrays. Ann. Comb. **13**, 65–85 (2009)
7. Di Bucchianico, A., Loeb, D.: A selected survey of umbral calculus. Electron. J. Combin. **2**, Dynamic Survey 3, 28 pp. (updated 2000) (1995)
8. Di Nardo, E., Senato, D.: Umbral Nature of the Poisson Random Variables. Algebraic Combinatorics and Computer Science, pp. 245–266. Springer Italia, Milan (2001)
9. Di Nardo, E., Senato, D.: An umbral setting for cumulants and factorial moments. European J. Combin. **27**(3), 394–413 (2006)
10. Di Nardo, E., Guarino, G., Senato, D.: A unifying framework for k-statistics, polykays and their multivariate generalizations. Bernoulli. **14**(2), 440–468 (2008)
11. Di Nardo, E., Oliva, I.: On the computation of classical, Boolean and free cumulants. Appl. Math. Comp. **208**(2), 347–354 (2009)
12. Di Nardo, E., Petrullo, P. Senato, D.: Cumulants and convolutions via Abel polynomials. European J. Comb. (2010). doi:10.1016/j.ejc.2010.03.002
13. Gessel, I.M.: Applications of the classical umbral calculus. Algebra Universalis. **49**(4), 397–434 (2003)
14. Merlini, D., Sprugnoli, R., Verria, M.: Combinatorial sums and implicit Riordan arrays. Discrete Math. **309**(2), 475–486 (2009)
15. Niederhausen, H.: Sheffer polynomials for computing exact Kolmogorov-Smirnov and Rényi type distributions. Ann. Statist. **9**, 923–944 (1981)
16. Paule, P., Scherzer, O., Schoisswohl, A.: Wavelets with scale dependent properties. Lect. Notes Comput. Sci. **2630**, 255–265 (2003)

17. Pommeret, D.: Orthogonality of the Sheffer system associated to a Levy process. J. Statist. Plann. Inference **86**, 1–10 (2000)
18. Ray, N.: Universal constructions in umbral calculus. In: Mathematical Essays in Honor of Gian-Carlo Rota, pp. 343–357. Birkhäuser, Boston (1998)
19. Roman, S.M., Rota, G.-C.: The umbral calculus. Adv. Math. **27**, 95–188 (1978)
20. Rota, G.-C., Taylor, B.D.: The classical umbral calculus. SIAM J. Math. Anal. **25**(2), 694–711 (1994)
21. Rota, G.-C., Shen, J., Taylor, B.D.: All polynomials of binomial type are represented by Abel polynomials. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25**(1), 731–738 (1998)
22. Saliani, S., Senato, D.: Compactly supported wavelets through the classical umbral calculus. J. Fourier Anal. Appl. **12**(3–4), 27–36 (2006)
23. Schoutens, W.: Lévy Processes in Finance: Pricing Financial Derivatives. Wiley, New York (2003)
24. Sheffer, I.M.: Some properties of polynomial sets of type zero. Duke Math. J. **5**, 590–622 (1939)
25. Shen, J.: Combinatorics for wavelets: the umbral refinement equation. Stud. Appl. Math. **103**(2), 121–147 (1999)
26. Sprugnoli, R.: Riordan arrays and combinatorial sums. Discret. Math. **132**, 267–290 (1994)
27. Taylor, B.D.: Difference equations via the classical umbral calculus. Mathematical essays in honor of Gian-Carlo Rota, pp. 397–411. Birkhauser, Boston (1998)
28. Taylor, B.D.: Umbral presentations for polynomial sequences. Comput. Math. Appl. **41**, 1085–1098 (2001)
29. Tempesta, P.: On Appell sequences of polynomials of Bernoulli and Euler type. J. Math. Anal. Appl. **341**, 1295–1310 (2008)
30. Wang, W., Wang, T.: Generalized Riordan arrays. Discret. Math. **308**(24), 6466–6500 (2009)
31. Wang, W., Wang, T.: Identities on Bell polynomials and Sheffer sequences. Discret. Math. **309**, 1637–1648 (2009)