

# Unification of the three families of generalized Apostol type polynomials on the Umbral algebra

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## Abstract

The aim of this paper is to investigate and introduce some new identities related to the unification and generalization of the three families of generalized Apostol type polynomials, which are Apostol-Bernoulli, Apostol-Euler, and Apostol-Genocchi polynomials, on the modern theory of the Umbral calculus and algebra. We also introduce some operators. Recently, Ozden constructed generating function of the unification of the Apostol type polynomials (see Ozden [H. Ozden, *AIP Conf. Proc.* 1281, (2010), 1125-1227.]). By using this generating function, we derive many properties of these polynomials. We give relations between these polynomials and Stirling numbers.

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**Key Words and Phrases.** Bernoulli polynomials; Euler polynomials; Genocchi polynomials; Apostol-Bernoulli polynomials; Apostol-Euler polynomials; Apostol-Genocchi polynomials; Sheffer sequences; Appell sequences; Stirling numbers; Multiplication formula (Raabe-type multiplication formula); Recurrence formula, Umbral algebra.

## 1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Throughout of this paper, we use the following notations:

$$\mathbb{N} := \{1, 2, 3, \dots\} \text{ and } \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Here  $\mathbb{Z}$  denotes the set of integers,  $\mathbb{R}$  denotes the set of real numbers,  $\mathbb{R}^+$  denotes the set of positive real numbers and  $\mathbb{C}$  denotes the set of complex numbers. We also assume that  $\ln(z)$  denotes the *principal branch of the many-valued function*  $\ln(z)$  with the imaginary part  $\Im(\ln(z))$  constrained by  $-\pi < \Im(\ln(z)) \leq \pi$ .

$$\binom{m}{n} = \frac{m!}{n!(m-n)!},$$

$$\binom{m}{a, b, c} = \frac{m!}{a!b!c!} \text{ and } a + b + c = m,$$

$$0^j = \begin{cases} 0 & \text{if } j \neq 0 \\ 1 & \text{if } j = 0, \end{cases}$$

and

$$(x)_b = x(x-1)\dots(x-b+1),$$

where  $b \in \mathbb{N}$  and  $(x)_0 = 1$ .

Many generating functions of the Bernoulli, Euler and Genocchi type polynomials have been introduced by many authors. Recently, Ozden [26]-[27] constructed generating functions of the unification of the Apostol type polynomials  $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$ . These generating functions were investigated rather systematically by Ozden *et al.* [30].

The following generating function of the *unification* of the Apostol type Bernoulli, Euler and Genocchi polynomials  $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$  of higher order, which was recently defined by Ozden [27]:

$$\left( \frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right)^v e^{xt} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) \frac{t^n}{n!}, \quad (1.1)$$

where

$$\left| t + b \ln \left( \frac{\beta}{a} \right) \right| < 2\pi; \quad x \in \mathbb{R}$$

and  $k \in \mathbb{N}_0$ ;  $v \in \mathbb{N}$ ;  $a, b \in \mathbb{R}^+$ ;  $\beta \in \mathbb{C}$ .

Observe that

$$\mathcal{Y}_{n,\beta}^{(1)}(x; k, a, b) = \mathcal{Y}_{n,\beta}(x; k, a, b) \quad (n \in \mathbb{N}) \quad (1.2)$$

which is defined by Ozden [26]. Recently, Ozden *et al.* [30] introduced many properties of these polynomials.

**Remark 1.** *By substituting  $x = 0$  in the generating function (1.1), we obtain the corresponding unification and generalization of the generating functions of Bernoulli, Euler and Genocchi numbers of higher order. Thus, we have*

$$\mathcal{Y}_{n,\beta}^{(v)}(0; k, a, b) = \mathcal{Y}_{n,\beta}^{(v)}(k, a, b).$$

**Remark 2.** *By setting  $b = v = 1$  and  $\beta = 1$  into (1.2), we have  $\mathcal{Y}_{n,1}(x; k, a, 1) = D_n(x; a, k)$  (see, for details, [15]).*

**Remark 3.** *By substituting  $a = b = k = 1$  into (1.1), one has the Apostol-Bernoulli polynomials  $\mathcal{Y}_{n,\beta}^{(v)}(x; 1, 1, 1) = \mathcal{B}_n^{(v)}(x, \beta)$ , which are defined by means of the following generating function:*

$$\left( \frac{t}{\beta e^t - 1} \right)^v e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(v)}(x, \beta) \frac{t^n}{n!}, \quad (|t + \log \beta| < 2\pi)$$

(see, for details, [28], [26], [30], [21], [17], [35], [33], [34], [42]; see also the references cited in each of these earlier works).

**Remark 4.** If we substitute  $b = v = 1$ ,  $k = 0$  and  $a = -1$  into (1.1), then we have the Apostol-Euler polynomials  $\mathcal{Y}_{n,\beta}(x; 0, -1, 1) = \mathcal{E}_n(x, \beta)$ , which is defined by the following generating function:

$$\frac{2e^{xt}}{\beta e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x, \beta) \frac{t^n}{n!}, \quad (|t + \log \beta| < \pi)$$

(see, for details, [28], [26], [30], [21], [17], [35], [33], [42]; see also the references cited in each of these earlier works).

**Remark 5.** By substituting  $b = 1 = v$ ,  $k = 1$  and  $a = -1$  into (1.1), one has the Apostol-Genocchi polynomials  $\mathcal{Y}_{n,\beta}(x; 1, -1, 1) = \frac{1}{2}\mathcal{G}_n(x, \beta)$ , which is defined by the following generating function:

$$\frac{2te^{xt}}{\beta e^t + 1} = \sum_{n=0}^{\infty} \mathcal{G}_n(x, \beta) \frac{t^n}{n!}, \quad (|t + \log \beta| < \pi)$$

(see, for details, [20], [28], [26], [30], [18]; see also the references cited in each of these earlier works).

**Remark 6.** Substituting  $\beta = b = k = a = v = 1$  into (1.1), we have  $\mathcal{Y}_{n,1}(x; 1, 1, 1) = B_n(x)$ , where  $B_n(x)$  denotes the classical Bernoulli polynomials, which are defined by the following generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (|t| < 2\pi)$$

(see, for example, [3]-[42]; see also the references cited in each of these earlier works).

**Remark 7.** Likely, substituting  $\beta = b = k = v = 1$ , and  $a = -1$  into (1.1), we have

$$\mathcal{Y}_{n,1}(x; 0, -1, 1) = E_n(x),$$

where  $E_n(x)$  denotes the classical Euler polynomials, which are defined by the following generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi)$$

(see, for example, [3]-[42]; see also the references cited in each of these earlier works).

We use the following relations and identities of the umbral algebra and calculus which are given by Roman [32]:

Let  $P$  be the algebra of polynomials in the single variable  $x$  over the field complex numbers. Let  $P^*$  be the vector space of all linear functionals on  $P$ . Let  $\langle L | p(x) \rangle$  be the action of a linear functional  $L$  on a polynomial  $p(x)$ . Let  $F$  denotes the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k.$$

This kind of algebra is called an umbral algebra. Each  $f \in F$  defines a linear functional on  $P$  and for all  $k \geq 0$ ,  $a_k = \langle f(t) | x^k \rangle$ . The order  $o(f(t))$  of a power series  $f(t)$  is the smallest integer  $k$  for which the coefficient of  $t^k$  does not vanish. A series  $f(t)$  for which

$o(f(t)) = 1$  is called a delta series. And a series  $f(t)$  for which  $o(f(t)) = 0$  is called a invertible series.

Let  $f(t), g(t)$  be in  $F$ , we have

$$\langle f(t)g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)p(x) \rangle. \quad (1.3)$$

For all  $p(x)$  in  $P$ , we have

$$\langle e^{yt} \mid p(x) \rangle = p(y). \quad (1.4)$$

The Sheffer polynomials are defined by means of the following generating function

$$\sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k = \frac{1}{g(t)} e^{xt}. \quad (1.5)$$

(see, for details, [32]; and see also [3], [10]).

**Theorem 1.** (See, for details, Roman [32, p. 20, Theorem 2.3.6]) *Let  $f(t)$  be a delta series and let  $g(t)$  be an invertible series. Then there exist a unique sequence  $s_n(x)$  of polynomials satisfying the orthogonality conditions*

$$\langle g(t)f(t)^k \mid s_n(x) \rangle = n! \delta_{n,k} \quad (1.6)$$

for all  $n, k \geq 0$ .

**Remark 8.** *The sequence  $s_n(x)$  in (1.6) is the Sheffer polynomials for pair  $(g(t), f(t))$ , where  $g(t)$  must be invertible and  $f(t)$  must be delta series. The Sheffer polynomials for pair  $(g(t), t)$  is the Appell polynomials or the Appell sequences for  $g(t)$ . The Bernoulli polynomials, the Euler polynomials, the Genocchi polynomials, and the polynomials  $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$  are the Appell polynomials cf. [3]-[43].*

Let

$$s_n(x) = g(t)^{-1} x^n, \quad (1.7)$$

derivative formula

$$ts_n(x) = s'_n(x) = ns_{n-1}(x), \quad (1.8)$$

recurrence formula

$$s_{n+1}(x) = \left( x - \frac{g'(t)}{g(t)} \right) s_n(x), \quad (1.9)$$

and multiplication formula, for  $\alpha \neq 0$

$$s_n(\alpha x) = \alpha^n \frac{g(t)}{g\left(\frac{t}{\alpha}\right)} s_n(x). \quad (1.10)$$

Proof of equations (1.7)-(1.10) are given by Roman [32].

## 2. THE OPERATOR $\frac{1}{f(t)}$

In this section we introduce  $\frac{1}{f(t)}$  operator on the Umbral algebra. By applying this operator to the Sheffer sequences, we obtain new identities related to the family of the Sheffer sequences.

**Theorem 2.** *Let  $n \in \mathbb{N}_0$ . Let  $s_n(x)$  be Sheffer sequence for  $(g(t), f(t))$ . The following relationship holds true:*

$$\frac{1}{f(t)}s_n(x) = \frac{1}{n+1}s_{n+1}(x).$$

*Proof.* We set

$$\left\langle g(t)f(t)^k \mid \frac{1}{f(t)}s_n(x) \right\rangle = \left\langle g(t)f(t)^{k-1} \mid s_n(x) \right\rangle.$$

By using (1.6) in the above relation, we obtain

$$\left\langle g(t)f(t)^k \mid \frac{1}{f(t)}s_n(x) \right\rangle = n! \delta_{n,k-1}.$$

From the definition of the Kronecker  $\delta$ , we get

$$\left\langle g(t)f(t)^k \mid \frac{1}{f(t)}s_n(x) \right\rangle = \left\langle g(t)f(t)^k \mid \frac{1}{n+1}s_{n+1}(x) \right\rangle.$$

After some calculation in the above equation, we obtain the desired result. □

Taking  $f(t) = t$  in Theorem 2, we deduce the following corollary:

**Corollary 1.** *Let  $n \in \mathbb{N}_0$ . Let  $s_n(x)$  be the Appell sequence for  $g(t)$ . The following relationship holds true:*

$$\frac{1}{t}s_n(x) = \frac{1}{n+1}s_{n+1}(x). \tag{2.1}$$

## 3. SOME NEW IDENTITIES AND RELATIONS OF THE UNIFICATION OF THE BERNOULLI, EULER AND GENOCCHI POLYNOMIALS $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$ OF HIGHER ORDER

In this section, by using properties of the Sheffer sequences and the Appell sequences, we give some important properties of the polynomials  $\mathcal{Y}_{n,\beta}^{(v)}(x, k, a, b)$  and the proof of this properties.

By using (1.7) and (1.1), we arrive at the following lemma:

**Lemma 1.** *Let  $n \in \mathbb{N}_0$ . The following relationship holds true:*

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \left( \frac{2^{1-k}t^k}{\beta^b e^t - a^b} \right)^v x^n.$$

**Lemma 2.** *Let  $n \in \mathbb{N}_0$ . The following relationship holds true:*

$$\langle (\beta^b e^t - a^b)^j \mid \mathcal{Y}_{n,\beta}(x; k, a, b) \rangle = \sum_{m=0}^j \binom{j}{m} (-a)^{b(j-m)} \beta^{bm} \mathcal{Y}_{n,\beta}(m; k, a, b).$$

*Proof.*

$$\begin{aligned}
& \langle (\beta^b e^t - a^b)^j \mid \mathcal{Y}_{n,\beta}(x; k, a, b) \rangle \\
&= \left\langle \sum_{m=0}^j \binom{j}{m} \beta^{bm} e^{mt} (-a)^{b(j-m)} \mid \mathcal{Y}_{n,\beta}(x; k, a, b) \right\rangle \\
&= \sum_{m=0}^j \binom{j}{m} \beta^{bm} (-a)^{b(j-m)} \langle e^{mt} \mid \mathcal{Y}_{n,\beta}(x; k, a, b) \rangle.
\end{aligned}$$

Using (1.4) in this equation, we complete proof of this Lemma.  $\square$

**Remark 9.** *By substituting  $\beta = a = b = k = 1$  into Lemma 2, we arrive at a special case:*

$$\begin{aligned}
& \langle (e^t - 1)^j \mid \mathcal{Y}_{n,1}(x; 1, 1, 1) \rangle = \langle (e^t - 1)^j \mid B_n(x) \rangle \\
&= \sum_{m=0}^j \binom{j}{m} (-1)^{j-m} B_n(m).
\end{aligned}$$

**Remark 10.** *If we set  $\beta = b = 1$ ,  $k = 0$  and  $a = -1$  in Lemma 2, we arrive at the following result:*

$$\begin{aligned}
& \langle (e^t + 1)^j \mid \mathcal{Y}_{n,1}(x; 0, -1, 1) \rangle = \langle (e^t + 1)^j \mid E_n(x) \rangle \\
&= \sum_{m=0}^j \binom{j}{m} E_n(m).
\end{aligned}$$

**Remark 11.** *Substituting  $\beta = b = k = 1$ , and  $a = -1$  into Lemma 2, we obtain*

$$\begin{aligned}
& \langle (e^t + 1)^j \mid \mathcal{Y}_{n,1}(x; 1, -1, 1) \rangle = \left\langle (e^t + 1)^j \mid \frac{1}{2} G_n(x) \right\rangle \\
&= \frac{1}{2} \sum_{m=0}^j \binom{j}{m} G_n(m).
\end{aligned}$$

**Remark 12.** *For  $k = a = b = 1$  into Lemma 2, we arrive at the following result.*

$$\begin{aligned}
& \langle (\beta e^t - 1)^j \mid \mathcal{Y}_{n,\beta}(x; 1, 1, 1) \rangle = \langle (\beta e^t - 1)^j \mid \mathcal{B}_n(x) \rangle \\
&= \sum_{m=0}^j \binom{j}{m} (-1)^{j-m} \beta^m \mathcal{B}_n(m, \beta).
\end{aligned}$$

**Remark 13.** *By putting  $k = 0$ ,  $b = 1$ , and  $a = -1$  in Lemma 2, we have*

$$\begin{aligned}
& \langle (\beta e^t + 1)^j \mid \mathcal{Y}_{n,\beta}(x; 0, -1, 1) \rangle = \langle (\beta e^t + 1)^j \mid \mathcal{E}_n(x) \rangle \\
&= \sum_{m=0}^j \binom{j}{m} \beta^m \mathcal{E}_n(m, \beta).
\end{aligned}$$

**Remark 14.** In a special case when  $k = b = 1$ , and  $a = -1$ , Lemma 2 yields:

$$\begin{aligned} & \langle (\beta e^t + 1)^j \mid \mathcal{Y}_{n,\beta}(x; 1, -1, 1) \rangle = \left\langle (\beta e^t + 1)^j \mid \frac{1}{2} \mathcal{G}_n(x) \right\rangle \\ & = \frac{1}{2} \sum_{m=0}^j \binom{j}{m} \beta^m \mathcal{G}_n(m, \beta). \end{aligned}$$

**Lemma 3.** Then the following identity holds:

$$S(n, l) = \frac{1}{l!} \langle (e^t - 1)^l \mid x^n \rangle$$

where  $S(n, l)$  is the Stirling numbers of the second kind.

**Theorem 3.** If  $a \neq \beta$ , then we have

$$\begin{aligned} & \langle (\beta^b e^t - a^b)^j \mid \mathcal{Y}_{n,\beta}(x; k, a, b) \rangle \\ & = \frac{\beta^{b(j-1)} k!}{2^{k-1}} \binom{n}{k} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1 - \frac{a^b}{\beta^b}\right)^{j-l-1} S(n-k, l). \end{aligned}$$

If  $a = \beta$ , then we have

$$\langle (e^t - 1)^j \mid \mathcal{Y}_{n,a}(x; k, a, b) \rangle = \frac{k! (j-1)!}{2^{k-1} a^b} \binom{n}{k} S(n-k, j-1).$$

where  $S(u, v)$  denote the Stirling numbers of the second kind.

*Proof.* By applying Lemma 1 to  $\langle (\beta^b e^t - a^b)^j \mid \mathcal{Y}_{n,\beta}(x, k, a, b) \rangle$ , we obtain

$$\langle (\beta^b e^t - a^b)^j \mid \mathcal{Y}_{n,\beta}(x, k, a, b) \rangle = \left\langle (\beta^b e^t - a^b)^j \mid \frac{2^{1-k} t^k}{\beta^b e^t - a^b} x^n \right\rangle.$$

By using (1.3) and (1.8) in the above equation, we get

$$\langle (\beta^b e^t - a^b)^j \mid \mathcal{Y}_{n,\beta}(x; k, a, b) \rangle = 2^{1-k} \langle (\beta^b e^t - a^b)^{j-1} \mid t^k x^n \rangle.$$

Applying Lemma 4 in this equation, we obtain

$$\langle (\beta^b e^t - a^b)^j \mid \mathcal{Y}_{n,\beta}(x; k, a, b) \rangle = 2^{1-k} \langle (\beta^b e^t - a^b)^{j-1} \mid (n)_k x^{n-k} \rangle.$$

Hence we get

$$\begin{aligned} & \langle (\beta^b e^t - a^b)^j \mid \mathcal{Y}_{n,\beta}(x; k, a, b) \rangle \\ & = 2^{1-k} (n)_{k+1} \beta^{b(j-1)} \sum_{l=0}^{j-1} \binom{j-1}{l} \left(1 - \frac{a^b}{\beta^b}\right)^{j-l-1} \langle (e^t - 1)^l \mid x^{n-k} \rangle. \end{aligned}$$

Thus by Lemma 3, we obtain

$$\begin{aligned} & \langle (\beta^b e^t - a^b)^j \mid \mathcal{Y}_{n,\beta}(x; k, a, b) \rangle \\ & = 2^{1-k} (n)_{k+1} \beta^{b(j-1)} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1 - \frac{a^b}{\beta^b}\right)^{j-l-1} S(n-k, l). \end{aligned}$$

After elementary manipulations in the above equation, we complete proof of the theorem.  $\square$

By combining Lemma 2 and Theorem 3, we arrive at the following corollary:

**Corollary 2.** *The following relationship holds true:*

$$\begin{aligned} & \sum_{m=0}^j \binom{j}{m} (-a)^{b(j-m)} \beta^{bm} \mathcal{Y}_{n,\beta}(x; k, a, b) \\ &= \frac{\beta^{b(j-1)} k!}{2^{k-1}} \binom{n}{k} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1 - \frac{a^b}{\beta^b}\right)^{j-l-1} S(n-k, l). \end{aligned}$$

**Remark 15.** *Theorem 3 provides us with a generalized and unification of the linear operator  $\langle \cdot | \cdot \rangle$  related to the Apostol type polynomials.*

**Remark 16.** *By setting  $\beta = a = b = k = 1$  in Theorem 3, we arrive at the well-known relation which was proved by Roman [32, p. 94]:*

$$\begin{aligned} & \langle (e^t - 1)^j | \mathcal{Y}_{n,1}(x; 1, 1, 1) \rangle = \langle (e^t - 1)^j | B_n(x) \rangle \\ &= n(j-1)! S(n-1, j-1). \end{aligned}$$

And using Corollary 2, we get

$$\sum_{m=0}^j \binom{j}{m} (-1)^{j-m} B_n(m) = n(j-1)! S(n-1, j-1).$$

**Remark 17.** *By putting  $\beta = b = k = 1$  and  $a = -1$  into Theorem 3, we have*

$$\langle (e^t + 1)^j | \mathcal{Y}_{n,1}(x; 1, -1, 1) \rangle = \left\langle (e^t + 1)^j | \frac{1}{2} G_n(x) \right\rangle.$$

From this equation, we get

$$\langle (e^t + 1)^j | G_n(x) \rangle = n(j-1)! \sum_{l=0}^{j-1} \frac{2^{j-l}}{(j-l-1)!} S(n-1, l)$$

cf. [11, Theorem 2]. And using Corollary 2, we have

$$\sum_{m=0}^j \binom{j}{m} G_n(m) = 2n(j-1)! \sum_{l=0}^{j-1} \frac{2^{j-l}}{(j-l-1)!} S(n-1, l).$$

**Remark 18.** *Upon substituting  $\beta = b = 1$ ,  $k = 0$  and  $a = -1$  into Theorem 3, we obtain*

$$\begin{aligned} & \langle (e^t + 1)^j | \mathcal{Y}_{n,1}(x; 0, -1, 1) \rangle = \langle (e^t + 1)^j | E_n(x) \rangle \\ &= \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} 2^{j-l} S(n, l). \end{aligned}$$

And using Corollary 2, we have

$$\sum_{m=0}^j \binom{j}{m} E_n(m) = \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} 2^{j-l} S(n, l).$$



**Remark 19.** If we substitute  $a = b = k = 1$  into Theorem 3, we get a special case of the generalized Bernoulli polynomials  $\mathcal{Y}_{n,\beta}(x; k, a, b)$ , that is, the so-called Apostol-Bernoulli polynomials  $\mathcal{B}_n(x, \beta)$ :

$$\begin{aligned} & \langle (\beta e^t - 1)^j \mid \mathcal{Y}_{n,\beta}(x; 1, 1, 1) \rangle = \langle (\beta e^t - 1)^j \mid \mathcal{B}_n(x, \beta) \rangle \\ & = n\beta^{j-1} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1 - \frac{1}{\beta}\right)^{j-l-1} S(n-1, l). \end{aligned}$$

And using Corollary 2, we have

$$\sum_{m=0}^j \binom{j}{m} (-1)^{j-m} \beta^m \mathcal{B}_n(m, \beta) = n\beta^{j-1} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1 - \frac{1}{\beta}\right)^{j-l-1} S(n-1, l).$$

**Remark 20.** By putting  $b = k = 1$ , and  $a = -1$  in Theorem 3, we get a special case of the generalized Bernoulli polynomials  $\mathcal{Y}_{n,\beta}(x; k, a, b)$ , that is, the so-called Apostol-Genocchi polynomials  $\mathcal{G}_n(x, \beta)$ :

$$\begin{aligned} & \langle (\beta e^t + 1)^j \mid \mathcal{Y}_{n,\beta}(x; 1, -1, 1) \rangle = \left\langle (\beta e^t + 1)^j \mid \frac{1}{2} \mathcal{G}_n(x, \beta) \right\rangle \\ & = n\beta^{j-1} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1 + \frac{1}{\beta}\right)^{j-l-1} S(n-1, l). \end{aligned}$$

And using Corollary 2, we have

$$\sum_{m=0}^j \binom{j}{m} \beta^m \mathcal{G}_n(m, \beta) = 2n\beta^{j-1} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1 + \frac{1}{\beta}\right)^{j-l-1} S(n-1, l).$$

**Remark 21.** Taking  $b = 1$ ,  $k = 0$ , and  $a = -1$  into Theorem 3, we get a special case of the generalized Bernoulli polynomials  $\mathcal{Y}_{n,\beta}(x; k, a, b)$ , that is, the so-called Apostol-Euler polynomials  $\mathcal{E}_n(x, \beta)$ :

$$\begin{aligned} & \langle (\beta e^t + 1)^j \mid \mathcal{Y}_{n,\beta}(x; 0, -1, 1) \rangle = \langle (\beta e^t + 1)^j \mid \mathcal{E}_n(x, \beta) \rangle \\ & = 2\beta^{j-1} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1 + \frac{1}{\beta}\right)^{j-l-1} S(n-1, l). \end{aligned}$$

And using Corollary 2, we have

$$\sum_{m=0}^j \binom{j}{m} \beta^m \mathcal{E}_n(m, \beta) = 2\beta^{(j-1)} \sum_{l=0}^{j-1} \frac{(j-1)!}{(j-l-1)!} \left(1 + \frac{1}{\beta}\right)^{j-l-1} S(n, l).$$

By using (1.8), we arrive at the following lemma:

**Lemma 4.** We have

$$t\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = n\mathcal{Y}_{n-1,\beta}^{(v)}(x; k, a, b).$$

**Remark 22.** A second proof of Lemma 4 is also obtained from (1.1) by using derivative with respect to  $x$ .

By applying (2.1) to the polynomials  $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$ , we arrive at the following Lemma:

**Theorem 4.** *The following integral operator holds true:*

$$\frac{1}{t} \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \frac{1}{n+1} \mathcal{Y}_{n+1,\beta}^{(v)}(x; k, a, b). \quad (3.1)$$

**Theorem 5.** *The following integral formula holds true:*

$$\left\langle \frac{e^{tc} - 1}{t} \mid \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) \right\rangle = \begin{cases} \int_0^c \mathcal{Y}_{n,\beta}^{(v)}(u; k, a, b) du & (n \in \mathbb{N}) \\ 0 & (n = 0). \end{cases}$$

*Proof.* We set

$$\left\langle \frac{e^{tc} - 1}{t} \mid \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) \right\rangle = \left\langle (e^{tc} - 1) \mid \frac{1}{t} \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) \right\rangle.$$

By applying the operator in (3.1) and the functional in (1.4) to the above equation, we obtain the desired result.  $\square$

We give action of a linear operator  $(\beta^b e^t - a^b)$  on the polynomial  $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$  by the following lemma:

**Lemma 5.** *The following identity holds true:*

$$(\beta^b e^t - a^b) \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = 2^{1-k} (n)_k \mathcal{Y}_{n-k,\beta}^{(v-1)}(x; k, a, b).$$

*Proof.* By (1.1) and Lemma 1, we obtain

$$(\beta^b e^t - a^b) \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = (\beta^b e^t - a^b) \left( \frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right)^v x^n.$$

After some calculations, we get

$$(\beta^b e^t - a^b) \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = 2^{1-k} t^k \mathcal{Y}_{n,\beta}^{(v-1)}(x; k, a, b).$$

Using Lemma 4 in this equation, it is easy to obtain the desired result.  $\square$

**Theorem 6.** *The following identity holds true:*

$$\mathcal{Y}_{n,\beta}^{(v)}(x+1; k, a, b) = \frac{(n)_k}{2^{k-1} \beta^b} \mathcal{Y}_{n-k,\beta}^{(v-1)}(x; k, a, b) + \left( \frac{a}{\beta} \right)^b \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b).$$

*Proof.* By applying the following well-known operator

$$e^{at} p(x) = p(x+a) \text{ cf. [32, p. 14]} \quad (3.2)$$

to the polynomial  $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$ , we obtain

$$(\beta^b e^t - a^b) \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \beta^b \mathcal{Y}_{n,\beta}^{(v)}(x+1; k, a, b) - a^b \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b). \quad (3.3)$$

Combining (3.3) with Lemma 5, we arrive at the desired result.  $\square$

**Remark 23.** Taking  $\beta = a = b = k = 1$  in Lemma 5 and Theorem 6, respectively, we have

$$(e^t - 1) B_n^{(v)}(x) = nB_{n-1}^{(v-1)}(x) \quad [32, \text{p. 95, Eq-(4.2.5)}]$$

and

$$B_n^{(v)}(x+1) = nB_{n-1}^{(v-1)}(x) + B_n^{(v)}(x) \quad [32, \text{p. 95, Eq-(4.2.6)}].$$

**Remark 24.** Substituting  $\beta = b = 1$ ,  $k = 0$  and  $a = -1$  into Lemma 5 and Theorem 6, respectively, we obtain

$$(e^t + 1) E_n^{(v)}(x) = 2E_n^{(v-1)}(x) \quad [32, \text{p. 103}]$$

and

$$E_n^{(v)}(x+1) = 2E_n^{(v-1)}(x) - E_n^{(v)}(x) \quad [32, \text{p. 103, Eq-(4.2.11)}].$$

**Remark 25.** Putting  $\beta = b = k = 1$  and  $a = -1$  in Lemma 5 and Theorem 6, respectively, we obtain

$$(e^t + 1) G_n^{(v)}(x) = 2nG_{n-1}^{(v-1)}(x) \quad [11, \text{p. 5, Theorem 7}]$$

and

$$G_n^{(v)}(x+1) = 2nG_{n-1}^{(v-1)}(x) - G_n^{(v)}(x).$$

**Remark 26.** If we put  $a = b = k = 1$  in Lemma 5, and Theorem 6, respectively, we obtain

$$(\beta e^t - 1) \mathcal{B}_n^{(v)}(x, \beta) = n\mathcal{B}_{n-1}^{(v-1)}(x, \beta)$$

and

$$\mathcal{B}_n^{(v)}(x+1, \beta) = \frac{n}{\beta} \mathcal{B}_{n-1}^{(v-1)}(x, \beta) + \frac{1}{\beta} \mathcal{B}_n^{(v)}(x, \beta).$$

**Remark 27.** If we set  $b = 1$ ,  $k = 0$ , and  $a = -1$  in Lemma 5 and Theorem 6, respectively, we obtain

$$(\beta e^t + 1) \mathcal{E}_n^{(v)}(x, \beta) = 2\mathcal{E}_n^{(v-1)}(x, \beta)$$

and

$$\mathcal{E}_n^{(v)}(x+1, \beta) = \frac{2}{\beta} \mathcal{E}_n^{(v-1)}(x, \beta) - \frac{1}{\beta} \mathcal{E}_n^{(v)}(x, \beta).$$

**Remark 28.** By substituting  $b = k = 1$ , and  $a = -1$  into Lemma 5 and Theorem 6, respectively, we obtain

$$(\beta e^t + 1) \mathcal{G}_n^{(v)}(x, \beta) = 2n\mathcal{G}_{n-1}^{(v-1)}(x, \beta)$$

and

$$\mathcal{G}_n^{(v)}(x+1, \beta) = \frac{2n}{\beta} \mathcal{G}_{n-1}^{(v-1)}(x, \beta) - \frac{1}{\beta} \mathcal{G}_n^{(v)}(x, \beta).$$

We give action of a linear operator  $\frac{1}{(\beta^b e^t - a^b)}$  on the polynomial  $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$  by the following lemma:

**Lemma 6.** The following identity holds true:

$$\frac{1}{(\beta^b e^t - a^b)} \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \frac{\mathcal{Y}_{n+k,\beta}^{(v+1)}(x; k, a, b)}{2^{1-k} \prod_{j=1}^k (n+j)}.$$

*Proof.* By (1.1) and Lemma 1, we obtain

$$\frac{1}{(\beta^b e^t - a^b)} \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \frac{1}{(\beta^b e^t - a^b)} \left( \frac{2^{1-k} t^k}{\beta^b e^t - a^b} \right)^v x^n.$$

After some calculations in this equation, we get

$$\frac{1}{(\beta^b e^t - a^b)} \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \frac{1}{2^{1-k}} \frac{1}{t^k} \mathcal{Y}_{n,\beta}^{(v+1)}(x; k, a, b).$$

As a consequence, we obtain the desired result by using (3.1).  $\square$

**Remark 29.** *Substituting  $\beta = a = b = k = 1$  into Lemma 6, we obtain*

$$\frac{1}{(e^t - 1)} B_n^{(v)}(x) = \frac{1}{n+1} B_{n+1}^{(v+1)}(x).$$

**Remark 30.** *If we put  $\beta = b = 1$ ,  $k = 0$ , and  $a = -1$  in Lemma 6, we obtain*

$$\frac{1}{(e^t + 1)} E_n^{(v)}(x) = \frac{1}{2} E_n^{(v+1)}(x).$$

**Remark 31.** *Taking  $\beta = b = k = 1$  and  $a = -1$  in Lemma 6, we have*

$$\frac{1}{(e^t + 1)} G_n^{(v)}(x) = \frac{1}{2(n+1)} G_{n+1}^{(v+1)}(x), [11, p. 3, Lemma 3].$$

**Remark 32.** *Putting  $a = b = k = 1$  in Lemma 6, we obtain*

$$\frac{1}{(\beta e^t - 1)} \mathcal{B}_n^{(v)}(x) = \frac{1}{n+1} \mathcal{B}_{n+1}^{(v+1)}(x).$$

**Remark 33.** *Upon substituting  $b = 1$ ,  $k = 0$ , and  $a = -1$  into Lemma 6, we obtain*

$$\frac{1}{(\beta e^t + 1)} \mathcal{E}_n^{(v)}(x) = \frac{1}{2} \mathcal{E}_n^{(v+1)}(x).$$

**Remark 34.** *If we set  $b = k = 1$  and  $a = -1$  in Lemma 6, we obtain*

$$\frac{1}{(\beta e^t + 1)} \mathcal{G}_n^{(v)}(x) = \frac{1}{2(n+1)} \mathcal{G}_{n+1}^{(v+1)}(x).$$

If we use Lemma 5 and Lemma 6, then we arrive at the following corollary:

**Corollary 3.** *The following identity holds true:*

$$\frac{\beta^b e^t}{(\beta^b e^t - a^b)} \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) + \frac{\mathcal{Y}_{n+k,\beta}^{(v+1)}(x; k, a, b)}{2^{1-k} a^{-b} \prod_{j=1}^k (n+j)}. \quad (3.4)$$

**Theorem 7.** *The following recurrence relation holds true:*

$$\mathcal{Y}_{n+1,\beta}^{(v)}(x; k, a, b) = (x - v) \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) + \frac{vk \mathcal{Y}_{n+1,\beta}^{(v)}(x; k, a, b)}{n + 1} - \frac{va^b \mathcal{Y}_{n+k,\beta}^{(v+1)}(x; k, a, b)}{2^{1-k} \prod_{j=1}^k (n + j)}.$$

*Proof.* Setting

$$g(t) = \left( \frac{\beta^b e^t - a^b}{2^{1-k} t^k} \right)^v$$

in (1.9), we obtain

$$\mathcal{Y}_{n+1,\beta}^{(v)}(x; k, a, b) = \left( x - v \frac{\beta^b e^t - kt^{-1} \beta^b e^t + kt^{-1} a^b}{\beta^b e^t - a^b} \right) \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b).$$

After elementary manipulations in this equation, we get

$$\begin{aligned} \mathcal{Y}_{n+1,\beta}^{(v)}(x; k, a, b) &= x \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) - \frac{v \beta^b e^t}{(\beta^b e^t - a^b)} \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) \\ &\quad + vk \frac{1}{t} \mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b). \end{aligned}$$

Combining the above result with (3.1) and (3.4) gives the recurrence relation. □

**Remark 35.** *If we set  $\beta = a = b = k = 1$  in Theorem 7, we obtain*

$$B_{n+1}^{(v)}(x) = (x - v) B_n^{(v)}(x) - \frac{v}{n + 1} B_{n+1}^{(v+1)}(x) + \frac{v}{n + 1} B_{n+1}^{(v)}(x).$$

Nörlund [25, p. 145, Eq-(81)] also proved the following recurrence relations:

$$B_n^{(v+1)}(x) = \left( 1 - \frac{n}{v} \right) B_n^{(v)}(x) + \frac{n}{v} (x - v) B_{n-1}^{(v)}(x).$$

**Remark 36.** *If we put  $\beta = b = 1$ ,  $k = 0$ , and  $a = -1$  in Theorem 7, we have*

$$E_{n+1}^{(v)}(x) = (x - v) E_n^{(v)}(x) + \frac{v}{2} E_n^{(v+1)}(x),$$

which is proved by Nörlund [25, p. 145, Eq-(82)].

**Remark 37.** *By substituting  $\beta = b = k = 1$  and  $a = -1$  into Theorem 7, we have*

$$G_{n+1}^{(v)}(x) = (x - v) G_n^{(v)}(x) + \frac{v}{n + 1} G_{n+1}^{(v)}(x) + \frac{v}{2(n + 1)} G_{n+1}^{(v+1)}(x),$$

which is proved by the authors [11].

**Remark 38.** *If we set  $a = b = k = 1$  in Theorem 7, we obtain recurrence formula of the Apostol-Bernoulli polynomials:*

$$\mathcal{B}_{n+1}^{(v)}(x, \beta) = (x - v) \mathcal{B}_n^{(v)}(x, \beta) - \frac{v}{n + 1} \mathcal{B}_{n+1}^{(v+1)}(x, \beta) + \frac{v}{n + 1} \mathcal{B}_{n+1}^{(v)}(x, \beta).$$

**Remark 39.** Substituting  $b = 1$ ,  $k = 0$ , and  $a = -1$  into Theorem 7, we obtain recurrence formula of the Apostol-Euler polynomials:

$$\mathcal{E}_{n+1}^{(v)}(x, \beta) = (x - v) \mathcal{E}_n^{(v)}(x, \beta) + \frac{v}{2} \mathcal{E}_n^{(v+1)}(x, \beta).$$

**Remark 40.** Putting  $b = k = 1$  and  $a = -1$  in Theorem 7, we obtain recurrence formula of the Apostol-Genocchi polynomials:

$$\mathcal{G}_{n+1}^{(v)}(x, \beta) = (x - v) \mathcal{G}_n^{(v)}(x, \beta) + \frac{v}{n+1} \mathcal{G}_{n+1}^{(v)}(x, \beta) + \frac{v}{2(n+1)} \mathcal{G}_{n+1}^{(v+1)}(x, \beta).$$

Carlitz [6] established a generalization of the Raabe-type multiplication formulas for the Bernoulli and Euler polynomials. Subsequently, Ozden [26] and Ozden *et al.* [30] presented a unification and generalization of the not only Carlitz [6] result but also Karande *et al.* [15] result. Their proof of the unification and generalization of the Raabe-type multiplication formulas is based upon the generating function (1.1) with  $v = 1$ .

Here our demonstration of the multiplication formula in Theorem 8 and 3.5 based upon umbral calculus method.

The multiplication formulas for Bernoulli polynomials also drive from the Hurwitz zeta function. Raabe-type multiplication formulas are use in many branches of Mathematics and Mathematical Physics; for example to construct the Dedekind sums and the Hardy-Berndt sums. One may need to use these formulas to construct unification and generalization of this sums.

**Theorem 8.** Let  $m$  be a positive integer. The following multiplication formula (Raabe-type multiplication formula) for the polynomials  $\mathcal{Y}_{n,\beta}^{(v)}(mx; k, a, b)$  holds true:

$$\begin{aligned} & \mathcal{Y}_{n,\beta}^{(v)}(mx; k, a, b) \\ = & m^{n-kv} a^{bv(m-1)} \sum_{u_1, u_2, \dots, u_{m-1} \geq 0} \binom{v}{u_1, u_2, \dots, u_{m-1}} \left(\frac{\beta}{a}\right)^{bj} \mathcal{Y}_{n,\beta^m}^{(v)}\left(x + \frac{j}{m}, k, a^m, b\right), \end{aligned}$$

where

$$j = u_1 + 2u_2 + \dots + (m-1)u_{m-1}.$$

*Proof.* If we substitute

$$g(t) = \left(\frac{\beta^b e^t - a^b}{2^{1-kt^k}}\right)^v$$

into (1.10), we obtain

$$\mathcal{Y}_{n,\beta}^{(v)}(mx; k, a, b) = m^{n-kv} \frac{1}{a^{bv} \left(\frac{\beta^b}{a^b} e^{\frac{t}{m}} - 1\right)^v} (2^{1-kt^k})^v x^n,$$

From this equation, we obtain

$$\mathcal{Y}_{n,\beta}^{(v)}(mx; k, a, b) = \frac{2^{(1-k)v} m^{n-kv} a^{-bv} t^{kv}}{\left(\left(\frac{\beta}{a}\right)^{bm} e^t - 1\right)^v} \sum_{j=0}^{m-1} \left(\frac{\beta}{a}\right)^{bj} e^{\frac{tj}{m}} x^n.$$

Thus by applying (1.1) and Lemma 1 in the above equation, we deduce

$$\begin{aligned} & \mathcal{Y}_{n,\beta}^{(v)}(mx; k, a, b) \\ &= m^{n-kv} a^{bvm-bv} \sum_{u_1, u_2, \dots, u_{m-1} \geq 0} \binom{v}{u_1, u_2, \dots, u_{m-1}} \left(\frac{\beta}{a}\right)^{bj} e^{\frac{tj}{m}} \left(\frac{2^{1-kt}k}{\beta^{bm}e^t - a^{bm}}\right)^v x^n. \end{aligned}$$

By using (3.2) with Lemma 1 in the above equation, we arrive at the desired result.  $\square$

**Remark 41.** *Theorem 8 is also related to some special multiplication formulas. For example, if we set  $a = b = k = 1$  into assertion of Theorem 8, we arrive at a multiplication formula of the Apostol-Bernoulli and Apostol-Euler polynomials of higher order, which was defined by Luo [19, Theorem 2.1 and Theorem 3.1] and also Bayad and Simsek [1, Corollary 5].*

**Remark 42.** *If we set  $\beta = a = b = k = 1$  in assertion of Theorem 8, we arrive at the multiplication formula of the Bernoulli polynomials of higher order. If we take  $\beta = b = 1, k = 0$  and  $a = -1$  into assertion of Theorem 8, we arrive at the multiplication formula of the Euler polynomials of higher order (see, for details, [19, Corollary 2.1 and Corollary 3.1]).*

If we set  $v = 1$  in Theorem 8, we obtain obtain multiplication formula (Raabe formula) for the

**Corollary 4.** *Let  $m$  be a positive integer. The following multiplication formula holds true:*

$$\mathcal{Y}_{n,\beta}(mx; k, a, b) = m^{n-k} a^{b(m-1)} \sum_{j=0}^{m-1} \left(\frac{\beta}{a}\right)^{bj} \mathcal{Y}_{n,\beta^m}(x + \frac{j}{m}, k, a^m, b). \quad (3.5)$$

**Remark 43.** *If we substitute  $g(t) = \frac{\beta^b e^t - a^b}{2^{1-kt}k}$  into (1.10), we obtain*

$$\mathcal{Y}_{n,\beta}(mx; k, a, b) = \alpha^{n-k} a^{bm-b} \frac{\frac{\beta^b}{a^b} e^t - 1}{\frac{\beta^{bm}}{a^{bm}} e^{\frac{t}{m}} - 1} \mathcal{Y}_{n,\beta^m}(x; k, a^m, b),$$

*By using (3.2) into the above equation and after some calculations, we arrive at proof of (3.5).*

**Remark 44.** *In special case, when  $\beta = a = b = k = v = 1$  into assertion of Theorem 8, we have*

$$\mathcal{Y}_{n,1}(mx; 1, 1, 1) = B_n(mx) = m^{n-1} \sum_{j=0}^{m-1} B_n\left(x + \frac{j}{m}\right),$$

*(see, for details, [4]-[43]; see also the references cited in each of these earlier works).*

**Remark 45.** *In special case, if we take  $\beta = b = v = 1, k = 0$  and  $a = -1$  into assertion of Theorem 8, we obtain multiplication formulas for Euler polynomials:*

If  $m = 2d + 1, d \in \mathbb{N}$ , then

$$E_n((2d + 1)x) = (2d + 1)^n \sum_{j=0}^{2d} (-1)^j E_n\left(x + \frac{j}{2d + 1}\right)$$

(see, for details, [4]-[43]; see also the references cited in each of these earlier works).

If  $m = 2d$ ,  $d \in \mathbb{N}$ , then

$$\begin{aligned} E_n(2dx) &= (2d)^n (-1) \sum_{j=0}^{2d-1} (-1)^j \mathcal{Y}_{n,1}(x; 0, -1, 1) \\ &= -(2d)^n \sum_{j=0}^{2d-1} (-1)^j \frac{2}{t} B_n \left( x + \frac{j}{2d} \right) \end{aligned}$$

Using (3.1) in the right side of the above equation, we derive the following result:

$$E_n(2dx) = -\frac{2(2d)^n}{n+1} \sum_{j=0}^{2d-1} (-1)^j B_{n+1} \left( x + \frac{j}{2d} \right)$$

(see, for details, [4]-[43]; see also the references cited in each of these earlier works).

**Remark 46.** *Multiplication formulas of the Genocchi polynomials and the Apostol type polynomials are the special case of Corollary 4.*

The next theorem gives us the polynomials  $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$  are in terms of the Apostol type polynomials.

**Theorem 9.** *Each of the following relationships holds true:*

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \frac{\prod_{y=0}^{v(k-1)-1} (n-y)}{2^{v(k-1)} a^{vb}} \mathcal{B}_{n-v(k-1)}^{(v)} \left( x, \left( \frac{\beta}{a} \right)^b \right), \quad (3.6)$$

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = -\frac{\prod_{y=0}^{kv-1} (n-y)}{2^{kv} a^{bv}} \mathcal{E}_{n-kv}^{(v)} \left( x, -\left( \frac{\beta}{a} \right)^b \right), \quad (3.7)$$

and

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = -\frac{\prod_{y=0}^{v(k-1)-1} (n-y)}{2^{kv} a^{bv}} \mathcal{G}_{n-v(k-1)}^{(v)} \left( x, -\left( \frac{\beta}{a} \right)^b \right). \quad (3.8)$$

*Proof of (3.6).*

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \frac{2^{v(1-k)}}{a^{vb}} t^{v(k-1)} \mathcal{B}_n^{(v)} \left( x, \left( \frac{\beta}{a} \right)^b \right),$$

where

$$\mathcal{B}_n^{(v)} \left( x, \left( \frac{\beta}{a} \right)^b \right) = \left( \frac{t}{\left( \frac{\beta}{a} \right)^b e^t - 1} \right)^v x^n.$$

Applying the derivative operator in (1.8)  $v(k-1)$  times in this equation, we complete the proof of the (3.6).  $\square$



**Remark 47.** We note that proofs of (3.7) and (3.8) are similar to that of (3.6). Thus we omit them.

We now ready to give a relation between the Stirling numbers of the first kind and the polynomials  $\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b)$ .

**Theorem 10.** *The following relation holds true:*

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \sum_{j=0}^v \sum_{l=0}^j \sum_{h=0}^{kl} (-1)^{j-l} \binom{v}{j} \binom{j}{l} \frac{s(kl, h) \mathcal{Y}_{n+k(v-l),\beta}^{(v)}(x, k, a, b)}{2^{(k-1)(l+v)} n^{-h} \prod_{j=1}^{kv} (n - kl + j)},$$

where  $s(kl, h)$  is the Stirling numbers of the first kind.

*Proof.* By using Lemma 1, we find

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \sum_{j=0}^v \binom{v}{j} \frac{1}{(\beta^b e^t - a^b)^v} \sum_{l=0}^j \binom{j}{l} (-1)^{j-l} (2^{1-k} t^k)^l x^n.$$

By using derivative operator (1.8) into the above equation, we obtain

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \sum_{j=0}^v \sum_{l=0}^j \binom{v}{j} \binom{j}{l} \frac{1}{(\beta^b e^t - a^b)^v} 2^{l-kl} (-1)^{j-l} (n)_{kl} x^{n-kl}. \quad (3.9)$$

Also by using Lemma 1 with

$$(y)_k = \sum_{m=0}^k s(k, m) y^m \quad [32, p. 57]$$

where  $s(k, m)$  denotes the Stirling numbers of first kind, in the above equation, we arrive at the desired result.  $\square$

We set

$$\frac{1}{(\beta^b e^t - a^b)^v} = -\frac{1}{a^{vb}} \sum_{y=0}^{\infty} \binom{v+y-1}{y} \left( e^{\frac{t}{b}} \frac{\beta}{a} \right)^{yb}. \quad (3.10)$$

Observe that further specializing to  $v = 1$  the right hand side of equation (3.10) yields the geometric series expansion.

By substituting (3.10) into (3.9), and after some algebraic manipulations, we arrive at the following corollary:

**Corollary 5.** *The following relation holds true:*

$$\mathcal{Y}_{n,\beta}^{(v)}(x; k, a, b) = \sum_{y=0}^{\infty} \sum_{j=0}^v \sum_{l=0}^j \sum_{h=0}^{kl} (-1)^{j-l} \binom{v+y-1}{y} \binom{v}{j, l, j-l} \frac{s(kl, h) (x+y)^{h-kl}}{2^{l(k-1)} a^{b(v+y)} \beta^{-yb} n^{-h}},$$

where  $s(kl, h)$  is the Stirling numbers of the first kind.

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