POSTULATIONAL BASES FOR THE UMBRAL CALCULUS.*

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As the somewhat condensed treatment of the umbral calculus which I gave elsewhere [1] has been misunderstood [2] a fuller treatment than was given before is desirable. Incidentally, what follows validates the purely formal uses of this calculus, or of its special cases, which have appeared in the literature, when such uses give correct results. There are immediate generalizations to abstract commutative rings, obtainable by obvious modifications of the following; but as such generalizations seem to be of no use at present, it seems hardly worth while to develop them.

1. Rational operations on umbrae.

(1.1) Real, or complex, numbers are called *scalars*. The sign $\equiv$ denotes either definitional identity or identity as in algebra; which, will be clear from the context.

(1.2) Scalars are denoted by small Latin letters *with non-negative integer suffixes*, thus x_N $(N=0,1,\cdots)$, or by small Greek letters, $\alpha, \beta, \cdots$. As usual, the sum, product of any scalars α, β are $\alpha+\beta$, $\alpha\beta$, and $0, 1$ have their usual meanings.

(1.3) Latin capitals, $A, \cdots, N, \cdots$ denote non-negative integers.

(1.4) If x_N $(N=0,1,\cdots)$ are any scalars, the one-rowed matrix $(x_0, x_1, \cdots, x_N, \cdots)$ is denoted by x: $x \equiv (x_0, x_1, \cdots, x_N, \cdots)$.

(1.5) The $(N+1)$-th element, $N=0,1,\cdots$, of x in (1.4) is denoted by x^N:

$$x^N \equiv x_N \qquad (N=0,1,\cdots).$$

(1.6) The x in (1.4) is called an *umbra*; x is the umbra of $(x_0, x_1, \cdots, x_N, \cdots)$, or of the sequence x_N $(N=0,1,\cdots)$. Note that an umbra has neither exponent nor suffix.

(1.7) *Equality* of umbrae is matric equality: if x is as in (1.4), and

* Received April 8, 1940.

[1] "Algebraic arithmetic," *American Mathematical Society Publications*, vol. 7 (1927), pp. 146-159.

[2] G. Temple, *Journal of the London Mathematical Society*, vol. 12 (1937), p. 114. Professor Temple has seen the present note, and writes (Feb. 21, 1938) that it clears up the obscurity.

$y \equiv (y_0, y_1, \cdots, y_N, \cdots)$, '$x$ is equal to y,' written $x \doteq y$, if, and only if, $x_N = y_N$ $(N = 0, 1, \cdots)$. Hence

(1.71) $x \doteq x$.

(1.72) If $x \doteq y$, then $y \doteq x$.

(1.73) If $x \doteq y$, and $y \doteq z$, then $x \doteq z$.

(1.8) The coefficient of $x_1^{s_1} \cdots x_T^{s_T}$ in the expansion of $(x_1 + \cdots + x_T)^N$ by the multinomial theorem, is denoted by $M_{s_1, \ldots, s_T}$. Note that exponents and suffixes 0, 1 are to be indicated precisely in the same way as exponents and suffixes > 1.

The next refer to rational functions of umbrae, and define 'umbral scalar multiplication,' 'umbral addition,' etc. The qualification 'umbral' will be dropped, as it is taken care of in the notation.

(1.9) The *scalar product*, αx, of α and $x \equiv (x_0, \cdots, x_N, \cdots)$ is

$$\alpha x \equiv \alpha(x_0, \cdots, x_N, \cdots) \equiv (\alpha x_0, \cdots, \alpha x_N, \cdots).$$

By definition, $x\alpha \doteq \alpha x$.

Now αx is an umbra, by (1.6), and it is a compound symbol. To denote the $(N+1)$-th element of αx in accordance with (1.5), we write $\{\alpha x\}^N$; thus

(1.91) $$\{\alpha x\}^N \equiv \alpha x^N \equiv \alpha x_N.$$

Similarly, if * is any compound symbol of scalars and umbrae, and if * is an umbra, the $(N+1)$-th element of * is denoted by $\{*\}^N$.

(1.10) The *sum*, s, $s \equiv \alpha a \dot{+} \cdots \dot{+} \xi x$, of $\alpha a, \cdots, \xi x$, where

$$a \equiv (a_0, \cdots, a_N, \cdots), \cdots, \; x \equiv (x_0, \cdots, x_N, \cdots),$$

is

$$s \equiv (\alpha a_0 + \cdots + \xi x_0, \cdots, \; \alpha a_N + \cdots + \xi x_N, \cdots).$$

Hence

(1.101) $$\{\alpha a \dot{+} \cdots \dot{+} \xi x\}^N = \alpha a_N + \cdots + \xi x_N;$$

(1.102) Addition, $\dot{+}$, of umbrae is commutative and associative;

(1.103) There is a unique z, the *zero umbra*, such that $x \dot{+} z \doteq x$ for every x:

$$z \equiv (0, \cdots, 0, \cdots);$$

(1.104) For every x there is a unique y such that $x \dot{+} y \doteq z$; y is called the *negative* of x; $y \equiv (-1)x$, and is denoted by $\dot{-}x$;

(1.105) With respect to $\dot{+}$ the set of all umbrae is an abelian group; the inverse of x in the group is $\dot{-}x$, and the identity of the group is z.

(1.11) If no two of $a, \cdots, x$ are equal as defined in (1.7), $a, \cdots, x$ are said to be *distinct*. In (1.12)–(1.125), $a, \cdots, x$ are distinct.

(1.12) If $a, \cdots, x$ are T distinct umbrae, $(\alpha a \dotplus \cdots \dotplus \xi x)^N$ denotes the scalar p_N,

(1.120) $p_N \equiv (\alpha a \dotplus \cdots \dotplus \xi x)^N \equiv \Sigma M_{S_1, \ldots, S_T} \alpha^{S_1} \cdots \xi^{S_T} a_{S_1} \cdots x_{S_T}$,
(see (1.8)). In particular,

(1.121) $$p_0 = a_0 \cdots x_0.$$

Hence, by (1.5),

(1.122) $(\alpha a \dotplus \cdots \dotplus \xi x)^N = \Sigma M_{S_1, \ldots, S_T} \alpha^{S_1} \cdots \xi^{S_T} a^{S_1} \cdots x^{S_T}$

the left of which is called the N-th *power* of the sum $\alpha a \dotplus \cdots \dotplus \xi x$. Hence such powers are expanded by the multinomial theorem, and $\dotplus$ is replaced by $+$ in the result.

If p_N is as above defined, and $p \equiv (p_0, \cdots, p_N, \cdots)$, then $p^N = p_N$, by (1.5). Note the distinction, as shown in (1.101), (1.122), between

$$\{\alpha a \dotplus \cdots \dotplus \xi x\}^N, \qquad (\alpha a \dotplus \cdots \dotplus \xi x)^N,$$

only the second of which is a power; both are scalars.

By (1.121),

(1.123) $$(\alpha a \dotplus \cdots \dotplus \xi x)^0 = a_0 \cdots x_0.$$

In (1.122) replace N by $N + R$. The resulting scalar,

$$(\alpha a \dotplus \cdots \dotplus \xi x)^{N+R},$$

is called the *product*,

$$(\alpha a \dotplus \cdots \dotplus \xi x)^N \cdot (\alpha a \dotplus \cdots \dotplus \xi x)^R,$$

of

$$(\alpha a \dotplus \cdots \dotplus \xi x)^N, \qquad (\alpha a \dotplus \cdots \dotplus \xi x)^R:$$

(1.124) $(\alpha a \dotplus \cdots \dotplus \xi x)^N \cdot (\alpha x \dotplus \cdots \dotplus \xi x)^R \equiv (\alpha a \dotplus \cdots \dotplus \xi x)^{N+R}$.

It follows that this multiplication, $\cdot$, is commutative and associative, and that it has the 'identity' $(\alpha a \dotplus \cdots \dotplus \xi x)^0$. The right of (1.124) may be (and is) calculated from the left by expanding each of the factors $(\ )^N$, $(\ )^R$ by the multinomial theorem, multiplying the resulting (scalar) polynomials together as in common algebra and finally degrading all exponents of small Latin letters to suffixes. For example, noting that $\alpha^0 = \beta^0 = 1$, and $\alpha^1 = \alpha$, $\beta^1 = \beta$, since α, β are scalars, we have

$$\begin{aligned} &(\alpha a \dotplus \beta b)^1 \cdot (\alpha a \dotplus \beta b)^2 \\ &= (\alpha a^1 b^0 + \beta a^0 b^1) \cdot (\alpha^2 a^2 b^0 + 2\alpha\beta a^1 b^1 + \beta^2 a^0 b^2) \end{aligned}$$

$$
\begin{aligned}
&= \alpha^3 a^3 b^0 + 3\alpha^2\beta a^2 b^1 + 3\alpha\beta^2 a^1 b^2 + \beta^3 a^0 b^3,\\
&= \alpha^3 a_3 b_0 + 3\alpha^2\beta a_2 b_1 + 3\alpha\beta^2 a_1 b_2 + \beta^3 a_0 b_3,\\
&= (\alpha a \dot{+} \beta b)^3.
\end{aligned}
$$

As a mere convenience of notation we write

(1. 125) $$(\xi x)^N \cdot [(\alpha a)^M + (\beta b)^R + \cdots + (\gamma c)^S] \equiv (\xi x)^N \cdot (\alpha a)^M + (\xi x)^N \cdot (\beta b)^R + \cdots + (\xi x)^N \cdot (\gamma c)^S,$$

the (scalar) sum of scalars on the right defining the expression on the left. Similarly for an infinity of scalar summands.

All in this section (1. 12) refers only to the case in which the T umbrae $a, \cdots, x$ are distinct. The contrary case is equally important in applications of the calculus, and requires special consideration.

(1. 13) If in $\alpha x \dot{+} \cdots \dot{+} \xi x$ there are precisely T summands $\alpha x, \cdots, \xi x$, each of which is a scalar product of a scalar and x, we replace $(\rightarrow)$ the T x's by T distinct umbrae, say $a, \cdots, x$, in any order, and indicate this replacement by writing

(1. 131) $$\alpha x \dot{+} \cdots \dot{+} \xi x \rightarrow \alpha a \dot{+} \cdots \dot{+} \xi x.$$

Then $(\alpha a \dot{+} \cdots \dot{+} \xi x)^N$ is to be calculated by (1. 122), and the exponents are degraded, as in (1. 120). In the result, each of $a, \cdots, x$ is replaced $(\leftarrow)$ by x; the resulting polynomial is defined to be N-th *power* $(\alpha x \dot{+} \cdots \dot{+} \xi x)^N$ of the sum $\alpha x \dot{+} \cdots \dot{+} \xi x$.

For example,

$$
\begin{aligned}
&(\alpha x + \beta x)^3 \rightarrow (\alpha a + \beta x)^3;\\
&(\alpha a + \beta x)^3 = \alpha^3 a_3 x_0 + 3\alpha^2\beta a_2 x_1 + 3\alpha\beta^2 a_1 x_2 + \beta^3 a_0 x_3,\\
&\qquad\qquad \leftarrow \alpha^3 x_3 x_0 + 3\alpha^2\beta x_2 x_1 + 3\alpha\beta^2 x_1 x_2 + \beta^3 x_0 x_3;\\
&(\alpha x + \beta x)^3 = (\alpha^3 + \beta^3) x_0 x_3 + 3\alpha\beta(\alpha + \beta) x_1 x_2.
\end{aligned}
$$

The relation (1. 124) holds also for powers $(\alpha x \dot{+} \cdots \dot{+} \xi x)^N$ when therein the replacements $\rightleftarrows$ are made.

Similarly, if in a $(\dot{+})$ sum s there are precisely A summands each of which is a scalar product of a scalar and $x, \cdots$, precisely C summands each of which is a scalar product of a scalar and w, and if these summands exhaust s, the $S \equiv A + \cdots + C$ x's, $\cdots$, w's, are replaced $(\rightarrow)$ by S distinct umbrae, say $s \rightarrow t$. Then $(t)^N$ is calculated by (1. 122), (1. 120), and the final replacement $(\leftarrow)$ of the S distinct umbrae by those introduced by $(\rightarrow)$. These powers $(s)^N$ also satisfy (1. 124).

(1. 132) Hence (1. 124) holds for any umbrae $a, \cdots, x$, distinct or not.

(1. 14) x^N was defined in (1. 5); it denotes the scalar x_N. Hence, since

multiplication of scalars is indicated (as always) by mere juxtaposition, without any symbol denoting the operation of multiplication,

$$(1.141) \qquad x^N x^R = x_N x_R.$$

Since this multiplication is multiplication of scalars, it is commutative and associative.

In $(\alpha a \dotplus \cdots \dotplus \xi x)^N$, defined in (1.120), take $\xi = 1$ and each of the other scalars $= 0$. Then by (1.9), (1.103), $(0a \dotplus \cdots \dotplus 1x)^N = (x)^N$. Note that ( ) is not omitted on the right. By (1.120), $(x)^N = x^N$. Hence, by (1.5), $(x)^N = x_N$. By (1.124), $(x)^N \cdot (x)^R = (x)^{N+R}$, and hence, by what has just been shown, $x^N \cdot x^R = x^{N+R} = x_{N+R}$,

$$(1.142) \qquad x_N \cdot x_R = x_{N+R}.$$

Thus, unless $x_N x_R = x_{N+R}$, $x_N x_R \neq x_N \cdot x_R$. The 'dot multiplication,' $\cdot$, is an operation peculiar to the calculus, and will be explicitly indicated where there is any possibility of confusion.

Similarly, $(\alpha a \dotplus \cdots \dotplus \xi x)^N (\alpha x \dotplus \cdots \dotplus \xi x)^R$, without the dot, is the (scalar) product of the scalars $(\alpha a \dotplus \cdots \dotplus \xi x)^N$, $(\alpha x \dotplus \cdots \dotplus \xi x)^R$, which are defined in (1.222); and this scalar product is different from the dot product in (1.124). To see the difference in an example, we compare the example illustrating (1.124) with the following:

$$\begin{aligned}
&(\alpha a \dotplus \beta b)^1 (\alpha a \dotplus \beta b)^2, \\
= &(\alpha a_1 b_0 + \beta a_0 b_1)(\alpha^2 a_2 b_0 + 2\alpha\beta a_1 b_1 + \beta^2 a_0 b_2), \\
= &\alpha^3 a_1 a_2 b_0{}^2 + \alpha^2 \beta b_0 b_1 (2a_1{}^2 + a_0 a_2) + \alpha\beta^2 a_0 a_1 (2b_1{}^2 + b_0 b_2) + \beta^3 a_0{}^2 b_1 b_2, \\
&\neq (\alpha a \dotplus \beta b)^1 \cdot (\alpha a \dotplus \beta b)^2.
\end{aligned}$$

(1.15) A particular case of (1.120) occurs so frequently that a special notation is convenient. If $s \equiv \alpha x \dotplus \cdots \dotplus \alpha x$ is a sum of precisely A scalar products αx, we write

$$(1.151) \qquad A \cdot \alpha x \equiv s \equiv \alpha x \dotplus \cdots \dotplus \alpha x.$$

There can be no confusion between the dot in $A \cdot \alpha x$ and that in (1.124), since here the dot is between a scalar and an umbra, while in (1.124) it is between two scalars. If desired, the dot in (1.151) may be circled, thus $\odot$. It would be incorrect to write $A\alpha x$ instead of $A \cdot \alpha x$, since $A\alpha$ is a scalar, and hence, by (1.9), $A\alpha x$ is a scalar product.

(1.16) *Umbral* multiplication can be defined in many (actually, an infinity of) ways to yield algebras simply isomorphic with parts of the common algebra of scalars, for example rings. Here we need mention only that species of umbral multiplication which is directly applicable to the power series in § 2. It will not be used in the sequel.

(1.161) $$x \equiv (x_0/0!, x_1/1!, \cdots, x_N/N!, \cdots)$$

is said to be of *e-type* ($e \equiv$ 'exponential'). Hence, if y is of e-type, $y^N = y_N/N!$. If w is not of e-type, it is replaced by $\bar{w}$, in which $w^N \equiv \bar{w}^N/N!$, until after all calculations involving $\bar{w}$ have been completed, when $\bar{w}_N$ is replaced by $N!\,w_N$.

Let $x \equiv (x_0/0!, \cdots, x_N/N!, \cdots)$, $y \equiv (y_0/0!, \cdots, y_N/N!, \cdots)$ be of e-type. The *product, xy,* of x, y (in this order) is the matrix p which is such that

(1.162) $$p^N \equiv \frac{(x \dotplus y)^N}{N!};$$

(1.163) $$xy \equiv \left(\frac{(x \dotplus y)^0}{0!}, \quad \frac{(x+y)^1}{1!}, \cdots, \quad \frac{(x \dotplus y)^N}{N!}, \cdots\right).$$

Hence umbral multiplication is commutative and associative. Thus powers may be defined as usual; the A-th power of x is denoted by $x^{(A)}$, to distinguish it from x^A.

2. Power series. The set of all (formal) power series in the variable θ is closed under the four rational operations. Division is immediately referred to multiplication, and need not be separately discussed. Irrational functions of these power series also occur, but as they are of less interest than the rational functions, and are readily investigated if desired, they will not be considered here. The use of formal (disregard of convergence) power series can be justified in detail, if not obviously legitimate in the present connection (for example, as in my paper, *Transactions of the American Mathematical Society,* vol. 25, 1923, 135-54); however, there is sufficient generality in the set of all power series in θ convergent in the same domain $|\theta| > 0$ to show here how the definitions, etc., in § 1 give immediately the algorithms of Blissard's umbral calculus.

If $x \equiv (x_0, \cdots, x_N, \cdots)$ we write

(2.1) $$e^{x\theta} \equiv \sum_{N=0}^{\infty} x_N(\theta^N/N!),$$

where e has its usual meaning $(2.7\cdots)$. Thus, by (1.5),

(2.11) $$\xi e^{x\theta} = \xi \sum_{N=0}^{\infty} x^N(\theta^N/N!).$$

By either of these, $\xi e^{x\theta}$ is a scalar. Hence if $\Lambda(\xi, \cdots, \eta)$ is a polynomial in $\xi, \cdots, \eta$ with scalar coefficients, $\Lambda \equiv \Lambda(\xi e^{x\theta}, \cdots, \eta e^{y\theta})$ is a scalar, as is also the N-th derivative $\partial_\theta{}^N\Lambda$ of Λ with respect to θ. By writing Λ as a MacLaurin series in θ, we express it in the form $\tau e^{w\theta}$, and similarly for the derivative. For any Λ (or its derivative) the appropriate $\tau e^{w\theta}$ is built up by repeated applications of the elementary identities (2. 2)–(2. 4) in θ.

$$e^{x\theta}e^{y\theta} = e^{(x\dot{+}y)\theta} \equiv \sum_0^\infty (x \dot{+} y)^N(\theta^N/N!), \tag{2. 2}$$

which, by (1. 120), is merely the formal multiplication of two MacLaurin series to produce a third. Generally, for any number of factors on the left,

$$e^{\xi x\theta}\cdot\cdot\cdot e^{y\eta\theta} \equiv e^{(\xi x\dot{+}\cdots\dot{+}\eta y)\theta} \sum_0^\infty (\xi x \dot{+} \cdot\cdot\cdot \dot{+} \eta y)^N(\theta^N/N!). \tag{2. 21}$$

For addition, (1. 101) gives

$$e^{x\theta} + e^{y\theta} \equiv e\{^{x+y}\}^{\theta} \equiv \sum_0^\infty \{x + y\}^N(\theta^N/N!), \tag{2. 2}$$

with the obvious extension to any number of summands.

Powers are obtained directly from (2. 21), or more conveniently thence by (1. 151):

$$[e^{\xi x\theta}]^A \equiv e^{(A\cdot\xi x)\theta} \equiv \sum_0^\infty (A\cdot\xi x)^N(\theta^N/N!). \tag{2. 3}$$

For derivation, we have

$$\begin{aligned}
\partial_\theta{}^N e^{\xi x\theta} &\equiv \partial_\theta{}^N \sum_{M=0}^\infty \xi^M x_M(\theta^M/M!),\\
&= \sum_{M=0}^\infty \xi^{N+M}x_{N+M}(\theta^M/M!),\\
&\equiv \sum_{M=0}^\infty \xi^{N+M}x^{N+M}(\theta^M/M!) \qquad [\text{by } (1.\ 5)],\\
&\equiv \sum_{M=0}^\infty (\xi x)^N\cdot(\xi x)^M(\theta^M/M!) \quad [\text{by } (1.\ 124)],\\
&\equiv (\xi x)^N\cdot\sum_{M=0}^\infty (\xi x^M)(\theta^M/M!) \quad [\text{by } (1.\ 125)],\\
&\equiv (\xi x)^N\cdot e^{\xi x\theta}:
\end{aligned}$$

$$\partial_\theta{}^N e^{\xi x\theta} \equiv (\xi x)^N\cdot e^{\xi x\theta}, \tag{2. 4}$$

in complete formal analogy with derivatives of ordinary (scalar) exponential functions. From (2. 3), (2. 4),

$$\partial_\theta{}^N[e^{\xi x\theta}]^A \equiv (A\cdot\xi x)^N e^{(A\cdot\xi x)\theta}; \tag{2. 5}$$

and from (1. 101), (1. 120),

$$e^{\xi x\theta}[e^{\alpha a\theta} + \cdot\cdot\cdot + e^{\gamma c\theta}] \equiv e^{(\xi x\dot{+}\{\alpha a\dot{+}\cdots\dot{+}\gamma c\})\theta}. \tag{2. 6}$$

The coefficient of $\theta^N/N!$ in the MacLaurin expansion of the left of (2. 6) is in fact

$$\sum_{S=0}^{N} \binom{N}{S} \xi^{N-S} x_{N-S}[\alpha^S a_S + \cdot \cdot \cdot + \gamma^S c_S],$$
$$= \sum_{S=0}^{N} \binom{N}{S} (\xi x)^{N-S} \{\alpha a \dotplus \cdot \cdot \cdot \dotplus \gamma c\}^S,$$
$$= (\xi x \dotplus \{\alpha a \dotplus \cdot \cdot \cdot \dotplus \gamma c\})^N,$$

which is the coefficient of $\theta^N/N!$ on the right of (2.6).

Many of the more interesting applications to special sequences of numbers (like the Bernoulli or Euler numbers), arise in the following simple way. Let

$$\frac{\Lambda(\theta, \alpha, \cdot \cdot \cdot, \gamma)}{\Phi(\theta, \alpha, \cdot \cdot \cdot, \gamma)}$$

be a rational function of $\theta, \alpha, \cdot \cdot \cdot, \gamma$ in its lowest terms. Replace $\alpha, \cdot \cdot \cdot, \gamma$ by $\alpha e^{a\theta}, \cdot \cdot \cdot, \gamma e^{\gamma\theta}$, and let the MacLaurin expansion of the result be

$$\frac{\Lambda(\theta, \alpha e^{a\theta}, \cdot \cdot \cdot, \gamma e^{\gamma\theta})}{\Phi(\theta, \alpha e^{a\theta}, \cdot \cdot \cdot, \gamma e^{\gamma\theta})} \equiv \xi e^{x\theta},$$

thus defining the numbers x_N $(N = 0, 1, \cdot \cdot \cdot)$. Let the MacLaurin expansions of Λ, Φ be

$$\Lambda(\theta, \alpha e^{a\theta}, \cdot \cdot \cdot, \gamma e^{\gamma\theta}) \equiv \eta e^{y\theta}, \quad \Phi(\theta, \alpha e^{a\theta}, \cdot \cdot \cdot, \gamma e^{\gamma\theta}) \equiv \zeta e^{u\theta},$$

thus defining y_N, u_N. Hence

$$\eta e^{y\theta} = \xi\zeta e^{(x \dotplus u)\theta},$$
$$\eta y^N = \xi\zeta(x \dotplus u)^N.$$

Hence, if $F(\theta)$ is a polynomial in θ, or a power series, if convergent,

$$F(\theta \dotplus y) = \xi\zeta F(\theta \dotplus x \dotplus u),$$

in which, after expansion, exponents of y, x, u are degraded to suffixes.

In practice, the special notations $\{\ \}$, $\dotplus$, $(\ ) \cdot (\ )$, $A \cdot \alpha x$ are dropped, $+$, $(\ )(\ )$, $A\alpha x$ being written, as the notation is a sufficient guide to the correct use of the algorithms. There are many extensions, in particular one to multiple suffixes, as in $x_{A,B,\ldots,C}$, and the corresponding power series,

$$\sum_{A, B, \ldots, C} x_{A,B,\ldots,C}\, \alpha^A \beta^B \cdot \cdot \cdot \gamma^C.$$

Finally, everything down to (2.6) goes through unchanged if scalars in (1.1) are re-defined to be elements of any commutative ring with a modulus ($\equiv$ identity with respect to multiplication).

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