Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **7** (2013), 1–10. doi:10.2298/AADM130204003B

ON GESSEL-KANEKO'S IDENTITY FOR BERNOULLI NUMBERS

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The present work deals with Bernoulli numbers. Using Zeilberger's algorithm, we generalize an identity on Bernoulli numbers of Gessel-Kaneko's type. Appendix written by Ira M. Gessel offers a closely related formula via umbral calculus.

1. INTRODUCTION

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Bernoulli numbers B_n are rational numbers defined by

$$B_n = -\frac{1}{n+1}\sum_{j=0}^{n-1}\binom{n+1}{j}B_j,\ n \in \mathbb{Z}^+,$$

where $B_0 = 1$ and $B_{2n+1} = 0$ for $n \in \mathbb{Z}^+$. These numbers $1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, \ldots$ arise naturally in number theory (in connection with values of the Riemann Zeta function), combinatorics and special functions. It is well-known that the B_n have exponential generating function

$$\frac{z}{e^z - 1} = \sum_{n=0}^{+\infty} B_n \frac{z^n}{n!},\ \ |z| < 2\pi. \tag{1}$$

By the Cauchy integral formula we have the contour integral definition of B_n

$$B_n = \frac{n!}{2\pi i}\oint \frac{z}{e^z - 1}\frac{\mathrm{d}z}{z^{n+1}}, \tag{2}$$

2010 Mathematics Subject Classification. 11B68, 33F10, 05A19.
Keywords and Phrases. Bernoulli numbers, Kaneko identity, Zeilberger algorithm.

where the integration path is a simple closed curve surrounding the origin in the positive sense. In 1995, Kaneko [**5**] established and gave two proofs of the following identity

$$\sum_{k=0}^{n+1} \binom{n+1}{k} \widetilde{B}_{n+k} = 0, \tag{3}$$

where $\widetilde{B}_n = (n+1)\, B_n$. Srivastava and Miller in [**8**] presented an elementary proof using the known basic identities on Bernoulli numbers.

In 2003, using umbral calculus, Gessel [**3**] gives the following generalization

$$\frac{1}{n+1} \sum_{k=0}^{n+1} m^{n+1-k} \binom{n+1}{k} \widetilde{B}_{n+k} = \sum_{k=1}^{m-1} \big((2n+1)k - (n+1)m\big) k^n (k-m)^{n-1}. \tag{4}$$

The above identity reduces to (3) by setting $m = 1$. In 2007, Chen [**1**] (also see [**4**]), extended (3), for odd $r \in \mathbb{Z}^+$, as follows:

$$\sum_{k=0}^{n+r} \binom{n+r}{k} \binom{n+k+r}{r} B_{n+k} = 0. \tag{5}$$

For $r = 1$, we get Kaneko's identity. In 2009, Chen and Sun [**2**], obtained two identities by applying Zeilberger's algorithm

$$\sum_{k=0}^{n+3} \binom{n+3}{k} (n+k+3)(n+k+2) \widetilde{B}_{n+k} = 0, \tag{6}$$

$$\begin{aligned} \frac{1}{n+3} \sum_{k=0}^{n+3} m^{n+3-k} \binom{n+3}{k} (n+k+3)(n+k+2) \widetilde{B}_{n+k} \\ = \sum_{k=1}^{m-1} P(n,m,k) k^n (k-m)^{n-1}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} P(n,m,k) = {} & 2(n+2)(2n+3)(2n+5)k^3 - 2m(n+2)(2n+5)(3n+5)k^2 \\ & + 3m^2(n+2)(2n^2+7n+7)k - m^3(n+1)^2(n+2). \end{aligned} \tag{8}$$

Note that (5) reduces to (6) by setting $r = 3$ and (7) reduces to (6) by setting $m = 1$. Motivated by the results (5) and (7), we shall give a generalization of both (5) and (7).

2. MAIN RESULT

Our results are summarized in the following theorem.

Theorem 1. *Let $m, n \in \mathbb{N}$.* (i) *For any odd integer r, we have*

$$\frac{r!}{n+r}\sum_{k=0}^{n+r} m^{n+r-k}\binom{n+r}{k}\binom{n+k+r}{r}B_{n+k} = \sum_{k=1}^{m-1} P_r^{(m,k)}(n)k^n(k-m)^{n-1}, \tag{9}$$

where $P_r^{(m,k)}(n)$ is a sequence of polynomials of degree $2r$ in n, m and k, homogenous on m and k of degree r, satisfying the following recurrence relation

$$\begin{aligned} &P_1^{(m,k)}(n) = (2n+1)k - (n+1)m, \\ &P_{r+2}^{(m,k)}(n) = (n+r+1)(n+r)m^2 P_r^{(m,k)}(n) \\ &\qquad\qquad + 2k(k-m)(n+r+1)(2n+2r+3)P_r^{(m,k)}(n+1). \end{aligned} \tag{10}$$

(ii) *The polynomials $P_r^{(m,k)}(n)$, $r = 1, 3, \ldots,$ can be expressed as*

$$P_r^{(m,k)}(n) = \sum_{i=0}^{r}(-1)^i Q_{i,r}(n)m^i k^{r-i}, \tag{11}$$

where $Q_{i,r}(n)$ satisfy the recurrence relation

$$\begin{aligned} Q_{i,r+2}(n) &= (n+r+1)(n+r)Q_{i-2,r}(n) \\ &\quad + (2n+2r+3)(2n+2r+2)(Q_{i-1,r}(n+1) + Q_{i,r}(n+1)). \end{aligned} \tag{12}$$

(iii) *In the explicit form, the polynomials $Q_{i,r}(n)$, are given by*

$$\begin{aligned} Q_{i,r}(n) = (2n+2r-1)\sum_{j=0}^{\lfloor i/2\rfloor}\Bigg[&2^{\theta_j}\left(\binom{\theta_j}{i-2j} + \frac{n+\theta_j+1}{2n+2\theta_j+1}\binom{\theta_j}{i-(2j+1)}\right) \\ &\Delta_{j,r}(n)\prod_{s=\theta_j}^{r-2}(n+s+1)(2n+2s+1)\Bigg], \end{aligned} \tag{13}$$

where $\theta_j = \left\lfloor \frac{r}{2} \right\rfloor - j$, $\Delta_{0,r}(n) = 1$ and for $j \in \mathbb{Z}^+$

$$\Delta_{j,r}(n) = \frac{\binom{\lfloor r/2 \rfloor}{j}}{\prod_{s=0}^{j-1}(2n-2j+2s+r+2)(2n-2j+2s+2r+1)}.$$

Remark 1. It is easy to see that (4), (5) and (7) are cases of (9) for $r = 1$, $m = 1$ and $r = 3$, respectively.

Proof. (i) To prove formula (9), we proceed by induction on r over the sequence of odd natural numbers. The result clearly holds for $r = 1$ (see [**3**]). Suppose that (9) is true up to the step r; we need to prove that it holds for $r + 2$ in place of r.

Denote the left-hand side of (9) by $L_r^{(m)}(n)$ and the right-hand side by $R_r^{(m)}(n)$. By the contour integral formula (2), we have

$$L_r^{(m)}(n) = \frac{1}{2\pi i}\oint \frac{1}{e^z-1}\sum_{k=0}^{n+r} C_r^{(m,k)}(n)\mathrm{d}z, \tag{14}$$

where

$$C_r^{(m,k)}(n) = m^{n+r-k}\frac{r!\binom{n+r}{k}\binom{n+k+r}{r}(n+k)!}{(n+r)z^{n+k}}. \tag{15}$$

and let

$$S_r^{(m)}(n) = \sum_{k=0}^{n+r} C_r^{(m,k)}(n). \tag{16}$$

By Zeilberger's Maple package `EKHAD`, we construct the function

$$G(n,k) = -\frac{2m^2k(n+r)(n+r+1)(2n+2r+3)}{(n+r+1-k)(n+r+2-k)}C_r^{(m,k)}(n),$$

such that

$$\begin{aligned} G(n,k+1) - G(n,k) &= z^2 C_r^{(m,k)}(n+2) \\ -2(r+n+1)(2r+2n+3)C_r^{(m,k)}(n+1) &- (r+n+1)(r+n)m^2 C_r^{(m,k)}(n). \end{aligned} \tag{17}$$

By summing the telescoping equation (17) over k, we obtain the following recurrence relation

$$z^2 S_r^{(m)}(n+2) = (r+n+1)\big(2(2r+2n+3)S_r^{(m)}(n+1) + (r+n)m^2 S_r^{(m)}(n)\big). \tag{18}$$

Integrating the left-hand side of (18) with respect to z, we get

$$\begin{aligned} &\frac{1}{2\pi i}\oint \frac{z^2}{e^z-1} S_r^{(m)}(n+2)\mathrm{d}z \\ &= \frac{(r+2)!}{n+r+2}\sum_{k=0}^{n+r+2} m^{n+r-k+2}\binom{n+r+2}{k}\binom{n+k+r+2}{r+2}B_{n+k} = L_{r+2}^{(m)}(n). \end{aligned} \tag{19}$$

Now, integrating the right-hand side of (18), we obtain

$$
\begin{aligned}
&\frac{r+n+1}{2\pi i}\oint \frac{1}{e^z-1}\big(2(2r+2n+3)S_r^{(m)}(n+1)+(r+n)m^2S_r^{(m)}(n)\big)\mathrm{d}z\\
&=2(2r+2n+3)(r+n+1)L_r^{(m)}(n+1)+(r+n+1)(r+n)m^2L_r^{(m)}(n)\\
&=2(2r+2n+3)(r+n+1)R_r^{(m)}(n+1)+(r+n+1)(r+n)m^2R_r^{(m)}(n)\\
&=\sum_{k=1}^{m-1}(r+n+1)(r+n)m^2P_r^{(m,k)}(n)k^n(k-m)^{n-1}\\
&\quad+\sum_{k=1}^{m-1}2k(k-m)(r+n+1)(2r+2n+3)P_r^{(m,k)}(n+1)k^n(k-m)^{n-1}\\
&=\sum_{k=1}^{m-1}P_{r+2}^{(m,k)}(n)k^n(k-m)^{n-1}=R_{r+2}^{(m)}(n).
\end{aligned}
$$

This together with (19) ends the proof of (9) by induction.

From the last three lines we easily obtain the recurrence relation (10).

(ii) It follows from (10) and (11) that $Q_{i,r}(n)$ satisfy the relation (12).

(iii) We verify that the expression given by (13) is true for all odd r. We replace j by $j-1$ and using

$$\Delta_{j-1,r}(n)=j\frac{(2n+r+2)(2n+2r+1)(2n+2r+3)}{(\lfloor r/2\rfloor+1)(2n+2r-2j+3)}\Delta_{j,r+2}(n),$$

we get

$$
\begin{aligned}
(20)\quad &(n+r+1)(n+r)Q_{i-2,r}(n)\\
&=\sum_{j=0}^{\lfloor i/2\rfloor}\Bigg[2^{\theta_j+1}\left(\binom{\theta_j+1}{i-2j}+\frac{(n+1)+\theta_j+1}{2\,(n+1)+2\theta_j+1}\binom{\theta_j+1}{i-(2j+1)}\right)\\
&\quad j\frac{(2n+r+2)(2n+2r+3)}{(\lfloor r/2\rfloor+1)(2n+2r-2j+3)}\Delta_{j,r+2}(n)\prod_{s=\theta_j+1}^{r}(n+s+1)(2n+2s+1)\Bigg].
\end{aligned}
$$

Using

$$\Delta_{j,r}(n+1)=\frac{(\theta_j+1)(2n+2r+3)}{(\lfloor r/2\rfloor+1)(2n+2r-2j+3)}\Delta_{j,r+2}(n),$$

we obtain

$$
\begin{aligned}
(21)\quad &(2n+2r+3)(2n+2r+2)Q_{i,r}(n+1)\\
&=(2n+2r+3)\sum_{j=0}^{\lfloor i/2\rfloor}\Bigg[2^{\theta_j+1}\left(\binom{\theta_j}{i-2j}+\frac{(n+1)+\theta_j+1}{2\,(n+1)+2\theta_j+1}\binom{\theta_j}{i-(2j+1)}\right)\\
&\quad\frac{(\theta_j+1)(2n+2r+3)}{(\lfloor r/2\rfloor+1)(2n+2r-2j+3)}\Delta_{j,r+2}(n)\prod_{s=\theta_j+1}^{r}(n+s+1)(2n+2s+1)\Bigg].
\end{aligned}
$$

In the similar fashion, we have

$$(22)\quad (2n+2r+3)(2n+2r+2)Q_{i-1,r}(n+1) = (2n+2r+3) \times \sum_{j=0}^{\lfloor i/2\rfloor}\Bigg[2^{\theta_j+1}\left(\binom{\theta_j}{i-2j-1} + \frac{(n+1)+\theta_j+1}{2(n+1)+2\theta_j+1}\binom{\theta_j}{i-(2j+1)-1}\right) \frac{(\theta_j+1)(2n+2r+3)}{(\lfloor r/2\rfloor+1)(2n+2r-2j+3)}\Delta_{j,r+2}(n)\prod_{s=\theta_j+1}^{r}(n+s+1)(2n+2s+1)\Bigg].$$

Finally, add (20), (21) and (22) and after small rearrangements and using

$$\binom{\theta_j+1}{i-2j} = \binom{\theta_j}{i-2j} + \binom{\theta_j}{i-(2j+1)},$$

$$\binom{\theta_j+1}{i-(2j+1)} = \binom{\theta_j}{i-(2j+1)} + \binom{\theta_j}{i-(2j+2)},$$

we get $Q_{i,r+2}(n)$ as desired. This completes the proof.

Remark 2. For $n \geq 2$, we set $\widetilde{B}_n^r = (n+r)\cdots(n+1)\,B_n$. Identity (9) is equivalent to

$$(23)\qquad \sum_{k=0}^{n+r} m^k\binom{n+r}{k}\widetilde{B}^r_{2n+r-k} = (n+r)\sum_{u+v=m}(-1)^{n-1}\widehat{P}_r^{(u,v)}(n)u^n v^{n-1},$$

where $\widehat{P}_r^{(u,v)}(n) = P_r^{(u+v,v)}(n)$. We notice the symmetric aspect of the right-hand side of (23) which permits to us, as mentioned by Gessel (see for instance [**3**]), to compute only half of the terms appearing in the summand.

Acknowledgements. We would like to thank the referees and editors for their helpful comments on the manuscript. We are particularly grateful to Professor Ira M. Gessel for kindly allowing his approach to appear in this paper as the appendix.

APPENDIX

by Ira M. Gessel[1]

Here we give an umbral proof, following the method of [**3**], of a different but closely related formula for the Bernoulli number sum in Theorem 1.

Theorem 2. *Let ℓ, n, m, and r be nonnegative integers. Then*

$$(24)\quad \sum_{k=0}^{n+r} m^{n+r-k}\binom{n+r}{k}\binom{\ell+k+r}{r}B_{\ell+k} + (-1)^{\ell+n+r+1}\sum_{k=0}^{\ell+r} m^{\ell+r-k}\binom{\ell+r}{k}\binom{n+k+r}{r}B_{n+k}$$

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$$= (r+1)\sum_{k=1}^{m-1}\sum_{j=0}^{r+1}(-1)^{\ell+j-1}\binom{n+r}{j}\binom{\ell+r}{r+1-j}k^{\ell+j-1}(m-k)^{n+r-j}.$$

If r is odd then

$$\sum_{k=0}^{n+r} m^{n+r-k}\binom{n+r}{k}\binom{n+k+r}{r}B_{n+k} \tag{25}$$
$$= \frac{1}{2}(r+1)\sum_{k=1}^{m-1}\sum_{j=0}^{r+1}\binom{n+r}{j}\binom{n+r}{r+1-j}k^{j+n-1}(k-m)^{n+r-j}.$$

Proof. We begin by proving some binomial coefficient identities. We start with the identity

$$\sum_{k=0}^{a}(-1)^k\binom{a}{k}\binom{b+k}{c} = (-1)^a\binom{b}{c-a}. \tag{26}$$

Equation (26) may be proved by expanding $(-x)^a(1+x)^b$ in powers of x in two ways: the coefficient of x^c in $(-x)^a(1+x)^b$ is clearly $(-1)^a\binom{b}{c-a}$, but we also have

$$(-x)^a(1+x)^b = \big(1-(1+x)\big)^a(1+x)^b = \sum_{k=0}^{a}(-1)^k\binom{a}{k}(1+x)^{b+k}$$
$$= \sum_{k=0}^{a}\sum_{c=0}^{b+k}(-1)^k\binom{a}{k}\binom{b+k}{c}x^c,$$

so (26) holds.

Now let us define polynomials $A_{\ell,n,r}(x)$ by

$$A_{l,n,r}(x) = \sum_{k=0}^{n+r}\binom{n+r}{k}\binom{l+k+r}{r}x^{l+k}.$$

We first note that an easy computation gives

$$A'_{\ell,n,r}(x) = (r+1)A_{\ell-1,n-1,r+1}(x). \tag{27}$$

Next we prove the formula

$$A_{\ell,n,r}(-1-x) = (-1)^{\ell+n+r}A_{n,\ell,r}(x). \tag{28}$$

The coefficient of x^i in $A_{\ell,n,r}(-1-x)$ is

$$\sum_{k=0}^{n+r}(-1)^{\ell+k}\binom{n+r}{k}\binom{\ell+k+r}{r}\binom{\ell+k}{i}$$

$$= (-1)^{\ell}\binom{r+i}{i}\sum_{k=0}^{n+r}(-1)^k\binom{n+r}{k}\binom{\ell+k+r}{r+i} = (-1)^{\ell+n+r}\binom{r+i}{i}\binom{\ell+r}{i-n},$$

by (26), which is the coefficient of x^i in $(-1)^{\ell+n+r}A_{n,\ell,r}(x)$. This proves (28).

Next we show that

$$A_{\ell,n,r}(x) = \sum_{j=0}^{r}\binom{n+r}{j}\binom{\ell+r}{r-j}x^{\ell+j}(1+x)^{n+r-j}. \tag{29}$$

The coefficient of $x^{\ell+k}$ on the right-hand side of (29) is

$$\sum_{j=0}^{r}\binom{n+r}{j}\binom{\ell+r}{r-j}\binom{n+r-j}{k-j} = \binom{n+r}{k}\sum_{j=0}^{r}\binom{k}{j}\binom{\ell+r}{r-j}$$
$$= \binom{n+r}{k}\binom{k+\ell+r}{r},$$

by Vandermonde's theorem, and this is the coefficient of $x^{\ell+k}$ in $A_{\ell,n,r}(x)$. This proves (29).

One of the properties of the Bernoulli numbers (formula (7.8) of [**3**]) is that for any polynomial $f(x)$ and any positive integer m,

$$f(B+m) - f(-B) = \sum_{k=1}^{m-1} f'(k), \tag{30}$$

where after the left-hand side is expanded in powers of B, each B^j is replaced with the Bernoulli number B_j. If we set $x = B/m$ in (28) we get

$$A_{\ell,n,r}(-1-B/m) = (-1)^{\ell+n+r}A_{n,\ell,r}(B/m) \tag{31}$$

Now set $f(x) = A_{\ell,n,r}(-x/m)$ in (30), obtaining

$$A_{\ell,n,r}(-1-B/m) - A_{\ell,n,r}(B/m) = -\frac{1}{m}\sum_{k=1}^{m-1}A'_{\ell,n,r}(-k/m).$$

Applying (31) gives

$$(-1)^{\ell+n+r}A_{n,\ell,r}(B/m) - A_{\ell,n,r}(B/m) = -\frac{1}{m}\sum_{k=1}^{m-1}A'_{\ell,n,r}(-k/m)$$
$$= -\frac{r+1}{m}\sum_{k=1}^{m-1}A_{\ell-1,n-1,r+1}(-k/m),$$

by (27), which by (29) is equal to

$$-\frac{r+1}{m}\sum_{k=1}^{m-1}\sum_{j=0}^{r+1}\binom{n+r}{j}\binom{\ell+r}{r+1-j}(-k/m)^{\ell+j-1}(1-k/m)^{n+r-j}$$
$$=-\frac{r+1}{m^{\ell+n+r}}\sum_{k=1}^{m-1}\sum_{j=0}^{r+1}(-1)^{\ell+j-1}\binom{n+r}{j}\binom{\ell+r}{r+1-j}k^{\ell+j-1}(m-k)^{n+r-j}.$$

Multiplying by $-m^{\ell+n+r}$ gives

$$m^{\ell+n+r}A_{\ell,n,r}(B/m)-(-1)^{\ell+n+r}m^{\ell+n+r}A_{n,\ell,r}(B/m)$$
$$=(r+1)\sum_{k=1}^{m-1}\sum_{j=0}^{r+1}(-1)^{\ell+j-1}\binom{n+r}{j}\binom{\ell+r}{r+1-j}k^{\ell+j-1}(m-k)^{n+r-j},$$

which is equivalent to (24). If we set $\ell = n$ and take r to be odd in (24), then the two terms on the left become equal, and dividing by 2 gives (25). □

We note that the case $m = 1$, $r = 1$ of (24) was given by Momiyama [**6**]. He also gives a formula for the left-hand side of (24) for $r = 1$ and general m, but with a less explicit formula for the right-hand side.

Although (25) is simpler than Theorem 1, for each particular value of r, it seems to give a slightly more complicated formula. For example, for $r = 1$, Theorem 1 is the same as (4) but (25) gives

$$\sum_{k=0}^{n+1}m^{n+1-k}\binom{n+1}{k}\tilde{B}_{n+k}=\sum_{k=1}^{m-1}\Bigg[\binom{n+1}{2}k^{n-1}(k-m)^{n+1}+(n+1)^2k^n(k-m)^n+\binom{n+1}{2}k^{n+1}(k-m)^{n-1}\Bigg]. \tag{32}$$

In fact, (32) is a symmetrized version of (4) in the sense that if we set

$$S_1(n,m,k)=(n+1)\big((2n+1)k-(n+1)m\big)k^n(k-m)^{n-1}$$

and

$$S_2(n,m,k)=\binom{n+1}{2}k^{n-1}(k-m)^{n+1}+(n+1)^2k^n(k-m)^n+\binom{n+1}{2}k^{n+1}(k-m)^{n-1}$$

then $S_2(n,m,k)=\frac{1}{2}\big(S_1(n,m,k)+S_1(n,m,m-k)\big)$. This is easier to see if we write $S_1(n,m,k)$ as

$$(-1)^n(n+1)^2k^n(m-k)^n+2(-1)^{n-1}\binom{n+1}{2}k^{n+1}(m-k)^{n-1}.$$

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(Received July 31, 2011)
(Revised January 30, 2013)