RIORDAN ARRAYS AND APPLICATIONS VIA THE CLASSICAL UMBRAL CALCULUS

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Abstract. We use the classical umbral calculus to describe Riordan arrays. Here, a Riordan array is generated by a pair of umbrae, and this provides efficient proofs of several basic results of the theory such as the multiplication rule, the recursive properties, the fundamental theorem and the connection with Sheffer sequences. In particular, we show that the fundamental theorem turns out to be a reformulation of the umbral Abel identity. As an application, we give an elementary approach to the problem of extending integer powers of Riordan arrays to complex powers in such a way that additivity of the exponents is preserved. Also, ordinary Riordan arrays are studied within the classical umbral perspective and some combinatorial identities are discussed regarding Catalan numbers, Fibonacci numbers and Chebyshev polynomials.

1. Introduction

An exponential Riordan array is an infinite lower triangular matrix generated by two formal exponential series $A(z) = \sum_{n\geq 0} a_n \frac{z^n}{n!}$ and $B(z) = \sum_{n\geq 1} b_n \frac{z^n}{n!}$, with $a_0 \neq 0$ and $b_1 \neq 0$, such that, denoting this matrix by (A, B), its (n, k)-entry is given by

$$(A, B)_{n,k} = \left[\frac{z^n}{n!}\right] \left(A(z)\, \frac{B(z)^k}{k!}\right) .$$

Riordan arrays were introduced by Shapiro et.al. in [26] as a generalization of the well known Pascal triangle and other types of combinatorial arrays (Rogers also studied in [20] equivalent generalizations of the Pascal triangle under the name of renewal arrays). Note that the generators A and B used in [26] are ordinary formal power series instead of series of exponential type. In this case the (n, k)-entry of (A, B) is given by $(A, B)_{n,k} = [z^n] \left(A(z)\, B(z)^k\right)$. Riordan arrays form a group under matrix multiplication. They have been extensively studied and characterized in connection with combinatorial identities, recursion properties and walk problems (see for instance [5, 14, 25, 26, 28]).

The purpose of this paper is to give a promising symbolic treatment of the exponential Riordan group based on a renewed approach to umbral calculus initiated in 1994 by Rota and Taylor [23]. In the last decade, this approach has been continued mostly by Di Nardo, Niederhausen and Senato [7, 10, 11] and more recently by Petrullo [9, 17, 19]. We will refer to this new version as the *classical umbral calculus*, to distinguish it from the more established treatment of umbral calculus using operator theory, as presented for instance in Roman's book [21]. Under the old point of view of umbral calculus, Riordan arrays are also known as *recursive matrices* (see Barnabei, Brini and Nicoletti [2]).

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The classical umbral calculus consists of formal operations over symbols, called umbrae and usually denoted by greek letters, whose powers represent sequences of numbers. A main feature of the classical umbral calculus is that the same sequence of numbers can be represented by distinct umbrae. That allows to suitable encode a broad family of generating functions. Nonetheless, more than an elegant method to encode and describe well-known concepts involving generating functions, the renewed umbral setting provides remarkable computational simplifications and conceptual clarifications in several contexts. For instance, let $\big(s_n(x)\big)$ be a Sheffer polynomial sequence, classically defined by (see [21])

$$1+\sum_{n\geq 1} s_n(x)\frac{z^n}{n!} = A(z)\, e^{x\,B(z)}\,.$$

A key observation is that a sequence $\big(s_n(x)\big)$ is of Sheffer type if and only if

$$s_n(x)=\sum_{k=0}^{n}(A,B)_{n,k}\,x^k\,. \tag{1}$$

Formula (1) shows that Riordan arrays essentially provide a different point of view on Sheffer sequences. In [7], Sheffer sequences are described in terms of umbrae and, as an application, an umbral representation for Riordan arrays is derived in [8]. In this work we elaborate more on Riordan arrays using a normalized version of such umbral representation (see Definition 18 in Section 3.1 and compare with Corollary 5.5 in [8]). Tipically, the type of Sheffer sequences used in applications are those associated to classical orthogonal polynomials, where $s_0(x)=1$. Taking this fact into consideration, we concentrate on *normalized* Riordan arrays; that is, those corresponding to *monic* Sheffer sequences ($s_0(x)=1$). This is not a great loss of generality, as lots of the arrays considered in enumerative problems (Pascal, Stirling of first and second kind, and many others) satisfy the condition $(A,B)_{n,n}=1$; that is $a_0=1=b_1$. Besides, often the Riordan group is defined for normalized Riordan arrays (see for instance [12, 26]).

In the context of combinatorial identities, the classical Abel identity provides a deep generalization of the well-known binomial identity [6]. Its umbral counterpart is written as (see Section 2.2)

$$(\gamma+\sigma)^n \simeq \sum_{k=0}^{n}\binom{n}{k}(\gamma+k.\alpha)^{n-k}\sigma(\sigma-k.\alpha)^{k-1}\,. \tag{2}$$

It turns out that the fundamental theorem of Riordan arrays (Theorem 3.1) is simply a reformulation of the umbral Abel identity. In particular, formula (1) is stated as (Theorem 3.3)

$$(\gamma+x.\beta.\alpha_{\mathcal{D}})^n \simeq \sum_{k=0}^{n}\binom{n}{k}(\gamma+k.\alpha)^{n-k}\,x^k\,. \tag{3}$$

As an application of the fundamental theorem, we give a short umbral proof of the multiplication rule for Riordan arrays (Theorem 3.2). Note that all known results about Riordan arrays can be restated in the umbral language using generating functions. However, more than doing a plain translation, we emphasize that the goal of this paper is to derive these results working within the umbral syntax in the first place, underlying in this way the computational convenience of following the umbral approach. As an instance, we give an elementary treatment to the subject of generalized powers of Riordan arrays, which are used to describe one-parameter subgroups in combinatorial physics [3, 12].

This paper is organized as follows. In Section 2 we recall the main features of the classical umbral calculus and review the umbral Abel identity and the Lagrange inversion formula. In Section 3, we unfold our umbral approach to exponential Riordan arrays, putting special emphasis on the fundamental theorem, from which short umbral proofs can be given for the multiplication rule of Riordan arrays and for the isomorphism between the Sheffer and the Riordan groups. We also give an umbral description of several important Riordan subgroups. The simplicity of the umbral notation allows us to give a short presentation of the main recursive properties of Riordan arrays in Section 4. In Section 5, a few Riordan arrays are shown for which we make explicit some of the features already discussed. In Section 6, we give an elementary approach to generalized powers of Riordan arrays that recovers some of the results in [12]. Section 7 is devoted to ordinary Riordan arrays. Finally, in Section 8 we apply the umbral syntax to the Catalan triangle introduced by Shapiro [24]. Moreover, we recover some known facts and point out connections with ballot numbers, Fibonacci numbers and the Chebyshev polynomials of the first kind.

2. The classical umbral calculus

This section briefly recalls the basics of umbral calculus underlying much of this paper. Umbral Abel polynomials and the umbral Abel identity are also reviewed. For more information on this we refer the reader to [10, 11, 16, 19].

2.1. **Preliminaries.**

The basic ingredients to perform classical umbral calculus are:

(i) A commutative integral domain R with identity 1 (we will usually set $R = \mathbb{C}$ or $R = \mathbb{C}[x, y]$ for simplicity).

(ii) An *alphabet* $A = \{\alpha, \beta, \gamma, \ldots\}$, whose elements are called *umbrae*.

(iii) A linear functional $E\colon R[A] \to R$ called *evaluation* such that
   - $E[1] = 1$ and
   - $E[x^n y^m \, \alpha^i \beta^j \cdots \gamma^k] = x^n y^m \, E[\alpha^i] E[\beta^j] \cdots E[\gamma^k]$ (this property is called *uncorrelation*).

(iv) Two special umbrae: ε (*augmentation*) and υ (*unity*) such that $E[\varepsilon^n] = \delta_{0,n}$ and $E[\upsilon^n] = 1$ for all $n \geq 0$.

An umbra α represents a sequence $(a_n) = (a_n)_{n \in \mathbb{N}}$ in R in the sense that $E[\alpha^n] = a_n$ for all $n \geq 0$ (this forces $a_0 = 1$, which is not a great loss of generality). We refer to $E[\alpha^n]$ as the n-th moment of α and we assume that each sequence (a_n) is represented by infinite many distinct umbrae, sometimes denoted by α, α', α'', ... (this property is named *saturation*). An element $p \in R[A]$ is called umbral polynomial and we define the support of p to be the set of all umbrae that appear in p. Two umbral polynomials p and q in $R[A]$ are said to be *uncorrelated* if $E[pq] = E[p]E[q]$. Therefore, distinct umbrae are always uncorrelated. On A we define two equivalence relations:

Umbral equivalence: $\alpha \simeq \gamma$ if and only if $E[\alpha] = E[\gamma]$, and
Similarity: $\alpha \equiv \gamma$ if and only if $E[\alpha^n] = E[\gamma^n]$, for all $n \geq 0$.

Both $\simeq$ and $\equiv$ are extended to the whole of $R[A]$ in the obvious way. A very useful device in the classical umbral calculus is the generating function of an umbra. Given an umbra α we may consider the formal exponential series

$$e^{\alpha z} := 1 + \sum_{n=1}^{\infty} \alpha^n \frac{z^n}{n!} \;\in\; R[A]\,[[z]]\,. \tag{4}$$

The evaluation E can be linearly extended to such a formal series so that

$$f_\alpha(z) := E[e^{\alpha z}] = 1 + \sum_{n=1}^{\infty} E[\alpha^n] \frac{z^n}{n!} \;\in\; R[[z]]\,. \tag{5}$$

We shall write $e^{\alpha z} \simeq f_\alpha(z)$ to mean $E[e^{\alpha z}] = f_\alpha(z)$, and we call $f_\alpha(z)$ the generating function (g.f. for short) of α. The g.f. of an umbral polynomial p is defined in an analogous way. Also, note that $\alpha \equiv \gamma$ if and only if $e^{\alpha z} \simeq e^{\gamma z}$. The table below gives the moments and the generating function of some important umbrae that will be useful throughout the paper.

name	**umbra**	**g.f.**	**moments**
augmentation	ε	1	$1, 0, 0, \ldots$
Bernoulli	ι	$\dfrac{z}{e^{z}-1}$	$1, b_1, b_2, \ldots$ (Bernoulli numbers)
unity	υ	e^{z}	$1, 1, 1, \ldots$
singleton	χ	$1+z$	$1, 1, 0, \ldots$
Bell	β	$e^{e^{z}-1}$	$1, B_2, B_3, \ldots$ (Bell numbers)
boolean unity	$\bar{\upsilon}$	$\frac{1}{1-z}$	$1, 2!, 3!, \ldots$

A key feature of the renewed approach to umbral calculus is given by the *dot product*. By starting from elements in $R[A]$, the dot product provides new symbols called *auxiliary umbrae*. Henceforth, when a new family F of auxiliary umbrae is defined, we will extend the alphabet A so that $F \subset A$. Often, the umbrae of A are used to make explicit the action of the evaluation on the powers of an auxiliary umbra. For instance, the action of E on the dot product $\gamma.\alpha$ is defined by

$$f_{\gamma.\alpha}(z) = f_{\gamma}\big(\log f_{\alpha}(z)\big)\,. \tag{6}$$

It readily follows from (6) that $\gamma \equiv \eta$ implies $\gamma.\alpha \equiv \eta.\alpha$ and $\alpha.\gamma \equiv \alpha.\eta$, for all α. In particular, we have $\varepsilon.\alpha \equiv \varepsilon \equiv \alpha.\varepsilon$ and $\upsilon.\alpha \equiv \alpha \equiv \alpha.\upsilon$, for all α. Also, β and χ satisfy $\chi.\beta \equiv \beta.\chi \equiv \upsilon$. The dot product $k.\alpha$, with $k \in \mathbb{Z}$, has g.f. $f_{k.\alpha}(z) = f_{\alpha}(z)^k$. If $k > 0$ and if $\alpha', \alpha'', \ldots, \alpha^{(k)}$ are k pairwise distinct auxiliary umbrae defined to be similar to α then we have $k.\alpha \equiv \alpha' + \alpha'' + \cdots + \alpha^{(k)}$, so that $k.\alpha$ can be thought of as a generalization of the usual notion of multiple of an umbra. The umbra $-1.\alpha$ is named the *inverse* of α and satisfies $\alpha + (-1.\alpha) \equiv 0$. The dot product $\alpha.\chi$ is called the *α-factorial umbra* (see [11]). Since $f_{\alpha.\chi}(z) = f_{\alpha}(\log(1+z)) \simeq e^{\alpha\log(1+z)} = (1+z)^{\alpha} = 1 + \sum_{n\geq 1}(\alpha)_n \frac{z^n}{n!}$, then we deduce

$$(\alpha.\chi)^n \simeq (\alpha)_n, \quad n \geq 1, \tag{7}$$

where $(\alpha)_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$. Analogously, since $\bar{\upsilon} \equiv -1.\chi.-1$ and $\alpha.c \equiv c\alpha$, it follows from (7) that $(\alpha.\bar{\upsilon})^n \simeq (\alpha.-1.\chi.-1)^n \simeq (-1)^n(-\alpha)_n$, and finally

$$(\alpha.\bar{\upsilon})^n \simeq (\alpha+n-1)_n, \quad n \geq 1. \tag{8}$$

In order to manipulate the formal composition of two g.f.'s we iterate the dot product by obtaining auxiliary umbrae of type $\eta.\gamma.\alpha$. Hence, the g.f. of $\gamma.\beta.\alpha$ is

$$f_{\gamma.\beta.\alpha}(z) = f_{\gamma}\Big(\log f_{\beta}\big(\log f_{\alpha}(z)\big)\Big) = f_{\gamma}\big(f_{\alpha}(z)-1\big)\,.$$

We write $\alpha^{\langle -1\rangle}$ to refer to a given symbol in A, if it exists, satisfying $\alpha^{\langle -1\rangle}.\beta.\alpha \equiv \chi \equiv \alpha.\beta.\alpha^{\langle -1\rangle}$. We call $\alpha^{\langle -1\rangle}$ the *compositional inverse* of α. Note that $E[\alpha] \neq 0$ for $\alpha^{\langle -1\rangle}$ to exist. The table below translates some operations among umbrae in terms of their corresponding g.f.'s.

umbrae	**g.f.'s**
$\alpha+\gamma$	Cauchy product
$\alpha\,\eta$	Hadamard product
$\alpha.\gamma$	log composition
$\alpha.\beta.\gamma$	composition

2.2. **Abel polynomials and the Lagrange inversion formula.** Let α be an umbra. The *derivative* of α is an auxiliary umbra $\alpha_{\mathcal{D}}$ whose powers satisfy $\alpha_{\mathcal{D}}^n \simeq n\alpha^{n-1}$ for $n \geq 1$. In particular $E[\alpha_{\mathcal{D}}] = 1$ and this implies that $\alpha_{\mathcal{D}}$ has compositional inverse $\alpha_{\mathcal{D}}^{\langle -1 \rangle}$. The derivative umbra provides a binomial-like expansion of the powers $(\gamma.\beta.\alpha_{\mathcal{D}})^n$ in terms of the umbrae α and γ. In fact, we have $(f_{\alpha_{\mathcal{D}}}(z) - 1)^n = z^n f_\alpha(z)^n \simeq z^n e^{(n.\alpha)z}$ and then

$$e^{(\gamma.\beta.\alpha_{\mathcal{D}})z} \simeq f_\gamma\big(z f_\alpha(z)\big) \simeq \sum_{m=0}^{\infty} \gamma^m \frac{z^m}{m!} e^{(m.\alpha)z} \simeq \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \gamma^m (m.\alpha)^k \frac{z^m}{m!}\frac{z^k}{k!}.$$

Therefore, by comparing the coefficients of $\frac{z^n}{n!}$ we recover

$$(9) \qquad (\gamma.\beta.\alpha_{\mathcal{D}})^n \simeq \sum_{k=0}^{n} \binom{n}{k} \gamma^k (k.\alpha)^{n-k}.$$

The umbral Abel polynomials $\mathfrak{a}_n(\gamma, \alpha) = \gamma(\gamma - n.\alpha)^{n-1}$ are straightforward generalizations of the classical Abel polynomials $x(x-na)^{n-1}$. They are powerful tools that allows one to understand the connections between an umbra and its compositional inverse. Such polynomials were first studied (in the special case $\gamma = x$) by Rota, Shen and Taylor [22] in connection with polynomials of binomial type. More recently, they have been applied to the cumulant theory by Di Nardo, Petrullo and Senato [9], and further generalized by Petrullo [17, 18, 19]. Abel polynomials satisfy the Abel Identity [6],

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} (y+ka)^{n-k} x(x-ka)^{k-1}.$$

It generalizes in a direct way to umbral Abel polynomials as follows [18],

$$(10) \qquad (\gamma+\sigma)^n \simeq \sum_{k=0}^{n} \binom{n}{k} (\gamma + k.\alpha)^{n-k} \sigma(\sigma - k.\alpha)^{k-1}.$$

One of the main consequences of (10) is a one-line proof of the Lagrange inversion formula. To this end, it is useful to introduce the umbra $\mathfrak{K}_{\gamma,\alpha}$ which satisfies

$$(11) \qquad \mathfrak{K}_{\gamma,\alpha}^n \simeq \gamma(\gamma - n.\alpha)^{n-1}, \quad n \geq 1.$$

Theorem 2.1 (Lagrange inversion formula)**.** *For all umbrae α, γ, we have*

$$(12) \qquad (\gamma.\beta.\alpha_{\mathcal{D}}^{\langle -1 \rangle})^n \simeq \gamma(\gamma - n.\alpha)^{n-1}, \quad n \geq 1,$$

Proof. By applying (9), then (11), and finally (10), we have $\mathfrak{K}_{\gamma,\alpha}.\beta.\alpha_{\mathcal{D}} \equiv \gamma$, that is $\mathfrak{K}_{\gamma,\alpha} \equiv \gamma.\beta.\alpha_{\mathcal{D}}^{\langle -1 \rangle}$. □

The translation of (12) in terms of generating functions gives the familiar identity

$$n[z^n] f_\gamma\big((z f_\alpha(z))^{\langle -1 \rangle}\big) = [z^{n-1}] f'_\gamma(z) \left(\frac{1}{f_\alpha(z)}\right)^n.$$

and the reader would be convinced that (12) is exactly the umbral coding for the Lagrange inversion formula. Thanks to equivalence (12) we may restate the Abel identity (10) in a different, but equivalent, way. Indeed, set $\eta \equiv \sigma.\beta.\alpha_{\mathcal{D}}^{\langle -1 \rangle}$. Hence, we have $\sigma \equiv \eta.\beta.\alpha_{\mathcal{D}}$ and, via (12), $\eta^n \simeq \sigma(\sigma - n.\alpha)^{n-1}$. This way, the Abel identity (10) can be rewritten as

$$(13) \qquad (\gamma + \eta.\beta.\alpha_{\mathcal{D}})^n \simeq \sum_{k=0}^{n} \binom{n}{k} (\gamma + k.\alpha)^{n-k} \eta^k.$$

By means of (9) and (13) it is easily checked that

$$(14) \qquad (\alpha + \gamma.\beta.\alpha_{\mathcal{D}})_{\mathcal{D}} \equiv \gamma_{\mathcal{D}}.\beta.\alpha_{\mathcal{D}}.$$

Now, let $\mathfrak{L}_{\gamma,\alpha}$ denote an umbra satisfying $\mathfrak{L}_{\gamma,\alpha} \equiv -1.\mathfrak{K}_{\gamma,\alpha}$, and then set $\mathfrak{K}_\alpha := \mathfrak{K}_{\alpha,\alpha}$ and $\mathfrak{L}_\alpha := \mathfrak{L}_{\alpha,\alpha}$. The umbrae $\mathfrak{K}_\alpha$ and $\mathfrak{L}_\alpha$ play a central role in the umbral theory of cumulants [9, 19]. Since $\mathfrak{L}_\alpha \equiv -1.\alpha.\beta.\alpha_{\mathcal{D}}^{\langle -1\rangle}$, via (14) we obtain $\chi \equiv \varepsilon_{\mathcal{D}} \equiv (\alpha + \mathfrak{L}_\alpha.\beta.\alpha_{\mathcal{D}})_{\mathcal{D}} \equiv (\mathfrak{L}_\alpha)_{\mathcal{D}}.\beta.\alpha_{\mathcal{D}}$. This says that

$$(\mathfrak{L}_\alpha)_{\mathcal{D}} \equiv \alpha_{\mathcal{D}}^{\langle -1\rangle}, \tag{15}$$

hence $\mathfrak{L}_{\mathfrak{L}_\alpha} \equiv \alpha$, and this explains why $\mathfrak{L}_\alpha$ is named the *Lagrange involution* of α. Such an umbra is also useful to recover a reformulation of (12) that will reveal useful for our aims. Let $k \geq 1$ and denote by $\delta^{(k)}$ an umbra with moments $(\delta^{(k)})^n \simeq \delta_{k,n}$, for $n \geq 1$. By virtue of (15) we have $\delta^{(k)}.\beta.(\mathfrak{L}_\alpha)_{\mathcal{D}} \equiv \delta^{(k)}.\beta.\alpha_{\mathcal{D}}^{\langle -1\rangle}$. Furthermore, we use (9) and (12) to get $\binom{n}{k}(k.\mathfrak{L}_\alpha)^{n-k} \simeq \binom{n-1}{k-1}(-n.\alpha)^{n-k}$, and then

$$n(k.\mathfrak{L}_\alpha)^{n-k} \simeq k(-n.\alpha)^{n-k}. \tag{16}$$

By taking into account $\mathfrak{L}_{\mathfrak{L}_\alpha} \equiv \alpha$ we may exchange α and $\mathfrak{L}_\alpha$ and obtain

$$n(k.\alpha)^{n-k} \simeq k(n.\mathfrak{K}_\alpha)^{n-k}. \tag{17}$$

Again, we stress that (16) is the umbral coding of the following identity on formal power series,

$$n[z^{n-k}]\big((zf_\alpha(z))^{\langle -1\rangle}\big)^k = k[z^{n-k}]\left(\frac{1}{f_\alpha(z)}\right)^n.$$

3. Exponential Riordan arrays

In this section we use the syntax developed in Section 2 to give an umbral coding of some fundamental facts of the theory of the (exponential) Riordan arrays. Umbral expressions for several important Riordan subgroups are given while the group axioms are efficiently translated in terms of the umbral algebra. Henceforth, we set $R = \mathbb{C}[x, y]$.

3.1. The exponential Riordan group.

Definition 18. Given two umbrae α and γ we denote by (γ, α) the infinite lower triangular complex matrix whose entries $(\gamma, \alpha)_{n,k}$ satisfy

$$(\gamma, \alpha)_{n,k} \simeq \binom{n}{k}(\gamma + k.\alpha)^{n-k} \quad \text{for} \quad n, k \geq 0\,. \tag{19}$$

We name (γ, α) the (exponential) Riordan array of α and γ. Riordan arrays (p, q), for $p, q \in R[A]$, are defined in an analogous way.

Riordan arrays are often introduced as a generalization of the Pascal triangle. Through (19) the umbral syntax highlights this circumstance even more. Riordan arrays act on a column array of complex numbers in the obvious way. We write $(\gamma, \alpha)\,\boldsymbol{a} = \boldsymbol{b}$ to express that the column array $\boldsymbol{b}$ is obtained by multiplying the column array $\boldsymbol{a}$ by (γ, α) on its left. Since we deal with a saturated umbral calculus, then there exist umbrae η and ω whose moments are the entries of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively. Hence, we may think of the Riordan action as an action on the alphabet A, and write $(\gamma, \alpha) \bullet \eta \equiv \omega$ instead of $(\gamma, \alpha)\,\boldsymbol{a} = \boldsymbol{b}$.

Theorem 3.1 (Fundamental Theorem)**.** *For all umbrae α, γ, η we have*

$$(\gamma, \alpha) \bullet \eta \equiv \gamma + \eta.\beta.\alpha_{\mathcal{D}}. \tag{20}$$

Proof. By virtue of (19) we have

$$\big((\gamma, \alpha) \bullet \eta\big)^n \simeq \sum_{k=0}^{n} \binom{n}{k}(\gamma + k.\alpha)^{n-k}\eta^k, \quad n \geq 0\,.$$

and the claim follows via (13). $\square$

Then, from an umbral point of view the Fundamental theorem of Riordan arrays is nothing but a reformulation of the Abel identity (10). One of the main consequence of this theorem is obtained by setting $\eta \equiv 1$. In fact, in this case the umbra $\gamma + \beta{\boldsymbol .}\alpha_{\scriptscriptstyle D}$ represents the column vector whose nth entry is the sum of all entries occurring in the nth row of (γ, α),

$$\big(\gamma + \beta{\boldsymbol .}\alpha_{\scriptscriptstyle D}\big)^n \simeq \sum_{k=0}^{n} (\gamma, \alpha)_{n,k} \,. \tag{21}$$

Consider two Riordan arrays (γ, α) and (σ, ρ). We denote by $(\gamma, \alpha)(\sigma, \rho)$ the array obtained by multiplying (γ, α) by (σ, ρ). Hence, the following theorem can be stated.

Theorem 3.2. *The set $\mathfrak{Rio}$ of all Riordan arrays is closed under matrix multiplication. In particular, we have*

$$(\gamma, \alpha)(\sigma, \rho) = (\gamma + \sigma{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\scriptscriptstyle D} \,,\, \alpha + \rho{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\scriptscriptstyle D}) \,. \tag{22}$$

Proof. From (20) and (14) we have

$$(\gamma, \alpha)(\sigma, \rho) \bullet \eta \equiv (\gamma, \alpha) \bullet (\sigma + \eta{\boldsymbol .}\beta{\boldsymbol .}\rho_{\scriptscriptstyle D}) \equiv (\gamma + \sigma{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\scriptscriptstyle D}) + \eta{\boldsymbol .}\beta{\boldsymbol .}(\alpha + \rho{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\scriptscriptstyle D})_{\scriptscriptstyle D} \,.$$

We may conclude that $(\gamma, \alpha)(\sigma, \rho) \bullet \eta \equiv (\gamma + \sigma{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\scriptscriptstyle D}, \alpha + \rho{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\scriptscriptstyle D}) \bullet \eta$ for all η, and then (22) holds. □

The array $(\varepsilon, \varepsilon)$ is the unity with respect to the matrix multiplication. Moreover, thanks to Theorem 2.1, and by virtue of (22) we may deduce that the inverse of (γ, α) is given by

$$(\gamma, \alpha)^{-1} = (\mathfrak{L}_{\gamma,\alpha}, \mathfrak{L}_{\alpha}) \,.$$

Hence, $\mathfrak{Rio}$ is a group with respect to matrix multiplication and we name it the (exponential) Riordan group.

3.2. **The Sheffer group.**

Definition 23. Given umbrae α and γ, the polynomial sequence $\big(\mathfrak{s}_n(x)\big)$ defined by

$$\mathfrak{s}_n(x) \simeq \big(\gamma + x{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\scriptscriptstyle D}\big)^n \quad \text{for} \quad n \geq 0 \,, \tag{24}$$

will be named the Sheffer sequence of (γ, α).

Sheffer sequences are classical polynomial sequences. They include several classes of orthogonal systems: Hermite, Charlier, Laguerre, Meixner of the first kind, and Meixner of the second kind. Sheffer sequences have been deeply studied within finite operator calculus by Roman [21]. Applications of umbrae to Sheffer sequences are in [18, 29], and more extensively in [8]. By comparing (20) and (24) the following relation among Sheffer sequences and Riordan arrays comes trivially.

Theorem 3.3. *The Sheffer sequence $\big(\mathfrak{s}_n(x)\big)$ of (γ, α) is represented by $(\gamma, \alpha) \bullet x$. Equivalently, the following expansion is possible*

$$\mathfrak{s}_n(x) = \sum_{k=0}^{n} (\gamma, \alpha)_{n,k} \, x^k \,. \tag{25}$$

Hence, by virtue of (25) the following fact is achieved: the Riordan array (γ, α) is nothing but the array of coefficients of the Sheffer sequence $\big(\mathfrak{s}_n(x)\big)$ in terms of the canonical basis (x^n). Note that, from (20) we obtain

$$(\gamma, \alpha) \bullet (\eta + \omega) \equiv (\gamma, \alpha) \bullet \eta + (\varepsilon, \alpha) \bullet \omega \,. \tag{26}$$

If $\big(\mathfrak{s}_n(x)\big)$ and $\big(\mathfrak{p}_n(x)\big)$ are the Sheffer sequences of (γ,α) and (ε,α) respectively, by setting $\eta = x$ and $\omega = y$ in (26), and by taking the nth moment of both sides, we may recover the well-known Sheffer identity,

$$\mathfrak{s}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{s}_k(x)\mathfrak{p}_{n-k}(y)\,.$$

Sheffer sequences are endowed with a binary operation called umbral composition. More precisely, if $\big(\mathfrak{s}_n(x)\big)$ and $\big(\mathfrak{r}_n(x)\big)$ are the Sheffer sequences of (γ,α) and (σ,ρ) respectively, then the umbral composition of $\big(\mathfrak{s}_n(x)\big)$ with $\big(\mathfrak{r}_n(x)\big)$ is the polynomial sequence $\big(\mathfrak{sr}_n(x)\big)$ defined by

$$\mathfrak{sr}_n(x) = \sum_{k=0}^{n} (\gamma,\alpha)_{n,k}\, \mathfrak{r}_k(x) \quad \text{for} \quad n \geq 0\,. \tag{27}$$

By expanding the $\mathfrak{r}_n(x)$'s according to (25) we may check that the coefficient of x^k in $\mathfrak{sr}_n(x)$ is the (n,k)-entry of the Riordan array $(\gamma,\alpha)(\sigma,\rho)$. This way, from (22) we obtain that $(\mathfrak{sr}_n(x))$ is the Sheffer sequence of $(\gamma + \sigma{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\mathcal{D}}, \alpha + \rho{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\mathcal{D}})$. Also, we deduce that $(\mathfrak{s}_n(x))$ and $(\mathfrak{r}_n(x))$ are inverses of each other if and only if $(\sigma,\rho) = (\mathfrak{L}_{\gamma,\alpha}, \mathfrak{L}_{\alpha})$, and that the Sheffer sequence (x^n) of $(\varepsilon,\varepsilon)$ is the unity with respect to the umbral composition. We can conclude that the set $\mathfrak{Sheff}$ of all Sheffer sequences is a group under the umbral composition, which we name the Sheffer group. The following theorem is now obvious.

Theorem 3.4. *The Sheffer group and the Riordan group are isomorphic.*

3.3. **Riordan subgroups.** We give umbral characterizations for some important Riordan subgroups.

The Appell subgroup. We denote by *Appell* the set of all arrays of type (γ,ε). Given any two elements (γ,ε) and (σ,ε) in *Appell*, it follows that

$$(\gamma,\varepsilon)(\sigma,\varepsilon) = (\gamma+\sigma,\varepsilon) \qquad \text{and} \qquad (\gamma,\varepsilon)^{-1} = (-1{\boldsymbol .}\gamma,\varepsilon)\;.$$

This shows that *Appell* is a subgroup. We can also see from the left formula above that it is an abelian subgroup. Being $\varepsilon_{\mathcal{D}} \equiv \chi$ and $\beta{\boldsymbol .}\chi \equiv \upsilon$, then Appell acts on the alphabet of umbrae following

$$(\gamma,\varepsilon) \bullet \eta \equiv \gamma + \eta\,.$$

Sheffer sequences relative to the Appell subgroup are named Appell sequences.

The Associated subgroup. The collection of arrays of the form (ε,α) is denoted by *Assoc*. Given any (ε,α) and (ε,ρ) in *Assoc*, we have

$$(\varepsilon,\alpha)(\varepsilon,\rho) = (\varepsilon, \alpha + \rho{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\mathcal{D}}) \qquad \text{and} \qquad (\varepsilon,\alpha)^{-1} = (\varepsilon,\mathfrak{L}_{\alpha})\,.$$

The action of *Assoc* on A is encoded by

$$(\varepsilon,\alpha) \bullet \eta \equiv \eta{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\mathcal{D}}\,.$$

The corresponding Sheffer sequences are named associated sequences or binomial sequences.

The Bell subgroup. Of particular interest are arrays of the form (α,α). We denote the collection of all such arrays by *Bell*. Given any (α,α) and (σ,σ) we have

$$(\alpha,\alpha)(\sigma,\sigma) = (\alpha + \sigma{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\mathcal{D}}, \alpha + \sigma{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\mathcal{D}}) \qquad \text{and} \qquad (\alpha,\alpha)^{-1} = (\mathfrak{L}_{\alpha},\mathfrak{L}_{\alpha})\,.$$

The *Bell*-action is encoded by

$$(\alpha,\alpha) \bullet \eta \equiv \alpha + \eta{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\mathcal{D}}\,.$$

The Stabilizer subgroups. Given any $\eta \in A$, the stabilizer $Stab(\eta)$ of η (with respect to the $\mathfrak{Rio}$-action) is

$$Stab(\eta) = \Big\{(\gamma,\alpha) \in \mathfrak{Rio} \,:\, \gamma + \eta\boldsymbol{.}\beta\boldsymbol{.}\alpha_{\mathcal{D}} \equiv \eta\Big\}\,.$$

Since $(\gamma,\alpha) \in Stab(\eta)$ if and only if $\gamma \equiv \eta - 1\boldsymbol{.}\eta\boldsymbol{.}\beta\boldsymbol{.}\alpha_{\mathcal{D}}$, then we have

$$\begin{aligned} Stab(\varepsilon) &= \Big\{(\gamma,\alpha) \in \mathfrak{Rio} \,:\, \gamma \equiv \varepsilon\Big\} = Assoc. \\ Stab(\upsilon) &= \Big\{(\gamma,\alpha) \in \mathfrak{Rio} \,:\, \gamma \equiv 1 - 1\boldsymbol{.}\beta\boldsymbol{.}\alpha_{\mathcal{D}}\Big\} = Stoch \quad \text{(Stochastic subgroup).} \\ Stab(\chi) &= \Big\{(\gamma,\alpha) \in \mathfrak{Rio} \,:\, \gamma \equiv \chi - 1\boldsymbol{.}\alpha_{\mathcal{D}}\Big\}. \end{aligned}$$

We give explicit umbral representations for the (n,k)-entry of Riordan arrays belonging to some of the subgroups mentioned above. The stabilizers case can be recover by applying (13) and (19).

Subgroup	$(\gamma,\alpha)_{n,k} \simeq \binom{n}{k}\big(\gamma + k\boldsymbol{.}\alpha\big)^{n-k}$
Appell (γ,ε)	$\binom{n}{k}\gamma^{n-k}$
Associated (ε,α)	$\binom{n}{k}\big(k\boldsymbol{.}\alpha\big)^{n-k}$
Bell (α,α)	$\binom{n}{k}\big((k+1)\boldsymbol{.}\alpha\big)^{n-k}$
$Stab(\eta)$ $(\eta - 1\boldsymbol{.}\eta\boldsymbol{.}\beta\boldsymbol{.}\alpha_{\mathcal{D}}, \alpha)$	$\binom{n}{k}\sum_{i=0}^{n-k}(\eta + k\boldsymbol{.}\alpha,\alpha)_{n-k,i}(-1\boldsymbol{.}\eta)^i$

4. Recursive properties of Riordan arrays

In this section the main recursive properties of a Riordan array (γ,α) are established.

Theorem 4.1. *The Riordan array (γ,α) satisfies the following recursive properties:*

$$(\gamma,\alpha)_{n,k} \simeq \frac{n}{k}\sum_{i=0}^{n-k}\binom{n-1}{i}\alpha^i\,(\gamma,\alpha)_{n-1-i,k-1} \quad n,k \geq 1\,, \tag{28}$$

and

$$(\gamma,\alpha)_{n,k} \simeq \frac{n}{k}\sum_{i=0}^{n-k}\binom{k-1+i}{i}\mathfrak{K}_{\alpha}^i\,(\gamma,\alpha)_{n-1,k-1+i} \quad n,k \geq 1\,. \tag{29}$$

Proof. From (19) we obtain

$$(\gamma,\alpha)_{n,k} \simeq \binom{n}{k}\big(\gamma + (k-1)\boldsymbol{.}\alpha + \alpha\big)^{n-k} \simeq \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}\big(\gamma + (k-1)\boldsymbol{.}\alpha\big)^{n-k-i}\alpha^i\,,$$

and then (28) follows thanks to elementary computations. Moreover by applying the Abel identity (10) we have

$$(\gamma,\alpha)_{n,k} \simeq \binom{n}{k}\big(\gamma + (k-1)\boldsymbol{.}\alpha + \alpha\big)^{n-k} \simeq \binom{n}{k}\sum_{i=0}^{n-k}\binom{n-k}{i}\big(\gamma + (k-1+i)\boldsymbol{.}\alpha\big)^{n-k-i}\alpha(\alpha - i\boldsymbol{.}\alpha)^{i-1}\,,$$

and (29) is a consequence of (11). $\square$

Equation (28) expresses the (n,k)-entry of (γ,α) as a linear combination of the entries in the preceding column up to the preceding row. Equation (29) gives the (n,k)-entry of (γ,α) as a linear combination of the entries in the preceding row starting from the preceding column (note that from column $n-1$ on all entries are zero). This result is the umbral equivalent version of Rogers' recursion formula for ordinary Riordan arrays [20] (see also [28]). The sequence of moments of $\mathfrak{K}_\alpha$ is the A-sequence of (γ,α). Since $\mathfrak{K}_\alpha{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\mathcal{D}} \equiv \alpha$ and $\mathfrak{K}_\alpha^n \simeq \alpha(\alpha - n{\boldsymbol .}\alpha)^{n-1}$, a further recurrence relation can be derived from (13) as follows.

Theorem 4.2. *The Riordan array* (γ,α) *satisfies the following recursive properties:*

$$(\gamma,\alpha)_{n,k} \simeq \binom{n}{k}\sum_{i=0}^{n-k}(k{\boldsymbol .}\mathfrak{K}_\alpha)^i\,(\gamma,\alpha)_{n-k,i} \quad n,k\geq 1\,. \tag{30}$$

Proof. From (19) and (13) we obtain

$$(\gamma,\alpha)_{n,k} \simeq \binom{n}{k}\big(\gamma + k{\boldsymbol .}\mathfrak{K}_\alpha{\boldsymbol .}\beta{\boldsymbol .}\alpha_{\mathcal{D}}\big)^{n-k} \simeq \binom{n}{k}\sum_{i=0}^{n-k}(k{\boldsymbol .}\mathfrak{K}_\alpha)^i(\gamma,\alpha)_{n-k,i}\,.$$

□

5. Examples

In this section we show a few examples of classical arrays that are in $\mathfrak{Rio}$. We derive some of their recursive properties and make explicit the related Sheffer sequences.

Identity array. The identity array is the Riordan array $\boldsymbol{I} = (\varepsilon,\varepsilon)$. Its Sheffer sequence is (x^n).

Pascal array. The Pascal array is the Riordan array $\boldsymbol{P} = (\upsilon,\varepsilon)$. So we have $\boldsymbol{P}_{n,k} = \binom{n}{k}$.

(1) Note that $\boldsymbol{P}\bullet\upsilon \equiv 2$ and then we recover the well-known identity

$$2^n = \sum_{k=0}^{n}\binom{n}{k}\,.$$

(2) From $\boldsymbol{P}\bullet x \equiv 1 + x$ we deduce that

$$(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^n\,.$$

Stirling arrays. The Stirling array of the second kind is defined by $\boldsymbol{S} = (\varepsilon, -1{\boldsymbol .}\iota)$. Its entries $\boldsymbol{S}_{n,k}$'s are named Stirling numbers of the second kind and are denoted by $S(n,k)$'s. Some fundamental properties of this array can be recovered as follows.

(1) Since $(-1{\boldsymbol .}\iota)_{\mathcal{D}} \equiv \upsilon$ then from (20) we obtain $\boldsymbol{S}\bullet\upsilon \equiv \beta$ and then the row-sum of this array gives the sequence of Bell numbers:

$$B_n = \sum_{k=0}^{n} S(n,k)\,.$$

(2) Let $\big(\phi_n(x)\big)$ be the binomial sequence represented by $x{\boldsymbol .}\beta$. The $\phi_n(x)$'s are called exponential polynomials [21]. Since $\boldsymbol{S}\bullet x \equiv x{\boldsymbol .}\beta$, we deduce that the Sheffer sequence of $\boldsymbol{S}$ is $\big(\phi_n(x)\big)$, hence

$$\phi_n(x) = \sum_{k=0}^{n} S(n,k)\,x^n\,.$$

(3) From (28) we recover

$$S(n,k) = \frac{1}{k} \sum_{i=0}^{n-k} \binom{n}{i+1} S(n-1-i, k-1)\,.$$

Since $\mathfrak{K}_{-1.\iota} \equiv -1.\iota.\beta.(-1.\iota)^{\langle -1 \rangle}_{\mathcal{D}}$, and being $(-1.\iota)^{\langle -1 \rangle}_{\mathcal{D}} \equiv \upsilon^{\langle -1 \rangle} \equiv \chi.\chi$, we obtain $\mathfrak{K}_{-1.\iota} \equiv -1.\iota.\chi$. Equivalence (29) gives

$$S(n,k) = \frac{n}{k} \sum_{i=0}^{n-k} \binom{k-1+i}{i} \mathscr{C}_i \, S(n-1-i, k-1)\,,$$

where the $\mathscr{C}_i$'s are sometimes called Cauchy numbers [15]. Finally from (30) we obtain

$$S(n,k) = \binom{n}{k} \sum_{i=0}^{n-k} \mathscr{C}_i^{(k)} \, S(n-k, i)\,,$$

with the $\mathscr{C}_i^{(k)}$'s denoting generalized Cauchy numbers, whose g.f. is the kth power of the g.f. of the $\mathscr{C}_i$'s.

The Stirling array of the first kind is defined by $\boldsymbol{s} = \boldsymbol{S}^{-1}$. Being $\mathfrak{L}_{-1.\iota} \equiv -1.\mathfrak{K}_{-1.\iota} \equiv \iota.\chi$, it immediately follows that $\boldsymbol{s} = (\varepsilon, \iota.\chi)$.

(1) Via generating functions it can be checked that $(\iota.\chi)_{\mathcal{D}} \equiv \chi.\chi$ and then we obtain $\boldsymbol{s} \bullet \upsilon \equiv \chi$. This way, as $s(0,0) = 1 = s(1,1)$ and $s(n,0) = 0$ for $n \geq 1$ then we have

$$0 = \sum_{k=0}^{n} s(n,k) \quad \text{for} \quad n \geq 2\,.$$

(2) From $\boldsymbol{s} \bullet x \equiv x.\chi$ we deduce that

$$(x)_n = \sum_{k=0}^{n} s(n,k)\, x^n\,.$$

and then the Sheffer sequence of $\boldsymbol{s}$ is the sequence of falling factorials $\big((x)_n\big)$.

(3) Note that $(\chi.\chi)^n \simeq (-1)^{n-1}(n-1)!$. Then, from (28) we recover

$$s(n,k) = \frac{1}{k} \sum_{i=0}^{n-k} \frac{(n)_{i+1}}{i+1} (-1)^i \, s(n-1-i, k-1)\,.$$

Since $\mathfrak{K}_{\iota.\chi} \equiv \iota$ then (29) gives

$$s(n,k) = \frac{n}{k} \sum_{i=0}^{n-k} \binom{k-1+i}{i} b_i \, s(n-1-i, k-1)\,.$$

Finally from (30) we obtain

$$s(n,k) = \binom{n}{k} \sum_{i=0}^{n-k} b_i^{(k)} \, s(n-k, i)\,,$$

with $b_i^{(k)}$ denoting a generalized Bernoulli number [6].

We conclude this example by noting that $\boldsymbol{s} \bullet (x.\chi) \equiv \big(\boldsymbol{s} \cdot \boldsymbol{S}\big) \bullet x \equiv x$. This says

$$x^n = \sum_{k=0}^{n} S(n,k)\, (x)_k\,,$$

and then $\boldsymbol{s}$ stores the transition matrix from $\big((x)_n\big)$ to (x^n).

6. Generalized powers in the Riordan group

Let $j \in \mathbb{N}$ and $(\gamma,\alpha) \in \mathfrak{Rio}$. In this section we give an elementary approach to the problem of extending the power j in $(\gamma,\alpha)^j$ to any complex number c, such that $(\gamma,\alpha)^c \in \mathfrak{Rio}$. This subject has been already discussed in [3, 12], where some physical insights are underlined. In our context, by $(\gamma,\alpha)^j$ we mean

$$(\gamma,\alpha)^j = \underbrace{\left(\gamma^{(1)},\alpha^{(1)}\right)\cdots\left(\gamma^{(j)},\alpha^{(j)}\right)}_{j}, \tag{31}$$

where the $\gamma^{(i)}$ and $\alpha^{(i)}$ are respectively umbrae similar to γ and α, for all $1 \leq i \leq j$. By denoting $(\beta.\alpha_{\mathcal{D}})^{*j} = \beta.\alpha_{\mathcal{D}}^{(j)}.\beta.\alpha_{\mathcal{D}}^{(j-1)}.\cdots.\beta.\alpha_{\mathcal{D}}^{(1)}$ and setting $(\beta.\alpha_{\mathcal{D}})^{*0} = \upsilon$, we can write (31) as

$$(\gamma,\alpha)^j = \left(\sum_{i=1}^{j}\gamma^{(i)}.(\beta.\alpha_{\mathcal{D}})^{*(i-1)}, \sum_{i=1}^{j}\alpha^{(i)}.(\beta.\alpha_{\mathcal{D}})^{*(i-1)}\right). \tag{32}$$

Formula (32) makes explicit the two umbrae this is a Riordan array of, so that we can write $(\gamma,\alpha)^j \in \mathfrak{Rio}$. In general, given an arbitrary $c \in \mathbb{C}$, in order to say that $(\gamma,\alpha)^c \in \mathfrak{Rio}$, we need to find umbrae σ and ρ such that $(\gamma,\alpha)^c = (\sigma,\rho)$. To achieve this goal, denote by $(\gamma,\alpha)+(\sigma,\rho)$ the componentwise matrix summation. Note that $(\gamma,\alpha)+(\sigma,\rho) \neq (\gamma+\sigma,\alpha+\rho)$ and in general $(\gamma,\alpha)+(\sigma,\rho)$ is not an element of $\mathfrak{Rio}$. For example, the matrix represented by $(\gamma,\alpha)-(\varepsilon,\varepsilon)$ is not in $\mathfrak{Rio}$ as the (n,k)-entry $\big((\gamma,\alpha)-(\varepsilon,\varepsilon)\big)_{n,k}$ equals 0 if $n=k$, and reduces to $(\gamma,\alpha)_{n,k}$ otherwise (note that $(\gamma,\alpha)\in\mathfrak{Rio}$ implies that $(\gamma,\alpha)_{n,n} \simeq 1$ for all $n \in \mathbb{N}$). Nevertheless, let us set

$$(\gamma,\alpha)^{\diamond c} := \sum_{j\geq 0}\binom{c}{j}\Big((\gamma,\alpha)-(\varepsilon,\varepsilon)\Big)^j. \tag{33}$$

It readily follows that by choosing $c = m \in \mathbb{N}$ the generalized power (33) matches the usual power $(\gamma,\alpha)^m$. However, it is not immediately clear that $(\gamma,\alpha)^{\diamond c} \in \mathfrak{Rio}$ for a generic $c \in \mathbb{C}$ (it is not even clear that $(\gamma,\alpha)^{\diamond c}$ is well defined). The next step is to find two auxiliary umbrae $\gamma^{\dagger c}$ and $\alpha^{\ddagger c}$ so that

$$(\gamma,\alpha)^{\diamond c} = (\gamma^{\dagger c},\alpha^{\ddagger c})$$

Obviously, if such umbrae $\gamma^{\dagger c}$ and $\alpha^{\ddagger c}$ exist, they must be given in terms of γ and α. Formula (32) suggests that the umbra $\gamma^{\dagger c}$ should depend on γ and α, whereas the umbra $\alpha^{\ddagger c}$ should depend only on α. Now, for $j \geq 1$ we have

$$\Big((\gamma,\alpha)-(\varepsilon,\varepsilon)\Big)^j = \underbrace{\Big(\big(\gamma^{(j)},\alpha^{(j)}\big)-\big(\varepsilon,\varepsilon\big)\Big)\cdots\Big(\big(\gamma^{(1)},\alpha^{(1)}\big)-\big(\varepsilon,\varepsilon\big)\Big)}_{j},$$

from where we get the following expansion,

$$\Big((\gamma,\alpha)-(\varepsilon,\varepsilon)\Big)^j_{n,k} \simeq \sum_{k<l_1<\cdots<l_{j-1}<n}\big(\gamma^{(j)},\alpha^{(j)}\big)_{n,l_{j-1}}\cdots\big(\gamma^{(1)},\alpha^{(1)}\big)_{l_1,k}.$$

By setting $l_0 = k$ and $l_j = n$ and applying (19), we obtain

$$\Big((\gamma,\alpha)-(\varepsilon,\varepsilon)\Big)^j_{n,k} \simeq \frac{1}{k!}\sum_{l_0<l_1<\cdots<l_{j-1}<l_j}\binom{n}{l_j-l_{j-1},\ldots,l_1-l_0}(\gamma^{(j)}+l_{j-1}.\alpha^{(j)})^{l_j-l_{j-1}}\cdots(\gamma^{(1)}+l_0.\alpha^{(1)})^{l_1-l_0},$$

where we use the multinomial coefficient

$$\binom{n}{l_j-l_{j-1},\ldots,l_1-l_0} = \frac{n!}{(l_j-l_{j-1})!\cdots(l_1-l_0)!}.$$

Therefore, from (33) we can write

$$\begin{aligned}(\gamma,\alpha)^{\diamond c}_{n,k} &\simeq (\varepsilon,\varepsilon)_{n,k} \quad + \\ + \quad &\sum_{j\geq 1}\binom{c}{j}\frac{1}{k!}\sum_{l_0<l_1<\cdots<l_{j-1}<l_j}\binom{n}{l_j-l_{j-1},\ldots,l_1-l_0}(\gamma^{(j)}+l_{j-1}.\alpha^{(j)})^{l_j-l_{j-1}}\cdots(\gamma^{(1)}+l_0.\alpha^{(1)})^{l_1-l_0}\,.\end{aligned}$$

Note that for $j > n-k$, the sum above is zero. This shows that the entries $(\gamma,\alpha)^{\diamond c}_{n,k}$ involve finite summations for any n and k; hence $(\gamma,\alpha)^{\diamond c}$ is well defined. In particular, when $n=k$ we get $(\gamma,\alpha)^{\diamond c}_{n,n}\simeq 1$ for all $n\geq 0$. Setting $k=0$, we define the auxiliary umbra $\gamma^{\dagger c}$ as the umbra having moments

$$(\gamma^{\dagger c})^n \simeq (\varepsilon,\varepsilon)_{n,0}+\sum_{j\geq 1}\binom{c}{j}\sum_{0<l_1<\cdots<l_{j-1}<n}\binom{n}{n-l_{j-1},\ldots,l_1}(\gamma^{(j)}+l_{j-1}.\alpha^{(j)})^{n-l_{j-1}}\cdots(\gamma^{(1)})^{l_1}\,. \tag{34}$$

Analogously, setting $k=1$ and taking $\gamma\equiv\varepsilon$, we define the auxiliary umbra $\alpha^{\ddagger c}$ as the umbra such that for $n\geq 1$, it satisfies

$$(\alpha^{\ddagger c})^{n-1} \simeq (\varepsilon,\varepsilon)_{n,1}+\sum_{j\geq 1}\binom{c}{j}\sum_{1<l_1<\cdots<l_{j-1}<l_j}\binom{n}{l_j-l_{j-1},\ldots,l_1-l_0}(l_{j-1}.\alpha^{(j)})^{l_j-l_{j-1}}\cdots(l_0.\alpha^{(1)})^{l_1-l_0}\,. \tag{35}$$

We emphasize that $\gamma^{\dagger c}$ and $\alpha^{\ddagger c}$ are well defined umbrae. Therefore $(\gamma^{\dagger c},\alpha^{\ddagger c})\in\mathfrak{Rio}$ and by (19), we have

$$(\gamma^{\dagger c},\alpha^{\ddagger c})_{n,k}\simeq\binom{n}{k}\big(\gamma^{\dagger c}+k.\alpha^{\ddagger c}\big)^{n-k}.$$

Note that $(\gamma,\alpha)^{\diamond c}_{n,0}\simeq(\gamma^{\dagger c})^n\simeq(\gamma^{\dagger c},\alpha^{\ddagger c})_{n,0}$. In fact, more is true.

Theorem 6.1. *For any $c\in\mathbb{C}$ and $(\gamma,\alpha)\in\mathfrak{Rio}$, the generalized power $(\gamma,\alpha)^{\diamond c}$ defined in (33) is in $\mathfrak{Rio}$. Moreover,*

$$(\gamma,\alpha)^{\diamond c}=(\gamma^{\dagger c},\alpha^{\ddagger c})\,,$$

where $\gamma^{\dagger c}$ and $\alpha^{\ddagger}$ are given by (34) and (35).

Proof. Let $j\in\mathbb{N}$. Using recursively Theorem 3.1, in particular Formula (13), we verify that

$$\left(\sum_{i=1}^{j}\gamma^{(i)}.(\beta.\alpha_{\mathcal{D}})^{*(i-1)}\right)^n\simeq\big(\gamma^{\dagger j}\big)^n\quad\text{and}\quad\left(\sum_{i=1}^{j}\alpha^{(i)}.(\beta.\alpha_{\mathcal{D}})^{*(i-1)}\right)^n\simeq\big(\alpha^{\ddagger j}\big)^n\qquad\text{for all}\quad n\geq 0.$$

Therefore, for all $j\in\mathbb{N}$ we have $(\gamma,\alpha)^{\diamond j}=(\gamma,\alpha)^j=(\gamma^{\dagger j},\alpha^{\ddagger j})$. Moreover, observe that $(\gamma,\alpha)^{c}_{n,k}$ and $\binom{n}{k}E[(\gamma^{\dagger c}+k.\alpha^{\ddagger c})^{n-k}]$ are polynomials in c that agree for all $c=j\in\mathbb{N}$. Hence, they are equal. □

In particular, when $\alpha=\varepsilon$ we have $\gamma^{\dagger c}\equiv c.\gamma$ and $\varepsilon^{\ddagger c}\equiv\varepsilon$, so that we can write $(\gamma,\varepsilon)^{\diamond c}=(c.\gamma,\varepsilon)$. Furthermore, for any $c_1,c_2\in\mathbb{C}$ note that

$$\big(\gamma,\varepsilon\big)^{\diamond(c_1+c_2)}=\big((c_1+c_2).\gamma,\varepsilon\big)=\big(c_1.\gamma+c_2.\gamma,\varepsilon\big)=\big(c_1.\gamma,\varepsilon\big)\big(c_2.\gamma,\varepsilon\big)=\big(\gamma,\varepsilon\big)^{\diamond c_1}\big(\gamma,\varepsilon\big)^{\diamond c_2}\,.$$

Indeed, the property above is true in general, as we can readily check that

$$\begin{aligned}\big(\gamma,\alpha\big)^{\diamond(c_1+c_2)}&=\sum_{j\geq 0}\binom{c_1+c_2}{j}\Big((\gamma,\alpha)-(\varepsilon,\varepsilon)\Big)^j\\ &=\left(\sum_{j\geq 0}\binom{c_1}{j}\Big((\gamma,\alpha)-(\varepsilon,\varepsilon)\Big)^j\right)\left(\sum_{j\geq 0}\binom{c_2}{j}\Big((\gamma,\alpha)-(\varepsilon,\varepsilon)\Big)^j\right)\\ &=\big(\gamma,\alpha\big)^{\diamond c_1}\big(\gamma,\alpha\big)^{\diamond c_2}.\end{aligned}$$

7. Ordinary Riordan arrays

In this section we present the umbral syntax to handle ordinary Riordan arrays. We name ordinary Riordan array an infinite lower triangular matrix $[\gamma,\alpha]$ defined by

$$[\gamma,\alpha]_{n,k} \simeq \frac{(\gamma+k.\alpha)^{n-k}}{(n-k)!}. \tag{36}$$

Since $n![\gamma,\alpha]_{n,k}=k!(\gamma,\alpha)_{n,k}$ then all results of Section 3 and the recursions in Section 4 can be easily adapted to these arrays. For instance, from the identity (13) we have

$$\frac{(\gamma+\eta.\beta.\alpha_{\mathcal{D}})^n}{n!} \simeq \sum_{k=0}^{n} \frac{(\gamma+k.\alpha)^{n-k}}{(n-k)!}\frac{\eta^k}{k!}, \tag{37}$$

and this explains how the Fundamental Theorem 3.1 applies to ordinary arrays. More explicitly, if $[\gamma,\alpha]\boldsymbol{a}=\boldsymbol{b}$ then the kth entry of the column vector $\boldsymbol{a}$ is $\frac{E[\eta^k]}{k!}$, and the nth entry of $\boldsymbol{b}$ is $\frac{E[(\gamma+\eta.\beta.\alpha_{\mathcal{D}})^n]}{n!}$. Moreover, elementary computations shows that

$$(\gamma,\alpha)=(\sigma,\rho)(\tau,\pi) \text{ if and only if } [\gamma,\alpha]=[\sigma,\rho][\tau,\pi],$$

so that the set of all ordinary arrays is a group under the usual matrix multiplication. The identity is $[\varepsilon,\varepsilon]$ and the inverse of an array is determined by means of

$$[\gamma,\alpha]^{-1}=[\mathfrak{L}_{\gamma,\alpha},\mathfrak{L}_{\alpha}].$$

If $\mathbf{1}$ has all the entries equal to 1 then $[\gamma,\alpha]\mathbf{1}$ stores the row-sums of $[\gamma,\alpha]$. Then, from $\bar{\upsilon}^n\simeq n!$ we obtain the following analogue of (21),

$$\frac{(\gamma+\bar{\upsilon}.\beta.\alpha_{\mathcal{D}})^n}{n!} \simeq \sum_{k=0}^{n}[\gamma,\alpha]_{n,k}.$$

The rules (28), (29) and (30) translate into the recursions below,

$$[\gamma,\alpha]_{n,k} \simeq \sum_{i=0}^{n-k}\frac{\alpha^i}{i!}[\gamma,\alpha]_{n-1-i,k-1} \quad n,k\geq 1\,, \tag{38}$$

$$[\gamma,\alpha]_{n,k} \simeq \sum_{i=0}^{n-k}\frac{\mathfrak{K}_{\alpha}^i}{i!}[\gamma,\alpha]_{n-1,k-1+i} \quad n,k\geq 1\,, \tag{39}$$

and

$$[\gamma,\alpha]_{n,k} \simeq \sum_{i=0}^{n-k}\frac{(k.\mathfrak{K}_{\alpha})^i}{i!}[\gamma,\alpha]_{n-k,i} \quad n,k\geq 1\,. \tag{40}$$

As in the exponential setting we may define the ordinary Sheffer sequence of $[\gamma,\alpha]$ to be the polynomial sequence $\big(\mathfrak{s}_n(x)\big)$ such that

$$\mathfrak{s}_n(x)=\sum_{k=0}^{n}[\gamma,\alpha]_{n,k}x^k.$$

By applying (37) we obtain

$$\mathfrak{s}_n(x)\simeq n!\,(\gamma+\bar{\upsilon}.x.\beta.\alpha_{\mathcal{D}})^n\,.$$

The umbral composition of the Sheffer sequences of $[\gamma,\alpha]$ with the Sheffer sequence of $[\sigma,\rho]$ is the Sheffer sequence of $[\gamma,\alpha][\sigma,\rho]$. The identity sequence (x^n) is the Sheffer sequence of $[\varepsilon,\varepsilon]$. In particular, the ordinary Riordan group $\mathfrak{Rio}'$ and the ordinary Sheffer group $\mathfrak{Sheff}'$ are isomorphic, and both of them are isomorphic to their exponential analogue. Note that the isomorphism between $\mathfrak{Rio}'$ and $\mathfrak{Sheff}'$ is one of the main results in [13] (see also [30]). The ordinary Appell sequences and the ordinary associated sequences correspond to arrays of type $[\gamma,\varepsilon]$ and $[\varepsilon,\alpha]$ respectively. Finally, the ordinary Sheffer identity says that

$$\mathfrak{s}_n(z)=\sum_{k=0}^{n}\mathfrak{s}_k(x)\mathfrak{p}_{n-k}(y),$$

whenever $z^k=\sum_{i=0}^{n}x^iy^{k-i}$ for all $k=1,2,\dots,n$, and with $\big(\mathfrak{p}_n(x)\big)$ denoting the sequence of $[\varepsilon,\alpha]$.

By applying (8) it is easily seen that the Pascal array can be also realized as the ordinary array $[\bar{\upsilon},\bar{\upsilon}]$. In the same fashion as in the exponential setting, where (28), (29) and (30) reduce to trivial identities, the equivalences (38), (39) and (40) provides well known recursive properties of the binomial coefficients. In fact, by comparing (7) and (8) we have $n(k\boldsymbol{.}\bar{\upsilon})^{n-k}\simeq k(n\boldsymbol{.}\chi)^{n-k}$. By virtue of (17) this provides $\mathfrak{K}_{\bar{\upsilon}}\equiv\chi$ and the recursive identities are

$$\binom{n}{k} = \sum_{i=0}^{n-k}\binom{n-1-i}{k-1}, \tag{41}$$

$$\binom{n}{k} = \binom{n-1}{k-1}+\binom{n-1}{k}, \tag{42}$$

$$\binom{n}{k} = \sum_{i=0}^{n-k}\binom{k}{i}\binom{n-k}{i}. \tag{43}$$

Moreover, from $\mathfrak{L}_{\bar{\upsilon}}\equiv -1\boldsymbol{.}\chi$ we obtain

$$[\bar{\upsilon},\bar{\upsilon}]^{-1}=[-1\boldsymbol{.}\chi,-1\boldsymbol{.}\chi]. \tag{44}$$

We now consider an important ordinary array that involves Catalan numbers. Let ς be an umbra with moments $\varsigma\simeq n!C_n$, $n\geq 1$, with $C_n=\frac{1}{n+1}\binom{2n}{n}$ denoting the nth Catalan number. From (16) we have $(n+1)(\mathfrak{L}_{\chi\boldsymbol{.}-1})^n\simeq\big((n+1)\boldsymbol{.}\bar{\upsilon}\big)^n\simeq(2n)_n$. This way, we have $\mathfrak{L}_{\chi\boldsymbol{.}-1}\equiv\varsigma$, then $\mathfrak{L}_{\varsigma}\equiv\chi\boldsymbol{.}-1$ and finally $\mathfrak{K}_{\varsigma}\equiv\bar{\upsilon}$. Via (28), (29) and (30) we obtain the recursive properties of the array $[\varsigma,\varsigma]$,

$$[\varsigma,\varsigma]_{n,k}=\sum_{i=0}^{n-k}C_i[\varsigma,\varsigma]_{n-1-i,k-1}, \tag{45}$$

$$[\varsigma,\varsigma]_{n,k}=\sum_{i=0}^{n-k}[\varsigma,\varsigma]_{n-1,k-1+i},$$

$$[\varsigma,\varsigma]_{n,k}=\sum_{i=0}^{n-k}\binom{k+i-1}{i}[\varsigma,\varsigma]_{n-k,i}.$$

Also, we have

$$[\varsigma,\varsigma]^{-1}=[\chi\boldsymbol{.}-1,\chi\boldsymbol{.}-1]. \tag{46}$$

Moreover, via (17) and (36) we obtain an explicit formula for $[\varsigma,\varsigma]_{n,k}$. In fact, from $(n+1)\big((k+1)\boldsymbol{.}\varsigma\big)^{n-k}\simeq(k+1)\big((n+1)\boldsymbol{.}\bar{\upsilon}\big)^{n-k}\simeq(k+1)(2n-k)_n$ we have

$$[\varsigma,\varsigma]_{n,k}=\frac{k+1}{n+1}\binom{2n-k}{n}. \tag{47}$$

These numbers are often called the ballot numbers and have interesting combinatorial insights [1]. The first rows of the array are

$$[\varsigma,\varsigma]=\begin{pmatrix}1&&&&&\\1&1&&&&\\2&2&1&&&\\5&5&3&1&&\\14&14&9&4&1&\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Since $\bar{\upsilon}\equiv\bar{\upsilon}_{\mathcal{D}}$, by means of (11) and (14) we have $(\varsigma+\bar{\upsilon}{\boldsymbol .}\beta{\boldsymbol .}\varsigma_{\mathcal{D}})_{\mathcal{D}}\equiv\bar{\upsilon}{\boldsymbol .}\beta{\boldsymbol .}\varsigma_{\mathcal{D}}\equiv\mathfrak{K}_{\varsigma}{\boldsymbol .}\beta{\boldsymbol .}\varsigma_{\mathcal{D}}\equiv\varsigma$. Hence, the row-sums of $[\varsigma,\varsigma]$ are given by

$$C_{n+1}=\sum_{k=0}^{n}[\varsigma,\varsigma]_{n,k}.$$

8. A Catalan triangle, the Fibonacci numbers and the Chebyshev polynomials

Applications of Riordan arrays mainly concern combinatorial problems [4, 5, 25, 28]. In this section we apply the umbral syntax to discuss some properties of the Riordan array $[2{\boldsymbol .}\varsigma,2{\boldsymbol .}\varsigma]$ whose combinatorial insights have been discussed in several papers [5, 24, 27]. We will recover most of their main properties by underlying some connections with ballot numbers, Fibonacci numbers and the Chebyshev polynomials of the second kind. First of all, note that from (47) we have $[\varsigma,\varsigma]_{n+1,1}=C_{n+1}$ and that by virtue of (38) with $k=1$ we recover the well-known convolution formula for the Catalan numbers,

$$C_{n+1}=\sum_{i=0}^{n}C_iC_{n-i}.$$

In particular, this says that $2{\boldsymbol .}\varsigma\equiv\varsigma+\bar{\upsilon}{\boldsymbol .}\beta{\boldsymbol .}\varsigma_{\mathcal{D}}$ so that

$$[2{\boldsymbol .}\varsigma,2{\boldsymbol .}\varsigma]=[\varsigma,\varsigma][\bar{\upsilon},\bar{\upsilon}].$$

Then, $[2{\boldsymbol .}\varsigma,2{\boldsymbol .}\varsigma]$ can be obtained by multiplying $[\varsigma,\varsigma]$ by the Pascal array $[\bar{\upsilon},\bar{\upsilon}]$ on its right,

$$[2{\boldsymbol .}\varsigma,2{\boldsymbol .}\varsigma]_{n,k}=\sum_{i=k}^{n}\binom{i}{k}[\varsigma,\varsigma]_{n,i}.$$

In particular, via (44) and (46) we obtain a nice expression for $[2{\boldsymbol .}\varsigma,2{\boldsymbol .}\varsigma]^{-1}$,

$$[2{\boldsymbol .}\varsigma,2{\boldsymbol .}\varsigma]^{-1}=[-1{\boldsymbol .}\chi,-1{\boldsymbol .}\chi][\chi{\boldsymbol .}-1,\chi{\boldsymbol .}-1].$$

However, a more explicit expression of $[2{\boldsymbol .}\varsigma,2{\boldsymbol .}\varsigma]^{-1}$ may be obtained as follows. Note that $(2n{\boldsymbol .}\chi)^{n-1}\simeq n(2{\boldsymbol .}\varsigma)^{n-1}$. Hence, by virtue of (17) we obtain $\mathfrak{K}_{2.\varsigma}\equiv 2{\boldsymbol .}\chi$ and then $\mathfrak{L}_{2.\varsigma}\equiv -2{\boldsymbol .}\chi$. Now, by applying (17) again, we have $(n+1)\big((k+1){\boldsymbol .}\varsigma\big)^{n-k}\simeq(k+1)\big((n+1){\boldsymbol .}2{\boldsymbol .}\chi\big)^{n-k}$ and the entries of the array are

$$[2{\boldsymbol .}\varsigma,2{\boldsymbol .}\varsigma]_{n,k}=\frac{k+1}{n+1}\binom{2n+2}{n-k}.$$

This allows us to compute the first rows of the array

$$[2{\boldsymbol .}\varsigma,2{\boldsymbol .}\varsigma]=\begin{pmatrix}1&&&&&\\2&1&&&&\\5&4&1&&&\\14&14&6&1&&\\42&48&27&8&1&\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}.$$

Thanks to (39), (38) and (40) we obtain the following recursions,

$$[2.\varsigma, 2.\varsigma]_{n,k} = [2.\varsigma, 2.\varsigma]_{n-1,k-1} + 2\,[2.\varsigma, 2.\varsigma]_{n-1,k} + [2.\varsigma, 2.\varsigma]_{n-1,k+1},$$

$$[2.\varsigma, 2.\varsigma]_{n,k} = \sum_{i=0}^{k} C_{i+1}\,[2.\varsigma, 2.\varsigma]_{n-1-i,k-1}$$

and

$$[2.\varsigma, 2.\varsigma]_{n,k} = \sum_{i=0}^{k} \binom{2k}{i} [2.\varsigma, 2.\varsigma]_{n-k,i}.$$

Now, we are interested in the Sheffer sequence $\big(\mathfrak{s}_n(x)\big)$ of $[2.\varsigma, 2.\varsigma]^{-1} = [-2.\chi, -2.\chi]$ satisfying

$$[2.\varsigma, 2.\varsigma] \begin{pmatrix} 1 \\ \mathfrak{s}_1(x) \\ \mathfrak{s}_2(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \end{pmatrix}. \tag{48}$$

Since $(-2(k+1).\chi)^{n-k} \simeq (-1)^{n-k}(n+k+1)_{n-k}$, we obtain

$$\mathfrak{s}_n(x) = \sum_{k=0}^{n} \binom{n+k+1}{n-k} (-1)^{n-k} x^k\,. \tag{49}$$

Such a polynomial sequence plays an important role in mathematics. In fact, from the umbral equivalence $n!\mathfrak{s}_n(x) \simeq (-2.\chi + \bar{\upsilon}.x.\beta.(-2.\chi)_{\mathcal{D}})^n$ we have the g.f.

$$1 + \sum_{n\geq 1} \mathfrak{s}_n(x) z^n = \frac{1}{(1+z)^2 - xz} = \frac{1}{1-(x-2)z+z^2}.$$

It is now easy to recognize that $\mathfrak{s}_n(x) = U_n\left(\frac{x-2}{2}\right)$ with $\big(U_n(x)\big)$ denoting the sequence of Chebyshev polynomials of the second kind [6]. It is not difficult to see that $\big(\mathfrak{s}_n(x)\big)$ satisfies the three-term recursion

$$\mathfrak{s}_n(x) = (x-2)\mathfrak{s}_{n-1}(x) - \mathfrak{s}_{n-2}(x), \quad n \geq 2,$$

with initial values $\mathfrak{s}_0(x) = 1$ and $\mathfrak{s}_1(x) = x - 2$. Such a polynomial sequence specializes to integers whenever x is integer, and then (48) specializes to interesting matrix identities. Since the change of variable $(x-2) \mapsto (2-x)$ is such that $\mathfrak{s}_n(x) \mapsto (-1)^n \mathfrak{s}_n(x)$, then it is sufficient to study the case $x - 2 \geq 0$.

Example 1 (Periodic sequences). If $x = 2$ then the recursion $\mathfrak{s}_n(1) = -\mathfrak{s}_{n-2}(1)$, with initial values $\mathfrak{s}_0(1) = 1$ and $\mathfrak{s}_1(1) = 0$, is solved by the period 4 sequence $1, 0, -1, 0$. More explicitly we have

$$\begin{pmatrix} 1 &&&&& \\ 2 & 1 &&&& \\ 5 & 4 & 1 &&& \\ 14 & 14 & 6 & 1 && \\ 42 & 48 & 27 & 8 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2^2 \\ 2^3 \\ 2^4 \\ \vdots \end{pmatrix}.$$

When $x = 3$ we have $\mathfrak{s}_n(3) = \mathfrak{s}_{n-1}(3) - \mathfrak{s}_{n-2}(3)$, with $\mathfrak{s}_0(3) = \mathfrak{s}_1(3) = 1$. It is not difficult to see that the recursion is satisfied by the period 6 sequence $1, 1, 0, -1, -1, 0$. We have

$$\begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 3^2 \\ 3^3 \\ 3^4 \\ \vdots \end{pmatrix}.$$

These two cases are better understood by recalling that $U_n(\cos y) = \frac{\sin(n+1)y}{\sin y}$ (see for instance [6]). Hence, we have $\mathfrak{s}_n(2) = U_n\left(\cos\frac{\pi}{2}\right)$ and $\mathfrak{s}_n(3) = U_n\left(\cos\frac{\pi}{3}\right)$ and this explains the periodic sequences.

Example 2 (Natural numbers). When $x = 4$ we have $\mathfrak{s}_n(4) = 2\mathfrak{s}_{n-1}(4) - \mathfrak{s}_{n-2}$, with $\mathfrak{s}_0(4) = 1$ and $\mathfrak{s}_1(4) = 2$. We obtain $\mathfrak{s}_n(4) = n + 1$ and then

$$\begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 4^2 \\ 4^3 \\ 4^4 \\ \vdots \end{pmatrix}.$$

In other terms, we have

$$4^n = \sum_{k=0}^{n} \frac{(k+1)^2}{n+1}\binom{2n+2}{n-k}.$$

Such an identity was proved by several authors and it is the motivation of the combinatorial approach of Chen et al. [5].

Example 3 (Fibonacci numbers). When $x = 5$ the recurrence $\mathfrak{s}_n(5) = 3\mathfrak{s}_{n-1}(5) - \mathfrak{s}_{n-2}$, with $\mathfrak{s}_0(5) = 1$ and $\mathfrak{s}_1(5) = 3$, is solved by the subsequence of Fibonacci numbers $\mathfrak{s}_n(5) = F_{2n+2}$. Then, the following identity relates Catalan numbers and Fibonacci numbers,

$$\begin{pmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 5 & 4 & 1 & & & \\ 14 & 14 & 6 & 1 & & \\ 42 & 48 & 27 & 8 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 8 \\ 21 \\ 55 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 5^2 \\ 5^3 \\ 5^4 \\ \vdots \end{pmatrix}.$$

Equivalently,

$$5^n = \sum_{k=0}^{n} \frac{k+1}{n+1}\binom{2n+2}{n-k} F_{2k+2}. \tag{50}$$

By taking into account (42), (49), we may invert (50) and obtain

$$F_{2n+2} = (-1)^n \sum_{k=0}^{n} \left(\binom{n+k+2}{n-k} - \binom{n+k+1}{n-k-1} \right) (-5)^k.$$

Moreover, being $F_{2n+3} = F_{2(n+1)+2} - F_{2n+2}$, then we deduce an analogous formula for the Fibonacci numbers of an odd index,

$$F_{2n+1} = (-1)^n \sum_{k=0}^{n} \left(\binom{n+k+2}{n-k} - \binom{n+k}{n-k-2} \right) (-5)^k.$$

Formulae relating Fibonacci numbers and binomial coefficients are already known (see for instance [4], where a reformulation of an identity of Andrews that is analogous to the formulae above is proved by means of Riordan arrays).

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