The Fibonacci Quarterly 1981 (19,5): 391-393

$$32L_{2t}^5F_n^5 - F_{n+2t}^5 - F_{n-2t}^5 = 10L_{2t}F_nF_{n-2t}F_{n+2t}(F_{n+2t}^2 + 2L_{2t}F_nF_{n-2t})$$
$$32L_{2t}^5L_n^5 - L_{n+2t}^5 - L_{n-2t}^5 = 10L_{2t}L_nL_{n-2t}L_{n+2t}(L_{n+2t}^2 + 2L_{2t}L_nL_{n-2t})$$

Type IV

$$32L_r^5F_n^5 - F_{n+r}^5 - F_{n-r}^5 = 10L_rF_nF_{n-r}F_{n+r}(F_{n+r}^2 + 2L_rF_nF_{n-r})$$
$$32L_r^5L_n^5 - L_{n+r}^5 - L_{n-r}^5 = 10L_rL_nL_{n-r}L_{n+r}(L_{n+r}^2 + 2L_rL_nL_{n-r})$$

14. (a) Fibonacci-Lucas identity used: $L_n^3 = 2F_{n-1}^3 + F_n^3 + 6F_{n+1}^2F_{n-1}$
(b) Type I extension: $b^3L_n^3 = 2F_{n-1}^3 + a^3F_n^3 + 6F_{n+1}^2F_{n-1}$
(c) Generalizations:

Type I

$$F_{2r+1}^3L_n^3 = 2F_{n-2r-1}^3 + L_{2r+1}^3F_n^3 + 6F_{n+2r+1}^2F_{n-2r-1}$$
$$D^3F_{2r+1}^3F_n^3 = 2L_{n-2r-1}^3 + L_{2r+1}^3L_n^3 + 6L_{n+2r+1}^2L_{n-2r-1}$$
$$F_{2r}^3L_n^3 = L_{2r}^3F_n^3 - 2F_{n-2r}^3 - 6F_{n+2r}^2F_{n-2r}$$
$$D^3F_{2r}^3F_n^3 = L_{2r}^3L_n^3 - 2L_{n-2r}^3 - 6L_{n+2r}^2L_{n-2r}$$

Type II

$$F_r^3L_n^3 = L_r^3F_n^3 - 2F_{n-r}^3 - 6F_{n+r}^2F_{n-r}$$
$$D^3F_r^3F_n^3 = L_r^3L_n^3 - 2L_{n-r}^3 - 6L_{n+r}^2L_{n-r}$$

Type III

$$4F_{2r+1}^3L_n^3 = F_{n-2r-1}^3 + 4L_{2r+1}^3F_n^3 + 3F_{n+2r+1}^2F_{n-2r-1}$$
$$4D^3F_{2r+1}^3F_n^3 = L_{n-2r-1}^3 + 4L_{2r+1}^3L_n^3 + 3L_{n+2r+1}^2L_{n-2r-1}$$
$$4F_{2r}^3L_n^3 = 4L_{2r}^3F_n^3 - F_{n-2r}^3 - 3F_{n+2r}^2F_{n-2r}$$
$$4D^3F_{2r}^3F_n^3 = 4L_{2r}^3L_n^3 - L_{n-2r}^3 - 3L_{n+2r}^2L_{n-2r}$$

Type IV

$$4F_r^3L_n^3 = 4L_r^3F_n^3 - F_{n-r}^3 - 3F_{n+r}^2F_{n-r}$$
$$4D^3F_r^3F_n^3 = 4L_r^3L_n^3 - L_{n-r}^3 - 3L_{n+r}^2L_{n-r}$$

Concluding Remarks

Following the suggestions of the referee and the editor, the proofs of the 14 identity sets have been omitted. They are tedious and do involve complicated, albeit fairly elementary, calculations. For some readers, the proofs would involve the use of composition algebras which are not developed in the article and which may not be well known.

The author has completed a supplementary paper giving, with indicated proof, the Type I, Type II, Type III, and Type IV composition algebras. After each composition albegra the corresponding identities using that algebra have been stated and proved. Copies of this paper may be obtained by request from the author.

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A FORMULA FOR TRIBONACCI NUMBERS

CARL P. McCARTY
LaSalle College, Philadelphia, PA 19141

In a recent paper [2], Scott, Delaney, and Hoggatt discussed the Tribonacci numbers T_n defined by

$$T_0 = 1,\ T_1 = 1,\ T_2 = 2 \quad \text{and} \quad T_n = T_{n-1} + T_{n-2} + T_{n-3},\ \text{for } n \geq 3,$$

and found its generating function, which is written here in terms of the complex variable z, to be

$$f(z) = \frac{1}{1 - z - z^2 - z^3} = \sum_{n=0}^{\infty} T_n z^n. \tag{1}$$

In this brief note, a formula for T_n is found by means of an analytic method similar to that used by Hagis [1].

Observe that

$$z^3 + z^2 + z - 1 = (z - r)(z - s)(z - \bar{s}), \tag{2}$$

where $r = .5436890127$,
$s = -.7718445064 + 1.115142580i$,
$|s| = 1.356203066$,
and
$|r - s| = 1.724578573$;

thus $f(z)$ is meromorphic with simple poles at the points $z = r$, $z = s$, and $z = \bar{s}$, all of which lie within an annulus centered at the origin with inner radius of .5 and outer radius of 2.

By the Cauchy integral theorem,

$$T_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_{|z| = .5} \frac{f(z)\,dz}{z^{n+1}},$$

and by the Cauchy residue theorem,

$$T_n = \frac{1}{2\pi i} \int_{|z| = R} \frac{f(z)\,dz}{z^{n+1}} - (R_1 + R_2 + R_3), \tag{3}$$

where $R \geq 2$ and R_1, R_2, and R_3 are the residues of $f(z)/z^{n+1}$ at the poles r, s, and $\bar{s}$, respectively.

In particular, since $f(z) = -1/((z - r)(z - s)(z - \bar{s}))$,

$$\begin{aligned} R_1 &= \lim_{z \to r} (z - r)f(z)/z^{n+1} = -1/((r - s)(r - \bar{s})r^{n+1}) \\ &= -1/(|r - s|^2 r^{n+1}), \end{aligned} \tag{4}$$

$$R_2 = \lim_{z \to s} (z - s)f(z)/z^{n+1} = -1/((s - r)(s - \bar{s})s^{n+1}), \tag{5}$$

and

$$R_3 = \lim_{z \to \bar{s}} (z - \bar{s})f(z)/z^{n+1} = -1/((\bar{s} - r)(\bar{s} - s)\bar{s}^{n+1}) = \bar{R}_2. \tag{6}$$

Along the circle $|z| = R \geq 2$ we have

$$|f(z)| = \frac{1}{|z^3 + z^2 + z - 1|} \leq \frac{1}{||z|^3 - |z^2 + z - 1||} \leq \frac{1}{R^3 - R^2 - R - 1},$$

hence

$$\left| \frac{1}{2\pi i} \int_{|z| = R} \frac{f(z)\,dz}{z^{n+1}} \right| \leq \frac{1}{R\,(R^3 - R^2 - R - 1)}. \tag{7}$$

Now, if R is taken arbitrarily large, then from (3) and (7) it follows that

$$T_n = -(R_1 + R_2 + R_3). \tag{8}$$

One final estimate is needed to obtain the desired formula. From (5) we have for $n \geq 0$,

$$|R_2| = \frac{1}{|s - r||s - \overline{s}||s|^{n+1}} = \frac{1}{2|s - r||\mathrm{Im}\ s||s|^{n+1}} < .26/|s|^{n+1} < .2,$$

which along with (8) and (6) implies

$$T_n + R_1 = -R_2 - R_3,$$

so

$$|T_n + R_1| = |R_2 + R_3| \leq 2|R_2| < .4;$$

hence

$$T_n - .4 < -R_1 < T_n + .4$$

or, equivalently,

$$T_n < -R_1 + .4 < T_n + 1.$$

Substituting the value of R_1 from (4) into (9) we may rewrite (9) in terms of the greatest integer function and obtain the desired formula:

$$T_n = \left[\frac{1}{|r - s|^2 r^{n+1}} + .4\right].$$

REFERENCES

1. P. Hagis. "An Analytic Proof of the Formula for F_n," *The Fibonacci Quarterly* 2 (1964):267-68.
2. A. Scott, T. Delaney, & V. E. Hoggatt, Jr. "The Tribonacci Sequence." *The Fibonacci Quarterly* 15 (1977):193-200.

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POLYNOMIALS ASSOCIATED WITH GEGENBAUER POLYNOMIALS

A. F. HORADAM
University of New England, Armidale, N.S.W., Australia
S. PETHE
University of Malaya, Kuala Lumpur 22-11, Malaysia

1. INTRODUCTION

Chebyshev polynomials $T_n(x)$ of the first kind and $U_n(x)$ of the second kind are, respectively, defined as follows:

$$T_n(x) = \cos(n \cos^{-1}x) \qquad (|x| \leq 1),$$

$$U_n(x) = \frac{\sin[(n + 1)\cos^{-1}x]}{\sin(\cos^{-1}x)} \qquad (|x| \leq 1).$$

In 1974 Jaiswal [6] investigated polynomials $p_n(x)$ related to $U_n(x)$. In 1977 Horadam [5] obtained similar results for polynomials $q_n(x)$, associated with $T_n(x)$. The polynomials $p_n(x)$ and $q_n(x)$ are defined as follows:

$$\text{(1)} \quad \begin{cases} p_n(x) = 2xp_{n-1}(x) - p_{n-3}(x) \quad (n \geq 3) \text{ with} \\ p_0(x) = 0,\ p_1(x) = 1,\ p_2(x) = 2x \end{cases}$$

and

$$\text{(2)} \quad \begin{cases} q_n(x) = 2xq_{n-1}(x) - q_{n-3}(x) \quad (n \geq 3) \text{ with} \\ q_0(x) = 0,\ q_1(x) = 2,\ q_2(x) = 2x. \end{cases}$$