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Asymptotic Behavior of Block Toeplitz Matrices and Determinants

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1. INTRODUCTION

A Toeplitz matrix is one of the form (φ_{i-j}) where i and j run through some index set and the φ_j are complex numbers. A block Toeplitz matrix is similar except that the φ_j are themselves square matrices of a fixed order r. This paper is concerned with asymptotic properties of the finite block Toeplitz matrices

$$(\varphi_{i-j}) \qquad 0 \leqslant i, j \leqslant N \tag{1.1}$$

and their determinants.

We shall write

$$\varphi(z) = \sum_{j=-\infty}^{\infty} \varphi_j z^j \qquad (|z| = 1).$$

(This notation will be used consistently throughout the paper; if φ is a scalar- or matrix-valued function then φ_j denotes its jth Fourier, or Laurent, coefficient which is either a number or a matrix.) The corresponding block Toeplitz matrix (1.1) will be denoted by $T_N[\varphi]$ and its determinant by $D_N[\varphi]$; the semi-infinite block Toeplitz matrix (φ_{i-j}), $0 \leqslant i, j < \infty$ will be denoted by $T[\varphi]$.

The earliest asymptotic result for block Toeplitz determinants seems to have been that of Gyires [8] who showed that if $\varphi(z)$ is continuous and positive definite for $|z| = 1$ then

$$\lim_{N\to\infty} D_N[\varphi]^{1/N} = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\det\varphi(e^{i\theta})\,d\theta\right\}.$$

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Hirschman [11] investigated the asymptotic behavior of $T_N[\varphi]^{-1}$ (for not necessarily Hermitian φ) and his results imply that under certain conditions, including

$$\det \varphi(e^{i\theta}) \neq 0, \qquad \mathop{\Delta}_{0\leqslant\theta\leqslant 2\pi} \arg \det \varphi(e^{i\theta}) = 0, \tag{1.2}$$

the limit of $D_N[\varphi]/D_{N-1}[\varphi]$ exists and is also given by

$$G[\varphi] = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log \det \varphi(e^{i\theta})\, d\theta\right\}.$$

Here log denotes any continuous logarithm, whose existence is guaranteed by (1.2).

If we define $\tilde{\varphi}(z) = \varphi(z^{-1})$ then Hirschman's assumptions (aside from smoothness conditions) were equivalent to $T[\varphi]$ and $T[\tilde{\varphi}]$ being invertible as operators (acting on the left) on an appropriate sequence space. The conditions (1.2) are necessary, but in the matrix case not sufficient, for this.

In the present paper it will be shown that (again aside from smoothness assumptions) the conditions (1.2) imply the existence of the limit $E[\varphi] = \lim_{N\to\infty} D_N[\varphi]/G[\varphi]^{N+1}$ and that this limit is nonzero if and only if $T[\varphi]$ and $T[\tilde{\varphi}]$ are both invertible. The proof will be based on (what is really more fundamental) an asymptotic inversion formula for $T_N[\varphi]^{-1}$.

In the scalar case $E[\varphi]$ is given by the formula

$$E[\varphi] = \exp\left\{\sum_{j=1}^{\infty} j(\log \varphi)_j (\log \varphi)_{-j}\right\}. \tag{1.3}$$

This was proved first by Szegö [15] for positive φ and subsequently extended by many authors. We shall see that in the matrix case a formula similar to this,

$$E[\varphi] = \exp\left\{r^{-1} \sum_{j=1}^{\infty} j(\log \det \varphi)_j (\log \det \varphi)_{-j}\right\}, \tag{1.4}$$

is sometimes correct (it is if φ is a scalar function times a matrix function extending analytically to an invertible matrix function inside or outside the unit circle), but is unfortunately usually wrong. In some applications [12] block Toeplitz determinants arise where only finitely many of the coefficients φ_j are nonzero, and here $E[\varphi]$ can be evaluated explicitly. In fact it will be shown that if all the φ_j vanish for $j < -\alpha$ (or for $j > \alpha$) then $E[\varphi] = D_{\alpha-1}[\varphi^{-1}]\, G[\varphi]^{\alpha}$.

Although we have not been able to find a general expression for $E[\varphi]$, we have evaluated the Fréchet derivative of $\log E[\varphi]$. In the scalar case this is easily integrated to give (1.3). Even in the matrix case $\log E[\varphi]$ can be expressed as an integral, but this is not entirely satisfactory. Note that in the scalar case $\log E[\varphi]$ is a bilinear function of the two sequences

$$(\log \varphi)_j \qquad 1 \leqslant j < \infty$$
$$(\log \varphi)_{-j} \qquad 1 \leqslant j < \infty.$$

Such bilinearity, plus computation of simple special cases, is enough to deduce (1.3). Certain analogous bilinearity relation will be established in the matrix. These are used to establish (1.4) in the cases described and may perhaps give someone an idea of what $E[\varphi]$ is in general.

The last part of the paper is concerned with the limiting behavior of the eigenvalues of $T_N[\varphi]$ when all but finitely many of the φ_j vanish. In the scalar case Schmidt and Spitzer [14] found the set of limit points of the eigenvalues as $N \to \infty$ and Hirschman [10] refined this by describing the limiting distribution of the eigenvalues. We extend these results here. As will be seen the introduction of a modicum of potential theory permits a considerable simplification, even in the scalar case.

Added in proof. A general expression for $E[\varphi]$ will be derived in a forthcoming paper.

2. Factorization of Matrix Functions

The theory of semi-infinite block Toeplitz matrices was developed by Gohberg and Krein in [5]. The basic assumption on φ was that it belonged to l_1 in the sense that

$$\sum_{j=-\infty}^{\infty} \| \varphi_j \| < \infty.$$

Here what norm is used on the $r \times r$ matrices is irrelevant, but because of what comes later it is most convenient to use the Hilbert-Schmidt norm. The infinite matrix $T[\varphi]$ may be thought of as an operator, acting on the left, on various spaces of sequences

$$f = \{f_0, f_1, \ldots\}$$

of r-vectors, including l_1. It was shown in [5] that whichever of these spaces is taken $T[\varphi]$ is a Fredholm operator (the null space has finite dimension and the range is closed and has finite codimension) if and only if

$$\det \varphi(e^{i\theta}) \neq 0 \tag{2.1}$$

and that its index (dimension of its null space minus codimension of its range) is equal to

$$-\underset{0\leqslant\theta\leqslant 2\pi}{\Delta} \arg \det \varphi(e^{i\theta}).$$

Crucial to the investigation of invertibility was a certain factorization of matrix functions analogous to the Wiener-Hopf factorization for scalar functions. It was shown that any φ belonging to $\hat{l}_1$ and satisfying (2.1) possessed a "right standard factorization" of the form

$$\varphi(z) = u^-(z) \begin{bmatrix} z^{\kappa_1} & & 0 \\ & \ddots & \\ 0 & & z^{\kappa_r} \end{bmatrix} u^+(z);$$

here $u^+(z)^{\pm 1}$ and $u^-(z)^{\pm 1}$ belong to $\hat{l}_1$, all Fourier coefficients of $u^+(z)^{\pm 1}$ with negative values of the index vanish, and all Fourier coefficients of $u^-(z)^{\pm 1}$ with positive values of the index vanish. The "right exponents" $\kappa_1, \ldots, \kappa_r$ are (except for ordering) uniquely determined integers.

Similarly there is a left standard factorization

$$\varphi(z) = v^+(z) \begin{bmatrix} z^{\kappa_1'} & & 0 \\ & \ddots & \\ 0 & & z^{\kappa_r'} \end{bmatrix} v^-(z).$$

The left exponents $\kappa_1', \ldots, \kappa_r'$ are generally different from the right exponents. Indeed one set of exponents may vanish but not the other.

Once one has these factorizations it is easy to see that $T[\varphi]$ is invertible if and only if φ satisfies (2.1) and all the right exponents of φ vanish. Since the right exponents of $\tilde{\varphi}(z) = \varphi(z^{-1})$ equal the negatives of the left exponents of φ the simultaneous invertibility of $T[\varphi]$ and $T[\tilde{\varphi}]$ is equivalent to the vanishing of all right and left exponents of φ.

Now we shall be concerned with a smaller class of matrix functions φ. We write

$$\| \varphi \| = \sum_{j=-\infty}^{\infty} \| \varphi_j \| + \left\{ \sum_{j=-\infty}^{\infty} |j| \, \| \varphi_j \|^2 \right\}^{1/2}$$

$$A = \{\varphi : \| \varphi \| < \infty\}.$$

It is clear that A is a Banach space with norm as given. Moreover it is not hard to prove the identity

$$\sum_{j=-\infty}^{\infty} |j| \| \varphi_j \|^2 = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\| \varphi(e^{i\theta_1}) - \varphi(e^{i\theta_2})\|^2}{| e^{i\theta_1} - e^{i\theta_2} |^2} d\theta_1 \, d\theta_2$$

from which it can be deduced that A is even a Banach algebra under pointwise matrix multiplication.

Since $A \subset l_1$ any element of A satisfying (2.1) possesses right and left standard factorizations, and by a theorem of Gohberg [4] the eight functions

$$u^{\pm}(z)^{\pm 1}, \qquad v^{\pm}(z)^{\pm 1} \tag{2.2}$$

all belong to A.

If $T[\varphi]$ is invertible as an operator on a space of sequences of vectors then it is also invertible on the corresponding space of sequences of matrices. In particular if $\varphi \in A$ and all right exponents of φ vanish then $T[\varphi]$ is an invertible operator on the space of sequences of $r \times r$ matrices $F = \{F_0, F_1, \ldots\}$ satisfying

$$\sum_{j=0}^{\infty} \| F_j \| + \left\{ \sum_{j=1}^{\infty} j \| F_j \|^2 \right\}^{1/2} < \infty.$$

It is not hard to see that in the right standard factorization $\varphi = u^- u^+$, which is not unique, one may take $u^+(z)$ to be the inverse of the matrix function whose Fourier coefficients with negative values of the index vanish and whose sequence of Fourier coefficients with nonnegative values of the index is given by $T[\varphi]^{-1} \{I, 0, 0, \ldots\}$ where I denotes the $r \times r$ identity matrix. This implies that $u^{\pm}$, and similarly the other functions of (2.2), may be chosen so that they vary continuously with φ, a fact which will be useful later.

Let us define $A_0 = \{\varphi \in A\colon (1.2) \text{ holds}\}$, $A_1 = \{\varphi \in A_0 : \text{all right and left exponents of } \varphi \text{ vanish}\}$. Alternatively A_0 consists of those φ for which $T[\varphi]$ is Fredholm of index zero (as an operator, say, on l_1 sequences of r-vectors) and A_1 consists of those φ for which $T[\varphi]$ and $T[\tilde{\varphi}]$ are both invertible. These are both open subsets of A.

It does not seem possible to find a simple necessary and sufficient condition that an arbitrary $\varphi \in A_0$ belong to A_1. However if the Fourier series of φ is infinite on one side only there is such a condition. The following lemma proves its sufficiency; its necessity will be a consequence of Theorems 4.1 and 5.1(a). (Pattanayak found a similar but more complicated characterization in case the entries of φ were rational functions of z).

LEMMA 2.1. *Suppose* $\varphi \in A_0$ *and* $\varphi_j = 0$ *for* $j < -\alpha$ *or for* $j > \alpha$. *Then* $D_{\alpha-1}[\varphi^{-1}] \neq 0$ *implies* $\varphi \in A_1$.

Proof. Suppose $D_{\alpha-1}[\varphi^{-1}] \neq 0$ and $\varphi_j = 0$ for $j < -\alpha$ but that not all the right exponents of φ vanish. Then $T[\varphi]$ is a noninvertible Fredholm operator of index zero so it must have a nontrivial null space. If $f = \{f_0, f_1, ...\} \neq 0$ belongs to this null space and we set

$$f(z) = \sum_{j=0}^{\infty} f_j z^j$$

then the Fourier coefficients of $\varphi(z) f(z)$ with nonnegative values of the index all vanish. Together with the assumption that $\varphi_j = 0$ for $j < -\alpha$ this implies that $\varphi(z) f(z)$ is of the form $\varphi(z)f(z) = g_{-1}z^{-1} + \cdots + g_{-\alpha}z^{-\alpha}$. But then since $f(z) = \varphi(z)^{-1} (g_{-1}z^{-1} + \cdots + g_{-\alpha}z^{-\alpha})$ the Fourier coefficients of the function on the right with negative values of the index all vanish; in particular

$$\sum_{j=1}^{\alpha} (\varphi^{-1})_{j-i}\, g_{-j} = 0, \qquad i = 1,..., \alpha. \tag{2.3}$$

Since $f \neq 0$ not all the g_{-j} vanish. Thus (2.3) implies $D_{\alpha-1}[\varphi^{-1}] = 0$, a contradiction.

If we apply the immediately preceding argument to the matrix function $\varphi(z)^t$ (t = transpose) we deduce that it has all right exponents zero. Thus $\varphi(z)$ has all left exponents zero.

Finally if $\varphi_j = 0$ for $j > \alpha$ then by what has just been shown $\tilde{\varphi} \in A_1$, so also $\varphi \in A_1$.

It should be mentioned that the above proof used the equalities $D_{\alpha-1}[\varphi] = D_{\alpha-1}[\varphi^t] = D_{\alpha-1}[\tilde{\varphi}]$ which are easily established.

LEMMA 2.2. *The set* A_1 *is connected.*

Proof. Since the Laurent polynomials

$$\varphi(z) = \sum_{j=-\alpha}^{\alpha} \varphi_j z^j$$

are dense in A and A_1 is open, the Laurent polynomials in A_1 are dense in A_1. We shall show that each such polynomial may be joined to a constant invertible matrix function by a curve lying in A_1. (The constant invertible matrices themselves form a connected set.)

Let φ have right factorization u^-u^+. Note that the Fourier coefficients of $u^- = \varphi(u^+)^{-1}$ with index less than $-\alpha$ all vanish. For $|t| < 1$ define $\varphi(t) = \varphi(t, z) = u^-(t^{-1}z)\, u^+(tz)$. Clearly $t \to \varphi(t)$ is a continuous function from the unit disc to A and each $\varphi(t) \in A_0$. By the preceding lemma $\varphi(t) \in A_1$ if $D_{\alpha-1}[\varphi(t)^{-1}] \neq 0$. This determinant is an analytic function of t on the open unit disc and is nonzero for $t = 0$, when φ is a constant invertible matrix function. Hence there is a curve joining $t = 0$ to $t = 1$ at no point of which (except possibly $t = 1$ itself) the determinant is zero. The entire curve $\varphi(t)$ then belongs to A_1.

The next lemma shows that almost all sufficiently small punctured discs centered at a point of A_0 lie entirely in A_1.

LEMMA 2.3. *Let $\varphi \in A_0$. Then for every $\psi \in A$ with the exception of those belonging to a nowhere dense subset the matrix functions $\varphi + \epsilon\psi$ belong to A_1 for all sufficiently small nonzero ϵ.*

Proof. Consider first the special case

$$\varphi(z) = \begin{bmatrix} z^{\kappa_1} & & 0 \\ & \ddots & \\ 0 & & z^{\kappa_r} \end{bmatrix}$$

where, since we assume $\varphi \in A_0$,

$$\kappa_1 + \cdots + \kappa_r = 0. \tag{2.4}$$

Suppose $\psi \in A$ is such that $\varphi + \epsilon\psi$ has right exponents not all zero for a nonzero sequence $\epsilon \to 0$. For each ϵ of this sequence $T[\varphi + \epsilon\psi]$ has a nontrivial null space. Thus there are $f_\epsilon \in l_1$ such that

$$T[\varphi + \epsilon\psi]\, f_\epsilon = 0, \qquad f_\epsilon = \{f_{\epsilon,0}, f_{\epsilon,1}, \ldots\}, \quad \sum_{j=0}^{\infty} \|f_{\epsilon,j}\| = 1. \tag{2.5}$$

From (2.5) it follows that $\lim_{\epsilon\to0} T[\varphi] f_\epsilon = 0$ and so

$$\lim_{\epsilon\to0} \sum_{j=\kappa}^{\infty} \|f_{\epsilon,j}\| = 0, \qquad \kappa = \max(-\kappa_i).$$

By taking a subsequence if necessary we may assume each sequence $\{f_{\epsilon,j}\}$ with $0 \leqslant j < \kappa$ converges. Thus f_ϵ converges to some $f \in l_1$ satisfying $T[\varphi]f = 0, f \neq 0$. Moreover since each $T[\psi]f_\epsilon$ belongs to the range of $T[\varphi]$, by (2.5), and since $T[\varphi]$ has closed range, we deduce that $T[\psi]f$ belongs to the range of $T[\varphi]$.

Assume the diagonal entries of $\varphi(z)$ ordered so that $\kappa_1 < 0,..., \kappa_p < 0$, $\kappa_{p+1} \geqslant 0,..., \kappa_r \geqslant 0$. Then the most general element of the null space of $T[\varphi]$ is the sequence of Fourier coefficients of a vector function of the form

$$\begin{bmatrix} \pi_1(z) \\ \vdots \\ \pi_p(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.6}$$

where each π_i is a polynomial of degree less than $-\kappa_i = |\kappa_i|$, $\pi_i = a_{i,0} + \cdots + a_{i,|\kappa_1|-1}\, z^{|\kappa_1|-1}$. Suppose $\psi(z) = (\psi_{i,j}(z))$ $(1 \leqslant i, j \leqslant r)$. Then with f equal to the sequence of Fourier coefficients of (2.6), $T[\psi]f$ belongs to the range of $T[\varphi]$ if and only if each $\psi_{p+i,1}(z)\, \pi_1(z) + \cdots + \psi_{p+i,p}(z)\, \pi_p(z)$ $(i = 1,..., r - p)$ has vanishing Fourier coefficients for values of the index between 0 and $\kappa_{p+i} - 1$ (inclusive).

This gives a system of $\kappa_{p+1} + \cdots + \kappa_r$ homogeneous linear equations in the $a_{i,j}$ of which there are $|\kappa_1| + \cdots + |\kappa_p|$ and by (2.4) this equals $\kappa_{p+1} + \cdots + \kappa_r$. Thus the existence of a nonzero vector f in the null space of $T[\varphi]$ such that $T[\psi]f$ is in the range of $T[\varphi]$ is equivalent to the vanishing of a determinant whose entries are Fourier coefficients of entries of the matrix function $\psi(z)$, no two of the entries of the determinant being the same Fourier coefficient of the same entry of $\psi(z)$. The set of $\psi \in A$ for which such a determinant can vanish is clearly a closed nowhere dense subset of A.

Next consider an arbitrary $\varphi \in A_0$, and suppose it has a right standard factorization

$$\varphi(z) = u^-(z) \begin{bmatrix} z^{\kappa_1} & & 0 \\ & \ddots & \\ 0 & & z^{\kappa_r} \end{bmatrix} u^+(z).$$

Since $\varphi + \epsilon\psi$ has right exponents zero if and only if $(u^-)^{-1}(\varphi + \epsilon\psi)(u^+)^{-1}$ does, it follows from what has already been shown that $\varphi + \epsilon\psi$ has right exponents zero for all sufficiently small nonzero ϵ as long as $(u^-)^{-1}\psi(u^+)^{-1}$ does not belong to a certain nowhere dense set, or equivalently as long as ψ itself does not belong to a certain nowhere dense set. (Note that left resp. right multiplication by u^- resp. u^+ is a homeomorphism of A.)

Similarly $\varphi + \epsilon\psi$ has left exponents zero for all sufficiently small nonzero ϵ as long as ψ does not belong to some other nowhere dense set, and the lemma is established.

3. Inversion of $T_N[\varphi]$

In this section we show that if $\varphi \in A_1$ then $T_N[\varphi]$ is invertible for N sufficiently large and we give an approximation to $T_N[\varphi]^{-1}$. This approximation will be in the trace norm. Recall that the trace norm of an operator T on a Hilbert space ($T_N[\varphi]$ is an operator on a finite dimensional Hilbert space) is equal to the sum of the eigenvalues of $(T^*T)^{1/2}$. This norm is denoted by $\| T \|_1$ and dominates the Hilbert–Schmidt norm $\| T \|_2$ and so also the ordinary operator norm $\| T \|_\infty$. There are also the inequalities

$$\| T_1 T_2 \|_1 \leqslant \| T_1 \|_1 \| T_2 \|_\infty$$

$$\| T_1 T_2 \|_1 \leqslant \| T_1 \|_2 \| T_2 \|_2 .$$

All this may be found in [6, Chapter II], for example.

If φ belongs to A_1 then so does φ^{-1}. Therefore it has left and right standard factorizations

$$\varphi(z)^{-1} = u^+(z)\, u^-(z) = v^-(z)\, v^+(z). \tag{3.1}$$

This notation will be retained for the rest of the section. We define $U_N[\varphi]$ to be the block matrix whose i, j entry is

$$(\varphi^{-1})_{i-j} - \sum_{m=1}^{\infty} u^+_{i+m} u^-_{-j-m} - \sum_{m=1}^{\infty} v^-_{-N+i-m} v^+_{N-j+m} . \tag{3.2}$$

The motivation for this definition is that for i or j fairly far from N (3.2) is close to the i, j entry of the inverse of the semi-infinite block Toeplitz matrix (φ_{i-j}) $0 \leqslant j, j < \infty$ while for i or j fairly large (3.2) is close to the i, j entry of the inverse of the semi-infinite matrix (φ_{i-j}) $-\infty < i$, $j \leqslant N$.

THEOREM 3.1. *If $\varphi \in A_1$ then $T_N[\varphi]$ is invertible for N sufficiently large and $\lim_{N\to\infty} \| T_N[\varphi]^{-1} - U_N[\varphi]\|_1 = 0$. The convergence is uniform for φ belonging to any compact subset of A_1.*

Proof. We shall see that the block matrix $T_N[\varphi]\, U_N[\varphi] - I$ (where I is the identity matrix) has i, j entry

$$\sum_{n=1}^{\infty} \varphi_{i-n-N} \sum_{m=1}^{\infty} u^+_{N+n+m} u^-_{-j-m} + \sum_{n=1}^{\infty} \varphi_{i+n} \sum_{m=1}^{\infty} v^-_{-N-n-m} v^+_{N-j+m} . \tag{3.3}$$

Granting this for the moment let us deduce the assertions of the theorem.

Look at the first term of (3.3). It is the i, j entry of the product of three Hilbert-Schmidt matrices having respective Hilbert-Schmidt norms

$$\left\{ \sum_{\substack{0\leqslant i\leqslant N\\ 1\leqslant n<\infty}} \| \varphi_{i-n-N} \|^2 \right\}^{1/2} \leqslant \left\{ \sum_{k=1}^{\infty} k \| \varphi_{-k} \|^2 \right\}^{1/2} \leqslant \| \varphi \|$$

$$\left\{ \sum_{n,m=1}^{\infty} \| u^+_{N+n+m} \|^2 \right\}^{1/2} \leqslant \left\{ \sum_{k=N}^{\infty} k \| u_k^+ \|^2 \right\}^{1/2}$$

$$\left\{ \sum_{\substack{1\leqslant n<\infty\\ 0\leqslant j\leqslant N}} \| u^-_{-j-m} \|^2 \right\}^{1/2} \leqslant \left\{ \sum_{k=1}^{\infty} k \| u_k^- \|^2 \right\}^{1/2} \leqslant \| u^- \|.$$

The trace norm of the matrix whose i, j entry is given by the first term of (3.3) is therefore at most $\| \varphi \| \, \| u^- \| \, \{\sum_{k=N}^{\infty} k \| u_k^+ \|^2\}^{1/2}$ which tends to zero as $N \to \infty$.

Similarly the trace norm of the matrix whose i, j entry is given by the second term of (3.3) is at most $\| \varphi \| \, \| v^+ \| \, \{\sum_{k=N}^{\infty} k \| v^-_{-k} \|^2\}^{1/2}$ which also tends to zero. Therefore $\lim_{N\to\infty} \| T_N[\varphi]\, U_N[\varphi] - I \|_1 = 0$. It follows that $T_N[\varphi]$ is invertible for sufficiently large N and that its inverse is given by the Neumann series

$$T_N[\varphi]^{-1} = U_N[\varphi] + U_N[\varphi] \sum_{s=1}^{\infty} (I - T_N[\varphi]\, U_N[\varphi])^s.$$

Now $\| U_N[\varphi]\|_\infty$ is bounded as $N \to \infty$. In fact

$$\| T_N[\varphi^{-1}]\|_\infty \leqslant \sum_{j=-\infty}^{\infty} \|(\varphi^{-1})_j \| \leqslant \| \varphi^{-1} \|,$$

the second term of (3.2) is the i, j entry of a matrix having trace norm (and so operator norm) at most

$$\left\{\sum_{k=1}^{\infty} k \| u_k^+ \|^2\right\}^{1/2} \left\{\sum_{k=1}^{\infty} k \| u_{-k}^- \|^2\right\}^{1/2} \leqslant \| u^+ \| \| u^- \|$$

and similarly the last term of (3.2) is the i, j entry of a matrix having operator norm at most $\| v^- \| \| v^+ \|$.

Hence $U_N[\varphi] \sum_{s=1}^{\infty} (I - T_N[\varphi] U_N[\varphi])^s$ has trace norm tending to zero as $N \to \infty$ and this gives the first conclusion of the theorem.

To prove the asserted uniformity we use the fact, remarked in the last section, that the factors $u^{\pm}$, $v^{\pm}$ may be chosen so as to vary continuously with $\varphi \in A_1$. Then on any compact subset of A_1 the norms $\| \varphi^{\pm 1} \|$, $\| u^{\pm} \|$, $\| v^{\pm} \|$ are uniformly bounded, and by Dini's theorem the sequences

$$\sum_{k=N}^{\infty} k \| u_k^+ \|^2, \qquad \sum_{k=N}^{\infty} k \| v_{-k}^- \|^2$$

tend uniformly to zero as $N \to \infty$. The last statement of the theorem is now easily verified.

It remains to check that the i, j entry of $T_N[\varphi]\ U_N[\varphi] - I$ is given by (3.3). The i, j entry of $T_N[\varphi]\ U_N[\varphi]$ equals

$$\sum_{n=0}^{N} \varphi_{i-n}(\varphi^{-1})_{n-j} - \sum_{n=0}^{N} \varphi_{i-n} \sum_{m=1}^{\infty} u_{n+m}^+ u_{-j-m}^- - \sum_{n=0}^{N} \varphi_{i-n} \sum_{m=1}^{\infty} v_{-N+n-m}^- v_{N-j+m}^+ . \tag{3.4}$$

Since

$$\sum_{n=-\infty}^{\infty} \varphi_{i-n}(\varphi^{-1})_{n-j} = \delta_{i,j}$$

(the $r \times r$ zero matrix for $i \neq j$, the identity matrix for $i = j$) the first term of (3.4) is equal to

$$\delta_{i,j} - \sum_{n=-\infty}^{-1} \varphi_{i-n}(\varphi^{-1})_{n-j} - \sum_{n=N+1}^{\infty} \varphi_{i-n}(\varphi^{-1})_{n-j} . \tag{3.5}$$

It follows from (3.1) that

$$(\varphi^{-1})_{i-j} = \sum_{m=-\infty}^{\infty} u_{i+m}^{+} u_{-j-m}^{-} = \sum_{m=-\infty}^{\infty} v_{-j-m}^{-} v_{i+m}^{+} .$$

Substituting these expressions for the Fourier coefficients of φ^{-1} appearing in (3.5) and using the facts $u_k^+ = v_k^+ = 0$ for $k < 0$, $u_k^- = v_k^- = 0$ for $k > 0$, we see that (3.5) is equal to

$$\delta_{i,j} - \sum_{n=-\infty}^{-1} \varphi_{i-n} \sum_{m=1}^{\infty} u_{n+m}^{+} u_{-j-m}^{-} - \sum_{n=N+1}^{\infty} \varphi_{i-n} \sum_{m=1}^{\infty} v_{-N+n-m}^{-} v_{N-j+m}^{+} .$$

All this is equal to the first term of (3.4). Hence $T_N[\varphi]\, U_N[\varphi] - I$ has i, j entry

$$- \sum_{n=-\infty}^{N} \varphi_{i-n} \sum_{m=1}^{\infty} u_{n+m}^{+} u_{-j-m}^{-} - \sum_{n=0}^{\infty} \varphi_{i-n} \sum_{m=1}^{\infty} v_{-N+n-m}^{-} v_{N-j+m}^{+} . \tag{3.6}$$

Now

$$\sum_{n=-\infty}^{\infty} \varphi_{i-n} u_{n+m}^{+}$$

equals the $i + m$ Fourier coefficient of $\varphi(z)\, u^+(z) = u^-(z)^{-1}$ and so it vanishes whenever $m \geqslant 1$, $i \geqslant 0$. Similarly

$$\sum_{n=-\infty}^{\infty} \varphi_{i-n} v_{-N+n-m}^{-}$$

equals the $i - m - N$ Fourier coefficient of $u^+(z)^{-1}$ and so it vanishes whenever $m \geqslant 1$, $i \leqslant N$. This implies that for $0 \leqslant i \leqslant N$ the expressions (3.3) and (3.6) are equal, and this is what was to be established.

4. Existence and First Properties of $E[\varphi]$

We retain the notation of the last section.

THEOREM 4.1. *If $\varphi \in A_1$ the limit*

$$E[\varphi] = \lim_{N\to\infty} D_N[\varphi]/G[\varphi]^{N+1}$$

exists, is nonzero, and is a continuous function of φ. If $t \to \varphi(t)$ is a differentiable function from a real closed interval to A_1 then

$$\log E[\varphi(t)]$$

is a differentiable function of t with derivative equal to

$$-\mathrm{tr} \sum_{i=-\infty}^{\infty} \varphi_i' \sum_{m=1}^{\infty} m u_m^{+} u_{-i-m}^{-} - \mathrm{tr} \sum_{i=-\infty}^{\infty} \varphi_{-i}' \sum_{m=1}^{\infty} m v_{-m}^{-} v_{i+m}^{+} .$$

(The dependence on t of the last expression is not displayed. The prime denotes differentiation with respect to t.)

Proof. We begin with the well known fact [7] that for any matrix T depending differentiably on a real parameter t

$$\frac{d}{dt} \log \det T = \mathrm{tr}\, T' T^{-1}. \tag{4.1}$$

This gives in our case, for a differentiable function $t \to \varphi(t)$ into A_1,

$$\frac{d}{dt} \log D_N[\varphi] = \mathrm{tr}\, T_N[\varphi']\, T_N[\varphi]^{-1}.$$

It follows from Theorem 3.1, the inequality

$$|\, \mathrm{tr}\, T \,| \leqslant \|\, T \,\|_1 , \tag{4.2}$$

and the boundedness of $\|\, T_N[\varphi'] \|_\infty$ for all N and t that

$$\mathrm{tr}\, T_N[\varphi']\, T_N[\varphi]^{-1} - \mathrm{tr}\, T_N[\varphi']\, U_N[\varphi]$$

tends to zero as $N \to \infty$ uniformly in t.

Corresponding to the three terms comprising $U_N[\varphi]$ (see (3.2)) are three terms comprising $\mathrm{tr}\, T_N[\varphi']\, U_N[\varphi]$. The first of these is

$$\begin{aligned} \mathrm{tr} & \sum_{i,j=0}^{N} \varphi_{j-i}' (\varphi^{-1})_{i-j} \\ &= \mathrm{tr} \sum_{i=-N}^{N} (N + 1 - |\, i \,|)\, \varphi_i' (\varphi^{-1})_{-i} \\ &= \mathrm{tr}(N+1) \sum_{i=-\infty}^{\infty} \varphi_i' (\varphi^{-1})_{-i} - \mathrm{tr}(N+1) \sum_{|i|>N} \varphi_i' (\varphi^{-1})_{-i} \\ &\qquad - \mathrm{tr} \sum_{i=-\infty}^{\infty} |\, i \,|\, \varphi_i' (\varphi^{-1})_{-i} + \mathrm{tr} \sum_{|i|>N} |\, i \,|\, \varphi_i' (\varphi^{-1})_{-i} . \end{aligned}$$

The first term on the right is exactly

$$(N+1)\,\mathrm{tr}\,\frac{1}{2\pi}\int_0^{2\pi}\varphi'(e^{i\theta})\,\varphi(e^{i\theta})^{-1}\,d\theta$$

(recall that φ' denotes the derivative of $\varphi = \varphi(t, z)$ with respect to the parameter t) which equals

$$(N+1)\,\frac{d}{dt}\,\frac{1}{2\pi}\int_0^{2\pi}\log\det\varphi(e^{i\theta})\,d\theta$$

by (4.1) once again. The second term on the right has absolute value at most (using (4.2) again and recalling that we have been using the Hilbert-Schmidt norm on the $r \times r$ matrices)

$$\sum_{|i|>N} |\,i\,|\,\|\,\varphi_i'\,\|\,\|(\varphi^{-1})_{-i}\,\|$$

which tends to zero as $N \to \infty$ uniformly in t, by Dini's theorem. Similarly the last term tends to zero.

We have shown that the first of the three terms comprising tr $T_N[\varphi']$ $U_N[\varphi]$ equals

$$(N+1)\,\frac{d}{dt}\log G[\varphi] - \mathrm{tr}\sum_{i=-\infty}^{\infty} |\,i\,|\,\varphi_i'(\varphi^{-1})_{-i} \tag{4.3}$$

plus a function of t tending uniformly to zero as $N \to \infty$.

The second of the three terms is the trace of

$$-\sum_{i,j=0}^{N}\varphi_{j-i}'\sum_{m=1}^{\infty}u_{i+m}^{+}u_{-j-m}^{-}$$
$$= -\sum_{j=-N}^{N}\varphi_j'\sum_{i=\max(0,-j)}^{N-j}\;\sum_{m=1}^{\infty}u_{i+m}^{+}u_{-i-j-m}^{-}\,. \tag{4.4}$$

If $j < 0$ and $i < -j$ then

$$\sum_{m=1}^{\infty}u_{i+m}^{+}u_{-i-j-m}^{-} = \sum_{m=-\infty}^{\infty}u_{i+m}^{+}u_{-i-j-m}^{-} = (\varphi^{-1})_{-j}\,.$$

Therefore the right side of (4.4) equals

$$-\sum_{j=-N}^{N}\varphi_j'\sum_{i=0}^{N-j}\sum_{m=1}^{\infty}u_{i+m}^{+}u_{-i-j-m}^{-} + \sum_{j=-N}^{-1}|\,j\,|\,\varphi_j'(\varphi^{-1})_{-j}\,. \tag{4.5}$$

Since

$$\sum_{j=-\infty}^{\infty} \| \varphi_j' \| \sum_{i=0}^{\infty} \sum_{m=1}^{\infty} \| u_{i+m}^{+} \| \| u_{-i-j-m}^{-} \|$$

$$= \sum_{j=-\infty}^{\infty} \| \varphi_j' \| \sum_{m=1}^{\infty} m \| u_m{}^{+} \| \| u_{-j-m}^{-} \|$$

converges uniformly in t, and similarly for

$$\sum_{j=-\infty}^{-1} | j | \| \varphi_j' \| \|(\varphi^{-1})_{-j} \|,$$

the trace of (4.5) differs from

$$-\mathrm{tr} \sum_{j=-\infty}^{\infty} \varphi_j' \sum_{i=0}^{\infty} \sum_{m=1}^{\infty} u_{i+m}^{+} u_{-i-j-m}^{-} + \mathrm{tr} \sum_{j=-N}^{-1} | j | \varphi_j'(\varphi^{-1})_{-j}$$

$$= -\mathrm{tr} \sum_{j=-\infty}^{\infty} \varphi_j' \sum_{m=1}^{\infty} m u_m{}^{+} u_{-j-m}^{-} + \mathrm{tr} \sum_{j=-N}^{-1} | j | \varphi_j'(\varphi^{-1})_{-j}$$

by a function of t tending uniformly to zero as $N \to \infty$.

Similarly the last of the three terms comprising tr $T_N[\varphi'] \; U_N[\varphi]$ differs from

$$-\mathrm{tr} \sum_{j=-\infty}^{\infty} \varphi_{-j}' \sum_{m=1}^{\infty} m v_{-m}^{-} v_{j+m}^{+} + \mathrm{tr} \sum_{j=-N}^{-1} | j | \varphi_{-j}'(\varphi^{-1})_j$$

by a function tending uniformly to zero. Combining the results for each of the three terms comprising tr $T_N[\varphi'] \; U_N[\varphi]$ shows that this, and so also $(d/dt) \log D_N[\varphi]$, differs from

$$(N+1) \frac{d}{dt} \log G[\varphi] - \mathrm{tr} \sum_{i=-\infty}^{\infty} \varphi_i' \sum_{m=1}^{\infty} m u_m{}^{+} u_{-i-m}^{-}$$

$$- \mathrm{tr} \sum_{i=-\infty}^{\infty} \varphi_{-i}' \sum_{m=1}^{\infty} m v_{-m}^{-} v_{i+m}^{+}$$

by a quantity tending to zero uniformly in t. Hence

$$\lim_{N\to\infty} \frac{d}{dt} \log \frac{D_N[\varphi]}{G[\varphi]^{N+1}}$$

$$= -\mathrm{tr} \sum_{i=-\infty}^{\infty} \varphi_i' \sum_{m=1}^{\infty} m u_m{}^{+} u_{-i-m}^{-} - \mathrm{tr} \sum_{i=-\infty}^{\infty} \varphi_{-i}' \sum_{m=1}^{\infty} m v_{-m}^{-} v_{i+m}^{+} \qquad (4.6)$$

uniformly in t.

It follows from Lemma 2.2 that an arbitrary $\varphi \in A_1$ may be joined to the identity I of A by a piecewise linear curve $\varphi(t)$ lying entirely in A_1. Since

$$\lim_{N\to\infty} \log \frac{D_N[I]}{G[I]^{N+1}}$$

trivially exists, and since

$$\lim_{N\to\infty} \frac{d}{dt} \log \frac{D_N[\varphi(t)]}{G[\varphi(t)]^{N+1}}$$

exists uniformly on each linear piece of the curve it follows that

$$\lim_{N\to\infty} \log \frac{D_N[\varphi]}{G[\varphi]^{N+1}}$$

itself exists. We call it, of course, $\log E[\varphi]$. That $(d/dt) \log E[\varphi(t)]$ is what it is claimed to be follows from (4.6).

To prove that E, or $\log E$, is continuous at each point φ_0 of A_1, let φ_1 be a nearby point. Write $\varphi(t) = (1-t)\varphi_0 + t\varphi_1$ $(0 \leqslant t \leqslant 1)$ and apply the mean value theorem to $\log E[\varphi(t)]$ whose derivative we know. Since $\varphi'(t) = \varphi_1 - \varphi_0$ we find that

$$|\log E[\varphi_1] - \log E[\varphi_0]| \leqslant M \| \varphi_1 - \varphi_0 \|$$

for some constant M if φ_1 is sufficiently close to φ_0. This completes the proof of the theorem.

Carrying the immediately preceding argument a little further shows that $\log E$ is a Fréchet differentiable function on A_1 and that its derivative at φ is the linear functional on A given by

$$\psi \to -\mathrm{tr} \sum_{i=-\infty}^{\infty} \psi_i \sum_{m=1}^{\infty} m u_m^{+} u_{-i-m}^{-} - \mathrm{tr} \sum_{i=-\infty}^{\infty} \psi_{-i} \sum_{m=1}^{\infty} m v_{-m}^{-} v_{i+m}^{+}.$$

Equivalently if we define, for $\psi^1, \psi^2 \in A$,

$$\langle \psi^1, \psi^2 \rangle = \mathrm{tr} \sum_{j=1}^{\infty} j \psi_j^{1} \psi_{-j}^{2}$$

then the linear functional is $\psi \to -\langle u^+, u^-\psi \rangle - \langle v^+\psi, v^- \rangle$.

Suppose now that we are in the scalar case, so that in particular $v^+ = u^+$, $v^- = u^-$ and $A_1 = A_0$. If $\varphi \in A_0$ then any continuously determined $\log \varphi$ belongs to A, by the Arens-Calderón form of the Wiener-Lévy theorem [1]. Define

$$(\log \varphi)^+(z) = \sum_{j=0}^{\infty} (\log \varphi)_j \, z^j, \qquad (\log \varphi)^-(z) = \sum_{j=-\infty}^{-1} (\log \varphi)_j \, z^j.$$

Then as our factors $u^{\pm}$ we may take $u^+ = \exp\{-(\log \varphi)^+\}$, $u^- = \exp\{-(\log \varphi)^-\}$.

Differentiating the first of these relations with respect to z (for $|z| < 1$) and equating coefficients of equal powers of z give

$$-iu_i^+ = \sum_{j=0}^{\infty} j u_{i-j}^+ (\log \varphi)_j^+$$

from which it follows that for any $\psi \in A$, $-\langle u^+, \psi \rangle = \langle (\log \varphi)^+, u^+\psi \rangle$. Replacing ψ by $u^-\psi$ and using $u^+u^- = \varphi^{-1}$ give

$$-\langle u^+, u^-\psi \rangle = \langle (\log \varphi)^+, \varphi^{-1}\psi \rangle. \tag{4.7}$$

Similarly

$$-\langle u^+\psi, u^- \rangle = \langle \varphi^{-1}\psi, (\log \varphi)^- \rangle. \tag{4.8}$$

Now consider the function

$$\langle (\log \varphi)^+, (\log \varphi)^- \rangle \tag{4.9}$$

on A_0. This has Fréchet derivative $\psi \to \langle (\log \varphi)^+, \{(d/d\varphi)(\log \varphi)^-\}\psi \rangle + \langle \{(d/d\varphi)(\log \varphi)^+\}\psi, (\log \varphi)^- \rangle$. Since $\langle \psi^1, \psi^2 \rangle = 0$ whenever $\psi_j^1 = 0$ for all $j > 0$ or $\psi_j^2 = 0$ for all $j < 0$ the derivative may also be written

$$\begin{aligned}\psi \to {} & \langle (\log \varphi)^+, \{(d/d\varphi) \log \varphi\}\psi \rangle + \langle \{(d/d\varphi) \log \varphi\}\psi, (\log \varphi)^- \rangle \\ & = \langle (\log \varphi)^+, \varphi^{-1}\psi \rangle + \langle \varphi^{-1}\psi, (\log \varphi)^- \rangle.\end{aligned}$$

By (4.7) and (4.8) this is just $\psi \to -\langle u^+, u^-\psi \rangle - \langle u^+\psi, u^- \rangle$ which, as we have seen, is the Fréchet derivative of $\log E$.

Thus the function (4.9) has the same derivative as $\log E$. Since both functions are equal to zero at $\varphi = 1$ and A_0 is connected the functions are equal everywhere. This gives the identity (1.3). (The fact that (1.3) holds for $\varphi \in A_0$ was first proved by Hirschman [9].)

Let us now return to the matrix case and see what happens when φ belongs to A_0 but not necessarily to A_1.

THEOREM 4.2. *The limit*

$$E[\varphi] = \lim_{N\to\infty} D_N[\varphi]/G[\varphi]^{N+1}$$

exists for all $\varphi \in A_0$, *is a continuous function on* A_0, *and is nonzero if and only if* $\varphi \in A_1$.

Proof. To prove the first statement take any $\varphi \in A_0$ and let $\psi \in A$ be such that $\varphi + \epsilon\psi$ belongs to A_1 for sufficiently small nonzero ϵ. The existence of such a ψ is guaranteed by Lemma 2.3.

By Theorem 4.1

$$\lim_{N\to\infty} \frac{D_N[\varphi + \epsilon\psi]}{G[\varphi + \epsilon\psi]^{N+1}} = E[\varphi + \epsilon\psi]$$

uniformly for ϵ on the boundary of a sufficiently small disc centered at $\epsilon = 0$. But each $D_N[\varphi + \epsilon\psi]/G[\varphi + \epsilon\psi]^{N+1}$ is analytic on the entire disc. Therefore the limit $E[\varphi]$ exists at $\epsilon = 0$ also.

We also have, if ϵ is sufficiently small but positive,

$$E[\varphi] = \frac{1}{2\pi} \int_0^{2\pi} E[\varphi + \epsilon e^{i\theta}\psi]\, d\theta.$$

It follows from this, and the continuity of E on A_1, that E is continuous in a neighborhood of any $\varphi \in A_0$. Thus E is continuous on A_0.

To prove the last statement of the theorem it need only be shown that $E[\varphi] = 0$ whenever $\varphi \notin A_1$. Suppose for example that not all the right exponents of φ are zero so that φ^{-1} has a left standard factorization

$$\varphi(z)^{-1} = u^+(z) \begin{bmatrix} z^{\kappa_1} & & 0 \\ & \ddots & \\ 0 & & z^{\kappa_r} \end{bmatrix} u^-(z)$$

with some $\kappa_i \neq 0$. If n is sufficiently large

$$\sum_{j=-n}^{0} u_j^- z^j$$

will have nonvanishing determinant for $|z| \geqslant 1$. Define φ_n by

$$\varphi_n(z)^{-1} = u^+(z) \begin{bmatrix} z^{\kappa_1} & & 0 \\ & \ddots & \\ 0 & & z^{\kappa_r} \end{bmatrix} \sum_{j=-n}^{0} u_j^- z^j,$$

which is a left standard factorization. Then φ_n^{-1} does not belong to A_1 (since its left exponents are not all zero) and for any $\alpha \geqslant n + \max |\kappa_i|$ all the Fourier coefficients of φ_n^{-1} with values of the index less than $-\alpha$ will vanish. It follows from Lemma 2.1 that $D_{\alpha-1}[\varphi_n] = 0$. Since α is arbitrarily large this implies $E[\varphi_n] = 0$. Finally, letting $n \to \infty$ and using the continuity of E give $E[\varphi] = 0$.

5. Further Properties of $E[\varphi]$

We begin this section with a simple observation about block determinants. If we have any square block matrix of $r \times r$ matrices and any $r \times r$ matrix P then the determinant of the first matrix is unchanged if any block row is multiplied on the left by P and added to any other block row (since this amounts to left multiplying the given matrix by a matrix with determinant 1); similarly for columns if multiplication by P is on the right. Thus row and column operations are available if used with some care.

This is useful for the evaluation of block Toeplitz determinants because these operations lead to other block Toeplitz matrices. In fact consider the block Toeplitz matrix $T_N[\varphi]$ and denote its block rows by $r_0, \ldots, r_N$. If for each $i \geqslant \alpha$ we replace r_i by

$$r_i + P_1 r_{r-1} + \cdots + P_\alpha r_{i-\alpha} \tag{5.1}$$

then the resulting matrix has i, j entry $\varphi_{i-j} + P_1\varphi_{i-j-1} + \cdots + P_\alpha\varphi_{i-j-\alpha}$ for $i \geqslant \alpha$. This is exactly the i, j entry of the block Toeplitz matrix associated with $(I + P_1 z + \cdots + P_\alpha z^\alpha)\,\varphi(z)$. Next, if the columns of the new matrix are $c_0, \ldots, c_N$ and for each $j \geqslant \alpha$ the jth column is replaced by

$$c_j + c_{j-1}Q_{-1} + \cdots + c_{j-\alpha}Q_{-\alpha} \tag{5.2}$$

then the i, j entry of the resulting matrix is, for $i \geqslant \alpha$ and $j \geqslant \alpha$, just the i, j entry of the block Toeplitz matrix associated with $\psi(z) =$

$(I + P_1 z + \cdots + P_\alpha z^\alpha)\,\varphi(z)\,(I + Q_{-1}z^{-1} + \cdots + Q_{-\alpha}z^{-\alpha})$. Thus applying to $T_N[\varphi]$ these row and column operations yields a matrix of the form

$$\begin{bmatrix} T_{\alpha-1}[\varphi] & X \\ Y & T_{N-\alpha}[\psi] \end{bmatrix}$$

where X and Y are certain matrices.

Write

$$\begin{aligned} P(z) &= I + P_1 z + \cdots + P_\alpha z^\alpha, \\ Q(z) &= I + Q_{-1}z^{-1} + \cdots + Q_{-\alpha}z^{-\alpha} \end{aligned} \tag{5.3}$$

and let $P(z)$ and $Q(z)$ have inverses, as formal power series, $\sum_{i=0}^{\infty} p_i z^i$, $\sum_{i=0}^{\infty} q_{-i} z^{-i}$ respectively. These formal inverses always exist. We have $p_0 = q_0 = I$ and if we set $p_i = q_{-i} = 0$ for $i < 0$ then

$$\begin{aligned} p_i + P_1 p_{i-1} + \cdots + P_\alpha p_{i-\alpha} &= 0 \qquad (i > 0) \\ q_{-i} + q_{-i+1} Q_{-1} + \cdots + q_{-i+\alpha} Q_{-\alpha} &= 0 \qquad (i > 0). \end{aligned} \tag{5.4}$$

Now define U and V to be the rectangular block Toeplitz matrices

$$\begin{aligned} U &= (p_{i-j}) \qquad 0 \leqslant i \leqslant N,\ \ 0 \leqslant j \leqslant \alpha - 1 \\ V &= (q_{i-j}) \qquad 0 \leqslant i \leqslant \alpha - 1,\ \ 0 \leqslant j \leqslant N \end{aligned}$$

where $N \geqslant \alpha - 1$, and consider

$$\begin{bmatrix} U & T_N[\varphi] \\ 0 & V \end{bmatrix}. \tag{5.5}$$

Perform on this matrix exactly those row and column operations resulting in the operations (5.1) and (5.2) on $T_N[\varphi]$ as described above. Because of the identities (5.4) the new matrix has the form

$$\begin{bmatrix} \bar{U} & T_{\alpha-1}[\varphi] & X \\ 0 & Y & T_{N-\alpha}[\psi] \\ 0 & \bar{V} & 0 \end{bmatrix}$$

where $\bar{U}$ and $\bar{V}$ are the square block Toeplitz matrices

$$\bar{U} = (p_{i-j}), \qquad \bar{V} = (q_{i-j}) \qquad 0 \leqslant i \leqslant \alpha - 1,\ \ 0 \leqslant j \leqslant \alpha - 1.$$

Because $\bar{U}$ and $\bar{V}$ are block triangular the determinant of this matrix is easily reduced. One applies, 2α times, Laplace's theorem on the expansion of determinants by means of certain $r \times r$ minors and their conjugate minors. Recalling that $p_0 = q_0 = I$ one finds that the determinant is equal to the determinant of $T_{N-\alpha}[\psi]$, which is

$$D_{N-\alpha}[\psi] = D_{N-\alpha}[P\varphi Q].$$

This is one way of evaluating the determinant of (5.5). There is another way. Assume that $T_N[\varphi]$ is invertible and multiply the matrix (5.5) on the left by

$$\begin{bmatrix} I & 0 \\ -V & I \end{bmatrix}\begin{bmatrix} T_N[\varphi]^{-1} & 0 \\ 0 & I \end{bmatrix}$$

(here the various I's have appropriate orders), which has determinant $D_N[\varphi]^{-1}$. We obtain the matrix

$$\begin{bmatrix} T_N[\varphi]^{-1}U & I \\ -VT_N[\varphi]^{-1}U & 0 \end{bmatrix}$$

which has determinant det $V\,T_N[\varphi]^{-1}U$. Thus the determinant of (5.5) is also equal to $D_N[\varphi]$ det $VT_N[\varphi]^{-1}U$.

Since we have already shown the determinant of (5.5) to be equal to $D_{N-\alpha}[P\varphi Q]$, we have established the identity

$$\frac{D_{N-\alpha}[P\varphi Q]}{D_N[\varphi]} = \det VT_N[\varphi]^{-1}U. \tag{5.6}$$

Note that the determinant on the right is of order α.

We shall use the notations

$$A_0^+ = \{\varphi \in A_0 \colon \varphi_j = 0 \text{ for } j < 0\}$$
$$A_0^- = \{\varphi \in A_0 \colon \varphi_j = 0 \text{ for } j > 0\}.$$

Observe that if φ belongs to A_0^+ resp. A_0^- then the determinant of φ belongs to scalar valued ($r = 1$) A_0^+ resp. A_0^-, so the same is true of the inverse of this determinant. Consequently, the inverse of φ, which is computed using the determinant and cofactors, also belongs to A_0^+ resp. A_0^-. Thus $A_0^+ \cup A_0^- \subset A_1$.

In our applications of (5.6) the polynomials $P(z)$, $Q(z)$ will belong to A_0^+ resp. A_0^-. The formal series

$$\sum_{i=0}^{\infty} p_i z^i, \qquad \sum_{i=0}^{\infty} q_{-i} z^{-i}$$

are then just the Fourier series for $P(z)^{-1}$, $Q(z)^{-1}$ respectively.

LEMMA 5.1. *Suppose*

$$\varphi(z)^{-1} = u^+(z)\, u^-(z)$$

where $u^+ \in A_0{}^+$, $u^- \in A_0{}^-$ *and that*

$$Q(z) = Q_0 + Q_{-1}z^{-1} + \cdots + Q_{-\alpha}z^{-\alpha} \tag{5.7}$$

belongs to $A_0{}^-$. *Then we have the identity*

$$E[\varphi Q] = \frac{D_{\alpha-1}[Q^{-1}u^+]}{G[Q^{-1}u^+]^\alpha}\, E[\varphi]. \tag{5.8}$$

Proof. Suppose first that $Q_0 = I$ and $\varphi \in A_1$. Let us apply (5.6) with P equal to the identity matrix function. If we write $T_N[\varphi]^{-1} = (w_{i,j,N})$, $0 \leqslant i, j \leqslant N$, then we obtain

$$\frac{D_{N-\alpha}[\varphi Q]}{D_N[\varphi]} = \det\left(\sum_{i=0}^{N} q_{\mu-i} w_{i,\nu,N}\right)_{\mu,\nu=0,\ldots,\alpha-1}. \tag{5.9}$$

Now Theorem 3.1 tells us that

$$\lim_{N\to\infty} \| T_N[\varphi]^{-1} - U_N[\varphi] \|_1 = 0$$

and so certainly

$$\lim_{N\to\infty} \| T_N[\varphi]^{-1} - U_N[\varphi] \|_\infty = 0.$$

This implies that in the computation of the determinant on the right side of (5.9), replacing each $w_{i,j,N}$ by the i, j entry of $U_N[\varphi]$ results in an error tending to zero as $N \to \infty$.

Recall now the form (3.2) of $U_N[\varphi]$. For fixed μ, ν the sum

$$\sum_{i=0}^{N} q_{\mu-i} \sum_{m=1}^{\infty} v^-_{-N+i-m} v^+_{N-\nu+m}$$

has norm at most

$$\| q \| \, \| v^- \| \sum_{m=1}^{\infty} \| v^+_{N-\nu+m} \|$$

which tends to zero as $N \to \infty$. The other two terms of (3.2) contributing to the i, ν entry of $U_N[\varphi]$ are

$$(\varphi^{-1})_{i-\nu} - \sum_{m=1}^{\infty} u_{i+m}^{+} u_{-\nu-m}^{-}$$

$$= \sum_{m=-\infty}^{\infty} u_{i+m}^{+} u_{-\nu-m}^{-} - \sum_{m=1}^{\infty} u_{i+m}^{+} u_{-\nu-m}^{-} = \sum_{m=0}^{\infty} u_{i-m}^{+} u_{-\nu+m}^{-} .$$

Consequently the right side of (5.9) converges, as $N \to \infty$, to the determinant

$$\det \left(\sum_{i,m=0}^{\infty} q_{\mu-i} u_{i-m}^{+} u_{-\nu+m}^{-} \right)_{\mu,\nu=0,1,\ldots,\alpha-1} .$$

Since $u_{-\nu+m}^{-}$ vanishes for $m > \nu$, and so certainly for $m \geqslant \alpha$, we may write

$$\sum_{i,m=0}^{\infty} q_{\mu-i} u_{i-m}^{+} u_{-\nu+m}^{-} = \sum_{m=0}^{\alpha-1} \left(\sum_{i=0}^{\infty} q_{\mu-i} u_{i-m}^{+} \right) u_{m-\nu}^{-} .$$

Therefore the last determinant is equal to the product of the two $\alpha \times \alpha$ block determinants

$$\det \left(\sum_{i=0}^{\infty} q_{\mu-i} u_{i-m}^{+} \right)_{\mu,m=0,\ldots,\alpha-1}$$

$$\det(u_{m-\nu}^{-})_{m,\nu=0,\ldots,\alpha-1} .$$

The first of these is just $D_{\alpha-1}[Q^{-1}u^{+}]$ while the second is a triangular determinant equal to $(\det u_0^{-})^{\alpha} = G[u^{-}]^{\alpha}$.

We shave shown that as $N \to \infty$ the right side of (5.9) converges to $D_{\alpha-1}[Q^{-1}u^{+}]\, G[u^{-}]^{\alpha}$. Since $G[Q] = 1$ left side converges to $(E[\varphi Q]/E[\varphi])\, G[\varphi]^{-\alpha}$ and we obtain

$$E[\varphi Q]/E[\varphi] = D_{\alpha}[Q^{-1}u^{+}]\, G[\varphi u^{-}]^{\alpha} = D_{\alpha-1}[Q^{-1}u^{+}]/G[u^{+}]^{\alpha}.$$

This is equivalent to (5.8) since $G[Q] = 1$.

To remove the assumption on Q_0 observe that it is in any case invertible and that for any ψ both $D_N[\psi]$ and $G[\psi]^{N+1}$ are multiplied by the same factor $(\det Q_0)^{N+1}$ if ψ is multiplied, on either side, by the constant matrix Q_0. Hence neither side of (5.8) is changed if Q is multiplied on the right by Q_0, so the identity for general Q_0 follows from the identity for $Q_0 = I$.

Finally to remove the assumption that $\varphi \in A_1$ observe that both sides of (5.8) are continuous functions of $u^+ \in A_0{}^+$. Thus it suffices to show that for any fixed $u^- \in A_0{}^-$ the set

$$\{u^+ \in A_0{}^+ : u^+u^- \in A_1\} \tag{5.10}$$

is dense in $A_0{}^+$. Since the polynomials

$$u^+(z) = \sum_{k=0}^{\alpha} u_k{}^+ z^k$$

of $A_0{}^+$ are dense in $A_0{}^+$ it suffices to show that any such polynomial is the limit of polynomials in the set. If we define, for $|t| < 1$, $U^+(t) = U^+(t, z) = u^+(tz)$ then clearly each $U^+(t) \in A_0{}^+$. Moreover by Lemma 2.1 we shall have $U^+(t)\, u^- \in A_1$ if

$$D_{\alpha-1}[(U^+(t)u^-)^{-1}] \tag{5.11}$$

is nonzero. This determinant is analytic for $|t| < 1$ and at $t = 0$ equals $D_{\alpha-1}[(u_0{}^+u^-)^{-1}] = \det(u_0{}^+u_0{}^-)^{-\alpha} \neq 0$. Consequently the determinant (5.11) can only vanish for a discrete set of t in the unit disc and so there is a sequence $t_n \to 1$ such that each $U^+(t_n)$ belongs to the set (5.10). This completes the proof of the lemma.

THEOREM 5.1. *The function $E[\varphi]$, $\varphi \in A_0$, has the following properties:*

(a) *If all the φ_i vanish for $i < -\alpha$ or for $i > \alpha$ then $E[\varphi] = D_{\alpha-1}[\varphi^{-1}]\, G[\varphi]^\alpha$.*

(b) $E[\varphi^{-1}] = E[\varphi]$.

(c) *If φ, φ_1, $\varphi_2 \in A_0{}^+$ and ψ, ψ_1, $\psi_2 \in A_0{}^-$ then $E[\varphi_1\psi\varphi_2] = E[\varphi_1\psi]\, E[\psi\varphi_2]$ and $E[\psi_1\varphi\psi_2] = E[\psi_1\varphi]\, E[\varphi\psi_2]$.*

Proof of (a). Take the special case of (5.8) where $u^-(z) = I$. We obtain for $u^+ \in A_0{}^+$ and Q of the form (5.7),

$$E[(u^+)^{-1}Q] = (D_{\alpha-1}[Q^{-1}u^+]/G[Q^{-1}u^+]^\alpha)\, E[(u^+)^{-1}].$$

For ψ belonging to $A_0{}^+$ (or $A_0{}^-$) each $D_N[\psi]$ equals $(\det \psi_0)^{N+1} = G[\psi]^{N+1}$ and so $E[\psi] = 1$. Therefore the last identity may be written simply $E[(u^+)^{-1}Q] = D_{\alpha-1}[Q^{-1}u^+]/G[Q^{-1}u^+]^\alpha$.

Now take the given φ and suppose that $\varphi \in A_1$ and that it has left standard factorization $\varphi = \varphi^+\varphi^-$. Assume all φ_i vanish for $i < -\alpha$. Then

$\varphi^- = (\varphi^+)^{-1}\varphi$ also has vanishing Fourier coefficients for values of the index less than $-\alpha$. Hence we may apply the last identity with $u^+ = (\varphi^+)^{-1}$, $Q = \varphi^-$ and $E[\varphi] = D_{\alpha-1}[\varphi^{-1}]/G[\varphi^{-1}]^\alpha$ results.

If φ does not belong to A_1 then $E[\varphi] = 0$ by Theorem 4.2 and $D_{\alpha-1}[\varphi^{-1}] = 0$ by Lemma 2.1. The equality therefore holds trivially in this case.

Proof of (b). This follows immediately from (a) if, say, $\varphi(z)$ is a Laurent polynomial

$$\sum_{i=-n}^{n} \varphi_i z^i.$$

For then we obtain $E[\varphi] = D_{\alpha-1}[\varphi^{-1}]/G[\varphi^{-1}]^\alpha$ for all $\alpha \geqslant n$ and we just let $\alpha \to \infty$. The identity for general $\varphi \in A_0$ follows by the usual density and continuity argument.

Proof of (c). Go back to the identity (5.8) and let $\alpha \to \infty$. We obtain, using (b) twice, $E[Q^{-1}u^+u^-] = E[Q^{-1}u^+]\, E[u^+u^-]$. This is the special case of the second of the asserted identities with $\psi_1^{-1} = Q$ of the form (5.7). The general case is obtained from this as usual.

Remark. Theorems 4.1 and 5.1(a) implies that if $\varphi_j = 0$ for all $j < -\alpha$ or all $j > \alpha$ and if $\varphi \in A_1$ then $D_{\alpha-1}[\varphi^{-1}] \neq 0$. Thus this condition is both necessary and sufficient for φ to belong to A_1.

Next we prove the validity of the identity

$$\log E[\varphi] = r^{-1} \sum_{j=1}^{\infty} j(\log \det \varphi)_j(\log \det \varphi)_{-j} \tag{5.12}$$

for a certain class of matrix functions. We shall say that a matrix function belonging to A_1 is *semi-scalar* if one of its left or right factors is a scalar valued function times the identity matrix. In this case of course the left and right standard factorizations coincide.

THEOREM 5.2. *If φ belongs to A_1 and is semi-scalar then* (5.12) *holds.*

Proof. Let φ have standard factorization $\varphi = \varphi^+(\varphi^- I)$ where φ^- is a scalar valued function. We are to prove

$$\log E[\varphi] = \sum_{j=1}^{\infty} j(\log \det \varphi^+)_j(\log \varphi^-)_j\,. \tag{5.13}$$

We may assume $G[\varphi^+] = G[\varphi^-] = 1$ since neither side of (5.13) is changed if φ^+ or φ^- is multiplied by a constant.

Consider first the special case $\varphi^-(z) = (1 - \zeta z^{-1})$, $|\zeta| < 1$. Then the hypothesis of Theorem 5.1(a) is satisfied with $\alpha = 1$ and so

$$E[\varphi] = D_0[\varphi^{-1}] = \det \frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta})^{-1}\, d\theta.$$

But

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi(e^{i\theta})^{-1}\, d\theta = \frac{1}{2\pi i} \int_{|z|=1} (z - \zeta)^{-1}\, \varphi^+(z)^{-1}\, dz = \varphi^+(\zeta)^{-1}$$

by the Cauchy integral formula. (The matrix function $\varphi^+(z)^{-1}$ extends analytically throughout the unit disc.) Therefore

$$\log E[\varphi] = -\log \det \varphi^+(\zeta) = -\sum_{j=1}^{\infty} (\log \det \varphi^+)_j\, \zeta^j.$$

(The term corresponding to $j = 0$ vanishes since $G[\varphi^+] = 1$.) Since

$$\log \varphi^-(z) = -\sum_{j=1}^{\infty} j^{-1} \zeta^j z^{-j}$$

this establishes (5.13) in the special case.

It follows from Theorem 5.1(c) that for arbitrary scalar functions $\varphi_1^-(z), \ldots, \varphi_n^-(z)$ belonging to A_0^- we have $E[\varphi^+\varphi_1^- \cdots \varphi_n^-] = E[\varphi^+\varphi_1^-] \cdots E[\varphi^+\varphi_n^-]$. Consequently (5.13) holds whenever φ^- is of the form

$$\varphi^-(z) = (1 - \zeta_1 z^{-1}) \cdots (1 - \zeta_n z^{-1}) \qquad |\zeta_1| < 1, \ldots, |\zeta_n| < 1.$$

Since functions of this form are dense in the scalar valued functions of A_0^- (with geometric mean 1) and since both sides of (5.13) are continuous functions of $\varphi^- \in A_0^-$ the identity holds in general.

Lest the reader doubt the lack of generality of (5.12), consider the 2×2 matrix function

$$\varphi(z) = \begin{bmatrix} z & a \\ a & z^{-1} \end{bmatrix}$$

with $a \neq 0, \pm 1$. Then $\varphi \in A_1$. Since $\det \varphi$ is constant formula (5.12), if it held, would give $E[\varphi] = 1$. But $E[\varphi]$ is easily computed by Theorem 5.1(a) with $\alpha = 1$ and in fact $E[\varphi] = a^2(a^2 - 1)^{-1}$.

Finally we state a formula which reduces the computation of $E[\varphi]$ to the case of matrix functions with determinant identically 1:

$$E[\varphi] = E[(\det \varphi)^{-1/r}\varphi] \exp \left\{ r^{-1} \sum_{j=1}^{\infty} j(\log \det \varphi)_j (\log \det \varphi)_{-j} \right\}.$$

This is established by writing down factorizations $(\det \varphi)^{-1/r} = \delta^+ \delta^-$, $\varphi = \varphi^+ \varphi^-$, applying Theorem 5.1(c) a few times to obtain $E[(\det \varphi)^{-1/r}\varphi] = E[\varphi]\, E[\delta^+\varphi^-]\, E[\delta^-\varphi^+]\, E[\delta^+ \delta^- I]$, and evaluating the last three factors by (5.12). The details are left to the reader.

6. Limiting Behavior of the Eigenvalues

Throughout this section $\varphi(z)$ will be a Laurent polynomial

$$\varphi(z) = \sum_{k=-\alpha}^{\beta} \varphi_k z^k \qquad (\alpha \geqslant 1, \beta \geqslant 0). \tag{6.1}$$

We denote by μ_N the discrete measure in the plane which assigns to each eigenvalue of $T_N[\varphi]$ its algebraic multiplicity multiplied by $(N+1)^{-1}$. Thus the total measure of μ_N is r. The limiting set of the eigenvalues will be denoted by Λ; a point λ belongs to Λ if

$$\lambda = \lim_{i \to \infty} \lambda_i$$

where λ_i is an eigenvalue of $T_{N_i}[\varphi]$ and $N_i \to \infty$.

We shall be concerned with the determination of the weak limit μ of μ_N (if it exists) and of the set Λ. As in [14] and [10] which dealt with the scalar case these things can be determined once there is enough information about the asymptotic behavior of $D_N[\varphi - \lambda I]$. The following lemma shows how. We denote by Δ the Laplacian in the sense of distributions and refer the reader to [3] for the little bit of potential theory which will be used from time to time in this section.

Lemma 6.1. *Let C be a compact set of two dimensional Lebesgue measure zero in the λ-plane, Γ its complement. Suppose that the relation*

$$\lim_{N \to \infty} D_N[\varphi - \lambda I]/g(\lambda)^{N+1} = e(\lambda)$$

holds uniformly on compact subsets of Γ, *where* $g(\lambda)$ *and* $e(\lambda)$ *are analytic in* Γ, $g(\lambda)$ *is nonzero, and* $e(\lambda)$ *has only isolated zeros. Then* $h(\lambda) = \log | g(\lambda)|$ *is locally integrable in the complex plane,* $\mu = (2\pi)^{-1} \Delta h$ *is a measure, and* μ_N *converges weakly to* μ. *If* h *cannot be continued harmonically to any point of* C *then the support of* μ *is exactly* C *and* Λ *is the union of* C *and the set of zeros of* $e(\lambda)$.

Proof. We have $(N+1)^{-1} \log | D_N(\varphi - \lambda I]| = \int \log | \zeta - \lambda | \, d\mu_N(\zeta)$ and all the μ_N are supported in a fixed compact set. Choose a subsequence $N_i \to \infty$ such that μ_{N_i} converges weakly to a measure μ. For each bounded set B in the plane $\int_B \log | \zeta - \lambda | \, d\lambda$ ($d\lambda$ denotes two-dimensional Lebesgue measure) is a continuous function of ζ so

$$\int d\mu_{N_i}(\zeta) \int_B \log | \zeta - \lambda | \, d\lambda \to \int d\mu(\zeta) \int_B \log | \zeta - \lambda | \, d\lambda.$$

Since $\log | \zeta - \lambda |$ is bounded above on the domains of integration we can interchange the orders of integration in the double integrals. We deduce that if B is any bounded set on which the convergence of $\int \log | \zeta - \lambda | \, d\mu_{N_i}(\zeta)$ to $h(\lambda)$ is uniform $\int_B h(\lambda) \, d\lambda = \int_B d\lambda \int \log | \zeta - \lambda | \, d\mu(\zeta)$. It follows that $h(\lambda) = \int \log | \zeta - \lambda | \, d\mu(\zeta)$ a.e. in Γ and so a.e. in the complex plane. Consequently h is locally integrable and $\Delta h = 2\pi\mu$.

Since μ was any weak limit of the μ_N the first statement of the lemma is established. The second follows from Hurwitz's theorem on the limits of zeros of sequences of analytic functions [16, §3.45] together with the fact that the support of μ is the smallest closed set on the complement of which $\int \log | \zeta - \lambda | \, d\mu(\zeta)$ is harmonic.

Theorem 5.1(a) provides enough information to determine μ and Λ in almost all cases. These will be worked out here in detail. Afterwards we shall derive an exact formula for $D_N[\varphi]$ from which the necessary asymptotic information can be derived even in those cases Theorem 5.1(a) could not handle.

For any φ of the form (6.1) the determinant $D_N[\varphi - \lambda I]$ is unchanged if $\varphi(z)$ is replaced by $\varphi(tz)$ for any positive number t. This replacement results in multiplying the ith block row of T_N by t^i and the jth block column by t^{-j} so that D_N is unaffected. Thus the condition $\varphi(z) - \lambda I \in A_0$ needed to apply the results of the preceding sections may be replaced by

$$\varphi(tz) - \lambda I \in A_0 \qquad \text{for some } t > 0. \tag{6.2}$$

If we write $\delta(z, \lambda) = \det [\varphi(z) - \lambda I]$ then (6.2) holds unless $\delta(z, \lambda)$ satisfies one of the following mutually exclusive conditions.

(i) $\delta(z, \lambda)$ extends to be analytic and zero at $z = 0$ or $z = \infty$;

(ii) $\delta(z, \lambda)$ has poles of order $p > 0$ at $z = 0$ and $q > 0$ at $z = \infty$ and if $z_1(\lambda),\dots, z_{p+q}(\lambda)$ are the zeros of $\delta(z, \lambda)$ ordered so that $| z_i(\lambda)| \leqslant | z_{i+1}(\lambda)|$ then $| z_p(\lambda)| = | z_{p+1}(\lambda)|$.

If neither (i) nor (ii) holds then $\delta(z, \lambda)$ has poles of order $p \geqslant 0$ at $z = 0$ and $q \geqslant 0$ at $t = \infty$ and (with the zeros ordered as before) $| z_p(\lambda)| < | z_{p+1}(\lambda)|$. In this case (6.2) holds for all t in the interval $I_\lambda = (| z_p(\lambda)|, | z_{p+1}(\lambda)|)$; if $p = 0$, $I_\lambda = (0, | z_1(\lambda)|)$, and if also $q = 0$, $I_\lambda = (0, \infty)$.

Write

$$G[\varphi, \lambda] = \exp \left\{\frac{1}{2\pi} \int_0^{2\pi} \log \delta(te^{i\theta}, \lambda)\, d\theta\right\} \qquad t \in I_\lambda .$$

Then Theorem 5.1(a) gives

$$\lim_{N\to\infty} D_N[\varphi - \lambda I]/G[\varphi, \lambda]^{N+1} = E[\varphi, \lambda] \tag{6.3}$$

where $E[\varphi, \lambda] = G[\varphi, \lambda]^\alpha D_{\alpha-1}[(\varphi(tz) - \lambda I)^{-1}]$. The convergence is locally uniform in λ. (It was not part of the statement of Theorem 4.2 but its proof could easily have been expanded slightly to give uniformity of convergence on compact subsets of A_0 .)

Our first assumption on φ will be that (i) or (ii) only holds for λ in a set of measure zero. If $\delta(z, \lambda) = \sum \delta_k(\lambda)z^k$ then each δ_k is a polynomial; δ_0 has degree exactly r and the other δ_k have lower degrees. Since (i) is equivalent to the simultaneous vanishing of $\delta_0(\lambda), \delta_{-1}(\lambda),\dots$ or of $\delta_0(\lambda), \delta_1(\lambda),\dots$ we see that (i) holds for at most r values of λ. Thus the set $C_1 = \{\lambda : \delta(z, \lambda) \text{ has property (i)}\}$ is finite.

Similarly we define $C_2 = \{\lambda : \delta(z, \lambda) \text{ has property (ii)}\}$. The structure of C_2 is more complicated. Any point λ_0 has a neighborhood whose intersection with C_2 is either empty, the entire neighborhood, or a finite set of analytic arcs emanating from λ_0. We assume that the second alternative never arises.

CONDITION A. *The interior of set C_2 is empty.*

Under this condition

$$C = C_1 \cup C_2 \tag{6.4}$$

is a finite union of analytic arcs and points.

Condition A may be violated. If

$$\varphi(z) = \begin{bmatrix} 0 & z^{-1} - z \\ 1 + z & z^{-1} + z^2 \end{bmatrix}$$

then $\delta(z, \lambda) = (z^{-1} - \lambda + 1)(z^2 - \lambda - 1)$ so that $| z_1(\lambda)| = | z_2(\lambda)| = | \lambda + 1 |^{1/2}$ whenever $| \lambda + 1 |^{1/2} < | \lambda - 1 |^{-1}$. In the scalar case however Condition A does hold without exception. This is easily deduced from Lemma 3.1 of [14].

Condition A enables us to use the asymptotic formula (6.3) for almost all λ. The function $E[\varphi, \lambda]$ is analytic in Γ, the complement of C. Moreover this function, when restricted to any connected component Γ_i of Γ, is algebraic. Thus on any component it either vanishes identically or has only finitely many zeros.

CONDITION B. *The function $E[\varphi, \lambda]$ does not vanish identically on any component of Γ.*

Since $E[\varphi, \lambda]$ is nonzero if λ is sufficiently large, it cannot vanish identically on the component of Γ containing infinity. In particular Condition B is safisfied if Γ is connected. Of course it is always satisfied in the scalar case. In the simple example

$$\varphi(z) = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$$

C is the unit circle and $E[\varphi, \lambda] = 0$ if $| \lambda | < 1$ so Condition B is violated.

THEOREM 6.1. *Suppose φ satisfies Conditions* A *and* B, *let the set* C *be defined by* (6.4), *and define $h(\lambda) = \log | G[\varphi, \lambda]|$, $\lambda \in \Gamma$. Then h is locally integrable in the complex plane, $\mu = (2\pi)^{-1} \Delta h$ is a measure with support exactly C, and μ_N converges weakly to μ. The limiting set Λ is the union of* C *and the set of zeros of the function $E[\varphi, \lambda]$.*

Proof. This will follow from Lemma 6.1 once it has been shown that $h(\lambda)$ cannot be continued harmonically to any point of C. Hirschman [10, §4] showed in the scalar case what in our notation can be described as follows: If c is any arc of C_2 and $h_1(\lambda)$ and $h_2(\lambda)$ are the values of $h(\lambda)$ on either side of the arc then each h_i continues harmonically to the other side of the arc but neither continuation is equal to the other function. The same argument applies in this case and will not be given here. The conclusion from this is that h cannot be continued harmonically to any point of C_2.

It remains to check that h cannot be continued harmonically to any isolated point of C, which necessarily belongs to C_1. Let λ_0 be such a point and suppose for example that $\delta(z, \lambda_0)$ has a zero at $z = 0$ of multiplicity $p > 0$. Suppose there are no other zeros inside or on the circle $|z| = t_0$. Then for all λ in some deleted neighborhood of λ_0 the function $\delta(z, \lambda)$ will have at $z = 0$ a pole of fixed order $q \geqslant 0$ and exactly $p + q$ zeros inside the circle $|z| = t_0$. We denote these zeros by $z_i(\lambda)$ and order them as usual so that $|z_1(\lambda)| \leqslant |z_2(\lambda)| \leqslant \cdots \leqslant |z_{p+q}(\lambda)|$. Each $z_i(\lambda)$ tends to zero as $\lambda \to \lambda_0$.

Since λ_0 is an isolated point of C we have for $\lambda \neq \lambda_0$, $|z_q(\lambda)| < |z_{q+1}(\lambda)|$. We apply the general Jensen formula [16, §3.62] to the function $\delta(z, \lambda)$ and each of the two circles $|z| = t_0$, $|z| = t$, where $|z_q(\lambda)| < t < |z_{q+1}(\lambda)|$, and substract. The result is

$$\frac{1}{2\pi}\int_0^{2\pi} \log|\delta(t_0 e^{i\theta}, \lambda)|\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} \log|\delta(te^{i\theta}, \lambda)|\, d\theta$$
$$= -\log|z_{q+1}(\lambda) \cdots z_{q+p}(\lambda)| - \log t_0{}^p.$$

Thus

$$h(\lambda) - \log|z_{q+1}(\lambda) \cdots z_{q+p}(\lambda)| = \frac{1}{2\pi}\int_0^{2\pi} \log|\delta(t_0 e^{i\theta}, \lambda)|\, d\theta + \log t_0{}^p$$

which is bounded as $\lambda \to \lambda_0$. Since each $z_i(\lambda)$ tends to zero as $\lambda \to \lambda_0$ it follows that $h(\lambda)$ is unbounded near λ_0. This completes the proof of the theorem.

A more concrete description of the measure μ can be obtained as follows. (A different method was used in [10] to obtain the same description in the scalar case.)

Let c be an arc of C_2 and take a little disc D that c cuts into two parts D_1 and D_2. Denote the restrictions of $h(\lambda)$ to these parts by $h_1(\lambda)$ and $h_2(\lambda)$. Then for $\lambda \notin \partial D_i$ we have

$$\frac{1}{2\pi}\int_{\partial D_i} \left\{\frac{\partial h_i(\zeta)}{\partial n_i} \log|\zeta - \lambda| - h_i(\zeta)\frac{\partial}{\partial n_i}\log|\zeta - \lambda|\right\} |d\zeta|$$
$$= \begin{cases} h_i(\lambda), & \lambda \in D_i \\ 0, & \lambda \notin D_i. \end{cases}$$

Here $|d\zeta|$ denotes arc length measure and $\partial/\partial n_i$ denotes the normal derivative at the point $\zeta \in \partial D_i$ in the direction interior to D_i.

If we take the sum of these two equalities and use the fact that $h_1 = h_2$ on c [10, §4] we find that

$$h(\lambda) - \frac{1}{2\pi} \int_c \{(\partial h_1(\zeta)/\partial n_1) + (\partial h_2(\zeta)/\partial n_2)\} \log | \zeta - \lambda | \, | d\zeta |$$

continues harmonically throughout D. Hence on c, $\mu = (2\pi)^{-1} \Delta h$ is absolutely continuous with respect to arc length and

$$(d\mu(\zeta)/| d\zeta |) = \frac{1}{2\pi} \{(\partial h_1(\zeta)/\partial n_1) + (\partial h_2(\zeta)/\partial n_2)\}. \tag{6.5}$$

Since $h(\lambda)$ is bounded in a neighborhood of each point C_2 no point where two or more arcs meet can have positive μ measure. Therefore (6.5) completely describes μ on C_2.

A more geometric interpretation of μ on c is the following. Let G_1, G_2 denote the limiting values of $G[\varphi, \lambda]$ from the two sides of c. Then G_1/G_2 maps c into the unit circle and μ is the measure on c induced by this mapping from normalized Lebesgue measure on the circle.

The description of μ on C_1 is easy. If $\lambda_0 \in C_1$

$$\mu(\{\lambda_0\}) = \lim_{\lambda \to \lambda_0} h(\lambda)/\log | \lambda - \lambda_0 |.$$

In case λ_0 is an isolated point of C then this number must be an integer since $G[\varphi, \lambda]$ is single valued and analytic in a deleted neighborhood of λ_0.

A slight variant of this shows that the measure of any connected component of C must be a positive integer. Since the total measure of μ is r there can be at most r components. This argument was used by Ullman [17] to prove connectedness in the scalar case.

We shall next derive an exact expression for $D_N[\varphi]$ which can be used to find a substitute for (6.3) when it holds for too small a λ-set (Condition A fails) or the right side is too often zero (Condition B fails). Two algebraic lemmas are needed.

LEMMA 6.2. *If $P(z)$ is a matrix with polynomial entries whose determinant has a simple zero at $z = \zeta$ then the matrix*

$$\rho = \lim_{z \to \zeta} (z - \zeta) \, P(z)^{-1}$$

has rank one.

Proof. The theory of matrices over Euclidean rings [18, §108] tells us that

$$P(z) = U(z) \begin{bmatrix} p_1(z) & & 0 \\ & \ddots & \\ 0 & & p_r(z) \end{bmatrix} V(z)$$

where U and V are invertible polynomial matrix functions and each p_i divides p_{i+1}. Since $\det P(z)$ has a simple zero at $z = \zeta$ each $p_i(\zeta)$ with $i < r$ must be nonzero and $p_r(z)$ must have a simple zero at $z = \zeta$. The desired conclusion follows.

LEMMA 6.3. *Suppose A_s ($s \in \mathscr{S}$) are $n \times n$ matrices of rank one and a_s are scalars. Then*

$$\det \sum_{s \in \mathscr{S}} a_s A_s = \sum_S \left(\prod_{s \in S} a_s \right) \det \sum_{s \in S} A_s$$

where S runs over all subsets of $\mathscr{S}$ containing n elements.

Proof. Denote the ith row of A_s by $A_s{}^i$. Then we have

$$\det \sum_{s \in \mathscr{S}} a_s A_s = \sum_f \det \begin{bmatrix} a_{f(1)} A^1_{f(1)} \\ \vdots \\ a_{f(n)} A^n_{f(n)} \end{bmatrix}$$

where f runs over all functions from $\{1,..., n\}$ to $\mathscr{S}$. Since each A_s has rank one the determinant on the right vanishes unless f is a one-one function. Therefore

$$\det \sum_{s \in \mathscr{S}} a_s A_s = \sum_S \sum_f \det \begin{bmatrix} a_{f(1)} A^1_{f(1)} \\ \vdots \\ a_{f(n)} A^n_{f(n)} \end{bmatrix}$$

where S runs over all subsets of $\mathscr{S}$ containing n elements and f runs over all one-one functions with range S. The inner sum is just

$$\left(\prod_{s \in S} a_s \right) \sum_f \det \begin{bmatrix} A^1_{f(1)} \\ \vdots \\ A^n_{f(n)} \end{bmatrix} = \left(\prod_{s \in S} a_s \right) \det \sum_{s \in S} A_s .$$

The formula for $D_N[\varphi]$ involves various expressions which we now introduce. As before $\delta(z) = \det \varphi(z)$. If $\delta(z)$ has a pole of order $p \geqslant 0$ at $z = 0$ and S is any set of p zeros of $\delta(z)$ we write

$$G_S[\varphi] = \exp \left\{ \frac{1}{2\pi i} \int_\sigma \log \delta(z)\, dz/z \right\}$$

$$D_S[\varphi^{-1}] = \det \left(\frac{1}{2\pi i} \int_\sigma z^{\mu-\nu} \varphi(z)^{-1}\, dz/z \right)_{\mu,\nu=0,\ldots,\alpha-1}.$$

Here σ is a simple closed curve enclosing $z = 0$, the points of S, but no other zeros of $\delta(z)$.

LEMMA 6.4. *If $S = \{z_s\}$ then*

$$G_S[\varphi] = (-1)^p \left(\prod z_s^{-1}\right) \lim_{z\to 0} z^p \delta(z).$$

Proof. Both $\delta(z)$ and $z^p \prod (z - z_s)^{-1}$ have continuously defined logarithms on σ. Cauchy's theorem applied to the interior of σ gives

$$\frac{1}{2\pi i} \int_\sigma \log \left[\delta(z)\, z^p \prod (z - z_s)^{-1} \right] dz/z$$
$$= \log \left[(-1)^p \left(\prod z_s^{-1} \right) \lim_{z\to 0} z^p \delta(z) \right]$$

and applied to the exterior of σ gives

$$\frac{1}{2\pi i} \int_\sigma \log \left[z^p \prod (z - z_s)^{-1} \right] dz/z = 0.$$

The assertion of the lemma follows.

We can now state and prove the formula for $D_N[\varphi]$.

THEOREM 6.2. *Assume $\delta(z)$ has a pole of order $p \geqslant 0$ at $z = 0$ and only simple zeros. Then for sufficiently large N*

$$D_N[\varphi] = \sum_S G_S[\varphi]^{N+\alpha+1} D_S[\varphi^{-1}] \tag{6.6}$$

where the summation extends over all sets S of p zeros of $\delta(z)$.

Proof. We first make the temporary assumption that φ_β and $\varphi_{-\alpha}$, the extreme coefficients of $\varphi(z)$, are both invertible. This implies that

$\delta(z)$ has poles of order αr at $z = 0$ and βr at $z = \infty$. The polynomial matrix function $z^\alpha \varphi(z)$ is invertible near $z = 0$ and so we may write $z^{-\alpha}\varphi(z)^{-1} = \psi_0 + \psi_1 z + \cdots$ for sufficiently small z. One has the identity $D_{N-1}[\varphi] = (-1)^{N\alpha r} \det \varphi_{-\alpha}^{N+\alpha} \det(\psi_{N+\mu-\nu})_{\mu,\nu=0,\ldots,\alpha-1}$. This was proved in the scalar case $r = 1$ by Baxter and Schmidt [2, §2] and the derivation is no different here.

It follows from our temporary assumption on φ that $z^{-\alpha}\,\varphi(z)^{-1}$ is analytic at $z = 0$ and vanishes at $z = \infty$. Therefore if $\{z_s\}$ is the set of zeros of $\delta(z)$ we may write $z^{-\alpha}\,\varphi(z)^{-1} = \sum(z - z_s)^{-1}\,\rho_s$ where ρ_s are certain $r \times r$ matrices. By Lemma 6.2 they all have rank one. Since $\psi_{N+\mu-\nu} = -\sum z_s^{-N-\mu+\nu-1}\,\rho_s$ an application of Lemma 6.3 with $n = \alpha r$ gives

$$\begin{aligned}
&\det(\psi_{N+\mu-\nu})_{\mu,\nu=0,\ldots,\alpha-1} \\
&\quad = (-1)^{\alpha r} \sum_S \left(\prod_{z_s \in S} z_s^{-N-\alpha}\right) \det\left(\sum_{z_s\in S} z_s^{\alpha-\mu+\nu-1}\rho_s\right)_{\mu,\nu=0,\ldots,\alpha-1} \\
&\quad = (-1)^{\alpha r} \sum_S \left(\prod_{z_s \in S} z_s^{-N-\alpha}\right) D_S[\varphi^{-1}]
\end{aligned}$$

where S runs over all sets of αr zeros of $\delta(z)$. Thus

$$\begin{aligned}
D_{N-1}[\varphi] &= (-1)^{(N+1)\alpha r} \sum_S \det \varphi_{-\alpha}^{N+\alpha} \left(\prod_{z_s\in S} z_s^{-N-\alpha}\right) D_S[\varphi^{-1}] \\
&= \sum_S G_S[\varphi]^{N+\alpha}\, D_S[\varphi^{-1}]
\end{aligned}$$

by Lemma 6.4 and the fact that

$$(N+1)\alpha \equiv (N+\alpha)\alpha \qquad (\mathrm{mod}\ 2).$$

To remove the temporary assumption of invertibility of $\varphi_{-\alpha}$ and φ_β let $\varphi_\epsilon(z) = \varphi(z) + \epsilon(z^\beta I - z^{-\alpha} A)$ where A is any invertible matrix with distinct eigenvalues a_i. As $\epsilon \to \infty$ (sic) $\delta_\epsilon(z) = \epsilon^r \prod_i (z^\beta - a_i z^{-\alpha}) + o(\epsilon^r)$ uniformly for z in any compact set not containing $z = 0$. It follows from Hurwitz's theorem that for ϵ sufficiently large $\delta_\epsilon(z)$ will have $(\alpha + \beta)r$ simple zeros near the $(\alpha + \beta)$th roots of the a_i. Since $\delta_\epsilon(z)$ is $z^{-\alpha r}$ times a polynomial of degree $(\alpha + \beta)r$ there are no other zeros.

A necessary and sufficient condition that a polynomial in z have only simple zeros is that its discriminant, which is a polynomial in the coefficients, be nonzero. Since the coefficients of $z^{\alpha r}\,\delta_\epsilon(z)$ are polynomials in ϵ and the discriminant does not vanish for large ϵ it can only vanish for finitely many ϵ. It follows that for ϵ in some deleted neighborhood of zero φ_ϵ satisfies the conditions under which (6.6) has already been established.

If we cut the deleted neighborhood along the negative real axis so that what remains is $\{\epsilon : 0 < |\epsilon| < \epsilon_0, |\arg \epsilon| < \pi\}$ then the zeros of $\delta_\epsilon(z)$ are analytic functions of ϵ. Some of these tend to zero as $\epsilon \to 0$, some tend to infinity, and the rest tend to the zeros of $\delta(z)$. Suppose S_ϵ is a set of zeros of $\delta_\epsilon(z)$ either not containing all zeros which tend to zero or else containing some zero which tends to infinity. Let $\zeta_i = \zeta_i(\epsilon)$ $(i = 1, \ldots, i_0)$ be the zeros of $\delta_\epsilon(z)$ tending to zero but not in S_ϵ and $\xi_j = \xi_j(\epsilon)$ $(j = 1, \ldots, j_0)$ those zeros in S_ϵ which tend to infinity. Then if σ_ϵ encloses $z = 0$ and S_ϵ but no other zeros of $\delta_\epsilon(z)$ we have

$$\int_{\sigma_\epsilon} \log \delta_\epsilon(z)\, dz/z$$
$$= \int_{\sigma_\epsilon} \log \left[\delta_\epsilon(z)\, z^{j_0} \prod (z - \zeta_i)^{-1} \prod (1 - z/\xi_j)^{-1}\right] dz/z$$
$$+ \int_{\sigma_\epsilon} \log \left[\prod (z - \zeta_i)\right] dz/z - \int_{\sigma_\epsilon} \log \left(z^{j_0} \prod (1 - z/\xi_j)^{-1}\right) dz/z.$$

The expression in brackets in the first integral on the right side is analytic and nonzero at all the ζ_i and ξ_j so the integral may be taken over a fixed closed contour σ. Therefore as $\epsilon \to 0$ the first integral tends to a finite limit.

The second integral on the right equals $2\pi i \log \prod(-\zeta_i)$ and the last equals $2\pi i \log \prod(-\xi_j)$. Hence $G_{S_\epsilon}[\varphi_\epsilon]$ is asymptotically a constant times $\prod\zeta_i / \prod\xi_j$ and this tends to zero as $\epsilon \to 0$ since $i_0 + j_0 > 0$.

Consider now the formula

$$D_N[\varphi_\epsilon] = \sum_{S_\epsilon} G_{S_\epsilon}[\varphi_\epsilon]^{N+\alpha+1} D_{S_\epsilon}[\varphi_\epsilon^{-1}]$$

where S_ϵ runs through all subsets of αr zeros of $\delta_\epsilon(z)$. If S_ϵ is as in the preceding paragraphs, i.e., it does not contain all zeros tending to zero or else contains some zero tending to infinity, then we have seen that $G_{S_\epsilon}[\varphi_\epsilon]$ tends to zero. Although $D_{S_\epsilon}[\varphi_\epsilon^{-1}]$ might at the same time tend to infinity they are both algebraic functions of ϵ, so the former must tend to zero at least as fast as some power of ϵ and the latter can tend to infinity no faster than some power of ϵ^{-1}. Therefore for N sufficiently large

$$\lim_{\epsilon \to 0} G_{S_\epsilon}[\varphi_\epsilon]^{N+\alpha+1} D_{S_\epsilon}[\varphi_\epsilon^{-1}] = 0$$

for these S_ϵ.

What remains are those subsets S_ϵ which contain all the $\alpha r - p$ zeros of $\delta_\epsilon(z)$ tending to zero, none tending to infinity, and p others tending to zeros of $\delta(z)$. It follows that $D_N[\varphi]$, which equals

$$\lim_{\epsilon \to 0} D_N[\varphi_\epsilon],$$

is exactly as given by the statement of the theorem.

Remark 1. The passage from the special case where (6.6) was established for all N to the general case was very crude. The formula may very well hold for all N in all cases. However the result as stated will be enough for applications once the following is pointed out: All the functions $G_{S_\epsilon}[\varphi_\epsilon]$ arising in the perturbation argument used for passage to the general case, which were seen to be $o(1)$ as $\epsilon \to 0$, are in fact $O(\epsilon^u)$ where $u > 0$ depends only on α, β and r; similarly all the $D_{S_\epsilon}[\varphi_\epsilon^{-1}]$ are $O(\epsilon^{-v})$ where $v > 0$ depends only on α, β and r. It follows that the "sufficiently large" of the statement of the theorem depends only on α, β and r and not on the specific φ. In particular if we apply the formula to $\varphi - \lambda I$ the same N works for all λ.

Remark 2. It follows from Lemma 6.4 that if S_1 and S_2 are two sets of p zeros of $\delta(z)$ then

$$G_{S_1}[\varphi]/G_{S_2}[\varphi] = \prod_{S_2 - S_1} z_s \Big/ \prod_{S_1 - S_2} z_s .$$

Consequently if there is a circle $|z| = t$ containing no zeros but enclosing a set S_0 of p zeros then $G_{S_0}[\varphi]$ is larger in absolute value than any other $G_S[\varphi]$ and so

$$\lim_{N \to \infty} D_N[\varphi]/G_{S_0}[\varphi]^{N+\alpha+1} = D_{S_0}[\varphi^{-1}].$$

This is just (6.3) once again, with $\lambda = 0$.

Let us consider now the two examples given earlier where Theorem 6.1 could not be applied. If

$$\varphi(z) = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$$

then Condition B is violated, but Theorem 6.2 gives easily $D_N[\varphi - \lambda I] = \lambda^{2(N+1)}$. Therefore, by Lemma 6.1 the limiting measure is a mass of 2 at $\lambda = 0$ and the limiting set is $\{0\}$. Of course these statements are hardly interesting since all the eigenvalues of $T_N[\varphi]$ are trivially zero.

More interesting is the example

$$\varphi(z) = \begin{bmatrix} 0 & z^{-1} - z \\ 1 + z & z^{-1} + z^2 \end{bmatrix}$$

where Condition A fails. We have here

$$\delta(z, \lambda) = (z^{-1} - \lambda + 1)(z^2 - \lambda - 1).$$

For $\lambda \neq 0, -1$ we have $p = 1$ and $\delta(z, \lambda)$ has the simple zeros $z_0 = (\lambda - 1)^{-1}$, $z_1 = (\lambda + 1)^{1/2}$, $z_2 = -(\lambda + 1)^{1/2}$. Write

$$C_0 = \{\lambda: |(\lambda + 1)^{1/2}(\lambda - 1)| = 1\}.$$

This consists of two mutually exterior simple closed curves, one surrounding $\lambda = -1$ and passing through $\lambda = 0$, the other surrounding $\lambda = 1$. For λ exterior to C_0 the zero z_0 has smaller absolute value than the others, $G_{\{z_0\}}[\varphi - \lambda I] = \lambda^2 - 1$, and so

$$\lim_{N\to\infty} D_N[\varphi - \lambda I]/(\lambda^2 - 1)^{N+2} = D_{\{z_0\}}[(\varphi - \lambda I)^{-1}]$$

uniformly on compact subsets of the exterior of C_0. For λ interior to C_0, but excluding $\lambda = -1$, the zeros z_1, z_2 have smaller absolute value than z_0, $G_{\{z_1\}}[\varphi - \lambda I] = (\lambda + 1)^{1/2}$, $G_{\{z_2\}}[\varphi - \lambda I] = -(\lambda + 1)^{1/2}$ and we find that

$$D_N[\varphi - \lambda I]/(\lambda + 1)^{\frac{1}{2}N+1} = D_{\{z_1\}}[(\varphi - \lambda I)^{-1}] + (-1)^N D_{\{z_2\}}[(\varphi - \lambda I)^{-1}] + o(1).$$

Now we apply Lemma 6.1 modified to allow the functions $g(\lambda)$ and $e(\lambda)$ to be multiple valued and to allow subsequences (in the present case the sequences of even and odd N). This is no problem and we deduce that the limiting distribution μ exists and has support $C = C_0 \cup \{-1\}$. If

$$h(\lambda) = \begin{cases} \log|\lambda^2 - 1| & \text{exterior to } C_0 \\ \frac{1}{2}\log|\lambda + 1| & \text{interior to } C_0 \end{cases}$$

then μ on C_0 is given by (6.5); and $\mu(\{-1\}) = \frac{1}{2}$. The limiting set Λ is C together with whatever zeros $D_{\{z_0\}}[(\varphi - \lambda I)^{-1}]$ may have exterior to C_0 and whatever zeros $D_{\{z_1\}}[(\varphi - \lambda I)^{-1}] \pm D_{\{z_2\}}[(\varphi - \lambda I)^{-1}]$ may have interior to C_0.

Note that C has three components. This appears to contradict an earlier statement that in the $r \times r$ case the support of the limiting distribution had at most r components. That statement however assumed the applicability of Lemma 6.1 with single valued $g(\lambda)$ as was the case when Conditions A and B held.

Finally it should be remarked that the methods of this section can be used to investigate the distribution of zeros of quite general determinants of the form $D_N[\varphi(\lambda)]$ where $\varphi(\lambda)$ is a Laurent polynomial depending analytically on the parameter λ.

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