ON A TOEPLITZ DETERMINANT IDENTITY OF BORODIN AND OKOUNKOV

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In this note we give two other proofs of an identity of A. Borodin and A. Okounkov which expresses a Toeplitz determinant in terms of the Fredholm determinant of a product of two Hankel operators. The second of these proofs yields a generalization of the identity to the case of block Toeplitz determinants.

The authors of the title proved in [2] an elegant identity expressing a Toeplitz determinant in terms of the Fredholm determinant of an infinite matrix which (although not described as such) is the product of two Hankel matrices. The proof used combinatorial theory, in particular a theorem of Gessel expressing a Toeplitz determinant as a sum over partitions of products of Schur functions. The purpose of this note is to give two other proofs of the identity. The first uses an identity of the second author [4] for the quotient of Toeplitz determinants in which the same product of Hankel matrices appears and the second, which is more direct and extends the identity to the case of block Toeplitz determinants, consists of carrying the first author's collaborative proof [1] of the strong Szegö limit theorem one step further.

We begin with the statement of the identity of [2], changing notation slightly. If ϕ is a function on the unit circle with Fourier coefficients ϕ_k then $T_n(\phi)$ denotes the Toeplitz matrix $(\phi_{i-j})_{i,j=0,\cdots,n-1}$ and $D_n(\phi)$ its determinant. Under general conditions ϕ has a representation $\phi = \phi_+\,\phi_-$ where ϕ_+ (resp. ϕ_-) extends to a nonzero analytic function in the interior (resp. exterior) of the circle. We assume that ϕ has geometric mean 1, and normalize $\phi_\pm$ so that $\phi_+(0) = \phi_-(\infty) = 1$. Define the infinite matrices U_n and V_n acting on $\ell^2(\mathbf{Z}^+)$, where $\mathbf{Z}^+ = \{0,\, 1, \cdots\}$, by

$$U_n(i,j) = (\phi_-/\phi_+)_{n+i+j+1}, \quad V_n(i,j) = (\phi_+/\phi_-)_{-n-i-j-1}$$

and the matrix K_n acting on $\ell^2(\{n,\, n+1, \cdots\})$ by

$$K_n(i,j) = \sum_{k=1}^{\infty} (\phi_-/\phi_+)_{i+k}\; (\phi_+/\phi_-)_{-k-j}.$$

Notice that K_n becomes $U_n V_n$ under the obvious identification of $\ell^2(\{n,\, n+1, \cdots\})$ with $\ell^2(\mathbf{Z}^+)$. It is easy to check that, aside from a factor $(-1)^{i+j}$ which does not affect its Fredholm determinant, the entries of K_n are the same as given by the integral formula (2.2) of [2]. The formula of Borodin and Okounkov is

$$D_n(\phi) = Z\; \det\,(I - K_n), \tag{1}$$

[1]Supported by National Science Foundation grant DMS–9970879.
[2]Supported by National Science Foundation grant DMS–9732687.

where

$$Z = \exp\{\sum_{k=1}^{\infty} k\,(\log\,\phi)_k\,(\log\,\phi)_{-k}\} = \lim_{n\to\infty} D_n(\phi).$$

(The last identity is the strong Szegö limit theorem.) This identity is especially useful for obtaining refined asymptotic results as $n \to \infty$.

Two versions of (1) were proved in [2]. One was algebraic and was an identity of formal power series and the other was analytic and assumed that the regions of analyticity of $\phi_\pm$ included neighborhoods of the unit circle although, as the authors point out, an approximation argument can be used to extend the range of validity. The requirements for our proofs are that $\log\,\phi_\pm$ be bounded and $\sum_{k=-\infty}^{\infty} |k|\,|(\log\,\phi)_k|^2 < \infty$. [3]

First proof

To state the relevant result of [4] we define the vectors $U_n\delta$ and $V_n\delta$ in $\mathbf{Z}^+$ by

$$U_n\delta(i) = (\phi_-/\phi_+)_{n+i}, \quad V_n\delta(i) = (\phi_+/\phi_-)_{-n-i}.$$

(These are not the results of acting on a vector δ by the operators U_n and V_n since $-1 \notin \mathbf{Z}^+$, but the notation suggests this.) The result is the following proposition.

If $I - U_n\,V_n$ is invertible then so is $T_n(\phi)$ and

$$\frac{D_{n-1}(\phi)}{D_n(\phi)} = 1 - \Big((I - U_n\,V_n)^{-1}U_n\delta,\, V_n\delta\Big)\,, \tag{2}$$

where the inner product denotes the sum of the products of the components.

The formula appears on p. 341 of [4] in different notation. To derive (1) from this we assume temporarily that $I - U_n\,V_n$ is invertible for all n (and therefore so is $I - V_n\,U_n$) and compute the upper-left entry of $(I - V_{n-1}\,U_{n-1})^{-1}$ in two different ways. This entry equals the upper-left entry of $(I - K_{n-1})^{-1}$, and Cramer's rule says that the inverse of the entry equals

$$\frac{\det\,(I - K_{n-1})}{\det\,(I - K_n)}.$$

On the other hand, there is a general formula which says that if one has a 2×2 block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ then the upper-left block of its inverse equals $(A - BD^{-1}C)^{-1}$. Here A and D are square and the various inverses are assumed to exist. In our case the large matrix is $I - V_{n-1}\,U_{n-1}$ and A is 1×1. It is easy to see that

$$A = 1 - (V_n\delta,\, U_n\delta), \quad D = I - V_n\,U_n, \quad C = -V_n\,U_n\delta, \quad B = -U_n\,V_n\delta,$$

[3] The bounded functions f satisfying $\sum_{k=-\infty}^{\infty} |k||f_k|^2 < \infty$ form a Banach algebra under a natural norm and for any such f the Hankel matrix (f_{i+j}) acting on $\ell^2(\mathbf{Z}^+)$ is Hilbert-Schmidt. Thus if $\log\phi_\pm$ belong to this algebra so do ϕ_-/ϕ_+ and ϕ_+/ϕ_- and it follows that U_n and V_n are Hilbert-Schmidt so K_n is trace class. Moreover the Szegö limit theorem holds for such ϕ. See [5] or, for this and a lot more, [3].

the last interpreted as a row vector. The formula says that the inverse of the upper-left entry of the inverse equals

$$1-(V_n\delta,\,U_n\delta)-\Big((I-V_nU_n)^{-1}\,V_n\,U_n\delta,\,U_n\,V_n\delta\Big)$$

$$=1-(V_n\delta,\,U_n\delta)-\Big(U_n\,(I-V_nU_n)^{-1}\,V_n\,U_n\delta,\,V_n\delta\Big)$$

$$=1-(V_n\delta,\,U_n\delta)-\Big(\Big[(I-U_nV_n)^{-1}-I\Big]\,U_n\delta,\,V_n\delta\Big)$$

$$=1-\Big((I-U_nV_n)^{-1}U_n\delta,\,V_n\delta\Big)\,,$$

which is the right side of (2). Thus we have established

$$\frac{D_{n-1}(\phi)}{D_n(\phi)}=\frac{\det\,(I-K_{n-1})}{\det\,(I-K_n)},$$

which shows that (1) holds for some constant Z. That Z is as stated follows by letting $n\to\infty$.

To remove the restriction that $I-U_n\,V_n$ be invertible for all n, we introduce a complex parameter λ and replace ϕ by $\phi^\lambda=\exp\,(\lambda\log\phi)$. Then both sides of (1) are entire functions of λ and are equal when λ is so small that $\|\,\phi_-^\lambda/\phi_+^\lambda-1\,\|_\infty<1$ and $\|\,\phi_+^\lambda/\phi_-^\lambda-1\,\|_\infty<1$, for then all U_n and V_n have operator norm less than 1 so all $I-U_n\,V_n$ are invertible. Since the two sides sides of (1) are equal for small λ they are equal for all λ.

Second proof

Denote by $T(\phi)$ the semi-infinite Toeplitz matrix $(\phi_{i-j})_{i,\,j\geq0}$. Then $T(\phi_-)$ and $T(\phi_+)$ are upper-triangular and lower-triangular resepectively. It follows that if P_n is the diagonal matrix whose first n diagonal entries are all 1 and whose other entries are 0 then

$$P_n\,T(\phi_+)=P_n\,T(\phi_+)\,P_n,\quad T(\phi_-)\,P_n=P_n\,T(\phi_-)\,P_n.$$

Observe that $T_n(\phi)$ is the upper-left $n\times n$ block of $P_n\,T_n(\phi)\,P_n$. Using the above, we can write [4]

$$P_n\,T(\phi)\,P_n=P_n\,T(\phi_+)\,T(\phi_+^{-1})\,T(\phi)\,T(\phi_-^{-1})\,T(\phi_-)\,P_n$$

$$=P_n\,T(\phi_+)\,P_n\,T(\phi_+^{-1})\,T(\phi)\,T(\phi_-^{-1})\,P_n\,T(\phi_-)\,P_n.$$

Now the upper-left blocks of $P_n\,T(\phi_\pm)\,P_n$ are $T_n(\phi_\pm)$, which are triangular matrices with diagonal entries all 1, by our assumed normalization. Therefore they have determinant one, so $D_n(\phi)$ equals the determinant of the upper-left block of $P_n\,T(\phi_+^{-1})\,T(\phi)\,T(\phi_-^{-1})\,P_n$. Set

$$T(\phi_+^{-1})\,T(\phi)\,T(\phi_-^{-1})=A.$$

[4] It is an easy general fact that if $\psi_1\in\overline{H^\infty}$ or $\psi_2\in H^\infty$ than $T(\psi_1\psi_2)=T(\psi_1)T(\psi_2)$. In particular $T(\phi_\pm)$ are invertible with inverses $T(\phi_\pm^{-1})$. Recall that H^∞ consists of all $\psi\in L^\infty$ such that $\psi_k=0$ when $k<0$.

Then the determinant of the upper-left block of P_nAP_n equals $\det(P_nAP_n + Q_n)$, where $Q_n = I - P_n$. Now A is invertible and differs from I by a trace class operator (we shall see this in a moment). Therefore

$$\det(P_nAP_n + Q_n) = \det A\ \det(A^{-1}P_nAP_n + A^{-1}Q_n)$$
$$= \det A\ \det(A^{-1}(I - Q_n)AP_n + A^{-1}Q_n) = \det A\ \det(P_n - A^{-1}Q_nAP_n + A^{-1}Q_n)$$
$$= \det A\ \det(P_n + A^{-1}Q_n)\ \det(I - Q_nAP_n),$$

since $P_nQ_n = 0$. The determinant of the operator on the right equals one, again since $P_nQ_n = 0$. Moreover

$$\det(P_n + A^{-1}Q_n) = \det(I - (I - A^{-1})Q_n) = \det(I - Q_n(I - A^{-1})Q_n).$$

We have shown

$$D_n(\phi) = \det A\ \det(I - Q_n(I - A^{-1})Q_n). \tag{3}$$

It remains to show that this is the same as (1). First, A is similar via the invertible operator $T(\phi_+)$ to $T(\phi)\,T(\phi_-^{-1})\,T(\phi_+^{-1})$. Therefore

$$\det A = \det\ T(\phi)\,T(\phi_-^{-1})\,T(\phi_+^{-1}) = \det\ T(\phi)\,T(\phi^{-1}). \tag{4}$$

This is a representation of the constant Z in the strong Szegö limit theorem [1, 3, 5].

Next

$$A^{-1} = T(\phi_-)\,T(\phi)^{-1}\,T(\phi_+) = T(\phi_-)\,T(\phi_+^{-1})\,T(\phi_-^{-1})\,T(\phi_+) = T(\phi_-/\phi_+)\,T(\phi_+/\phi_-). \tag{5}$$

Because ϕ_-/ϕ_+ and ϕ_+/ϕ_- are reciprocals we see that the i, j entry of this matrix equals

$$\delta_{i,j} - \sum_{k=1}^{\infty}(\phi_-/\phi_+)_{i+k}\,(\phi_+/\phi_-)_{-k-j},$$

and so $\det(I - Q_n(I - A^{-1})Q_n)$ equals $\det(I - K_n)$. (This also shows that A^{-1} differs from I by a trace class operator, so the same is true of A.) This gives (1) with $Z = \det T(\phi)\,T(\phi^{-1})$.

Let us see how to modify this argument for the case of block Toeplitz determinants, where ϕ is a matrix-valued function. We assume the factorization $\phi = \phi_+\phi_-$, the order of the factors being important now, where $\phi_\pm^{\pm1}$ belong to the algebra described in footnote 3 and $\phi_+^{\pm1} \in H^\infty$, $\phi_-^{\pm1} \in \overline{H^\infty}$. Then (3) is derived without change as is formula (4) for $\det A$ since $\phi^{-1} = \phi_-^{-1}\phi_+^{-1}$. But (5) no longer holds because it would require $\phi = \phi_-\phi_+$, which does not hold. But if we also assume a factorization $\phi = \psi_-\psi_+$, with $\psi_\pm$ having properties analogous to those of $\phi_\pm$, we can replace (5) by

$$A^{-1} = T(\phi_-)\,T(\psi_+^{-1})\,T(\psi_-^{-1})\,T(\phi_+) = T(\phi_-\psi_+^{-1})\,T(\psi_-^{-1}\phi_+).$$

Now $\phi_-\psi_+^{-1}$ and $\psi_-^{-1}\phi_+$ are mutual inverses and we deduce that in this case (1) holds with $Z = \det T(\phi)\,T(\phi^{-1})$ and K_n the matrix, thought of as acting on $\ell^2(\{n, n+1, \cdots\})$, with i, j entry

$$\sum_{k=1}^{\infty}(\phi_-\psi_+^{-1})_{i+k}\,(\psi_-^{-1}\phi_+)_{-k-j}.$$

Acknowledgement

The authors thank Alexander Its who, after seeing [2] and the first proof presented here, asked the first author whether there was a direct proof. The second proof was the result.

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AMS Subject Classification: 47B35