NORTH-HOLLAND

Generalized Inversion of Block Toeplitz Matrices

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ABSTRACT

An analog of a Wiener-Hopf factorization method is proposed for finite block Toeplitz matrices. For an arbitrary rational matrix polynomial, notions of essential indices and polynomials are introduced. A connection between these notions and a Wiener-Hopf factorization of some block triangular matrix functions is studied. A formula for a generalized (one-sided, two-sided) inversion of a block Toeplitz matrix is found in terms of indices and essential polynomials of its symbol. Well-known inversion formulas are obtained as special cases of this formula. © 1998 Elsevier Science Inc.

INTRODUCTION

A method of a Wiener-Hopf factorization was first applied to a study of convolution equations on a finite interval by M. P. Ganin [9]. In this work it was shown that solving of these equations is equivalent to solving of a Riemann boundary problem with a triangular 2×2 matrix function. Subsequently the method was developed in the works [22, 21], and others.

In the discrete case this idea was first used in [19]. It turned out that the inversion of a finite scalar Toeplitz matrix can also be obtained in terms of the Wiener-Hopf factorization of a triangular 2×2 matrix function. However, for this method one requires an explicit solution of the problem of the Wiener-Hopf factorization.

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In the present paper finite block Toeplitz matrices

$$\|a_{i-j}\|_{\substack{i=0,1,\dots,n\\ j=0,1,\dots,m}}$$

with $p \times q$ blocks are considered. The goal of the work is to propose an analog of the Wiener-Hopf factorization method and to find an explicit method for a generalized inversion of these matrices.

We obtain a connection between a generalized (one-sided, two-sided) inversion of such a matrix and a Wiener-Hopf factorization of an auxiliary block triangular $(p+q) \times (p+q)$ matrix function

$$A(t) = \begin{pmatrix} t^{-m-1} I_q & 0 \\ \sum_{k=-m}^{n} a_k t^k & t^{n+1} I_p \end{pmatrix}$$

(see Section 2). In order to find the generalized inverse G in an explicit form, we shall need an explicit method for a construction of the Wiener-Hopf factorization of $A(t)$. In the case $p = q = 1$ there exists the effective algorithm of G. N. Chebotarev [8] for a computation of the factorization indices of $A(t)$ and the factors $A_{\pm}(t)$. Another explicit method of the Wiener-Hopf factorization of $A(t)$ for this case was found in [1]. Since $A(t)$ is a rational matrix polynomial, in the common case there also exists an explicit solution of the factorization problem (see, e.g., [12]). This solution use finite block Toeplitz matrices formed from the moments of $A^{-1}(t)$ with respect to the unit circle $\mathbb{T}$.

In the present paper we obtain an explicit method for a construction of a generalized inverse of a block Toeplitz matrix directly in terms of the sequence $a_{-m}, \dots, a_0, \dots, a_n$. To do this, we study in detail a kernel structure of a family of block Toeplitz matrices and define notions of essential indices and polynomials (Section 3). These notions were first introduced in connection with an explicit construction of a Wiener-Hopf factorization for triangular 2×2 matrix functions [1]. In [2] the technique of indices and essential polynomials was developed for a sequence of square matrices, and a family of inversion formulas for block Toeplitz matrices with square blocks was obtained. Moreover, the technique can be used for an explicit solution of the factorization problem for meromorphic matrix functions [5]. The same notions (characteristic numbers and polynomials) were independently introduced for a scalar case in [17]. In this work the notion of indices was also defined in the more general case of Toeplitz-like operators. The specifics of the block Toeplitz case were discussed, not knowing about the paper [2], in [14] and [16].

For an application of the technique of essential polynomials one requires an essentialness criterion, which allows one to check that the given integers are indices and the given vector polynomials are essential polynomials of the given sequence of matrices (Section 4). Using this criterion, we obtain a formula for a generalized inverse G of a block Toeplitz matrix in terms of essential indices and polynomials of the sequence $a_{-m}, \dots, a_0, \dots, a_n$ (Section 5). Another method of generalized inversion in the more general case of Hankel and Toeplitz mosaic matrices was proposed in [15]. The same arguments as for Toeplitz operators allow us to find a formula for a generating polynomial of G (Section 6). Well-known inversion formulas and the formula for a generalized inversion of scalar Toeplitz matrices [3, 6] are special cases of our results (Section 7).

1. NOTATION AND USUAL DEFINITIONS

Let $\mathbb{C}^{p \times q}$ be the set of complex $p \times q$ matrices. For a matrix A we shall denote by $\ker_R A$ its right kernel and by $\ker_L A$ its left kernel:

$$\ker_R A = \{x \mid Ax = 0\}, \qquad \ker_L A = \{y \mid yA = 0\}.$$

By $[A]_j$ ($[A]^j$) denote the jth row (the jth column) of the matrix A. Let A be a block matrix with blocks in $\mathbb{C}^{p \times q}$, and let A has the block size $(n+1) \times (m+1)$. We partition the column $R \in \ker_R A$ into $m+1$ blocks (the size of the blocks is $q \times 1$):

$$R = \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_m \end{pmatrix},$$

and for R we define its generating vector polynomial in the variable t to be the polynomial

$$R(t) = r_0 + r_1 t + \cdots + r_m t^m.$$

Similarly, for a row in $\ker_L A$ we define the generating vector polynomial in t^{-1}.

Let $a_{-m}, \dots, a_0, \dots, a_n$ ($n \geqslant 0$, $m \geqslant 0$; n, m are not zero simultaneously) be a finite sequence of complex $p \times q$ matrices. Let us denote by $a(t) = \sum_{j=-m}^{n} a_j t^j$ the generating matrix polynomial in t and t^{-1} of this sequence. Using the terminology of the work [12], we shall call $a(t)$ a rational matrix polynomial. Let us form the block Toeplitz matrix

$$T_a = \begin{pmatrix} a_0 & a_{-1} & \cdots & a_{-m} \\ a_1 & a_0 & \cdots & a_{-m+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n-1} & \cdots & a_{n-m} \end{pmatrix}$$

consisting from the elements of the sequence. We note that an arbitrary matrix A can be considered as a block Toeplitz matrix with rectangular blocks. To do this, we can partition A into rows ($m = 0$) or into columns ($n = 0$).

In the sequel we shall consider T_a as the matrix of a finite section of a Toeplitz operator $\mathbb{T}_a$. Recall (see, e.g., [10]) that the infinite Toeplitz matrix

$$\begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

defines the *Toeplitz operator* $\mathbb{T}_a$ acting from the vector space $l^s_{q\times 1}$ into $l^s_{p\times 1}$ ($1 \leqslant s \leqslant \infty$). Here $\{a_j\}_{j=-\infty}^{\infty}$ is an infinite sequence of complex $p \times q$ matrices such that $\sum_{j=-\infty}^{\infty} |a_j| < \infty$ ($|\cdot|$ is a matrix norm on the set of $p \times q$ matrices). The matrix function $a(t) = \sum_{j=-\infty}^{\infty} a_j t^j$, $|t| = 1$, is called a *symbol* of the operator $\mathbb{T}_a$. Denote by P_i the projector onto the first i coordinates from the Banach space $l^1_{j\times 1}$, and by Q_i the complementary projector. It is easily seen that

$$P_i = \mathbb{I} - \mathbb{T}_{t^i I_j} \mathbb{T}_{t^{-i} I_j}, \qquad Q_i = \mathbb{T}_{t^i I_j} \mathbb{T}_{t^{-i} I_j}.$$

Here $\mathbb{I}$ is the identity operator and I_j is the $j \times j$ identity matrix. Then the block Toeplitz matrix T_a is the matrix of the operator $P_{n+1} \mathbb{T}_a P_{m+1} | \operatorname{Im} P_{m+1}$.

In complete analogy with the theory of Toeplitz operators, we shall say that the matrix polynomial $a(t)$ is the *symbol* of the block Toeplitz matrix T_a.

By $W_{p\times q}$ denote the Banach space of all $p \times q$ matrix functions of the form $a(t) = \sum_{j=-\infty}^{\infty} a_j t^j$, $|t| = 1$, $\{a_j\}_{j=-\infty}^{\infty} \in l_{p\times q}^1$; by $W_{p\times q}^+$ $[W_{p\times q}^-]$ denote the subspace of $W_{p\times q}$ consisting of all matrix functions of the form $a(t) = \sum_{j=0}^{\infty} a_j t^j$ $[a(t) = \sum_{j=-\infty}^{0} a_j t^j]$. If $p = q$, then $W_{p\times p}$ is a Banach algebra and $W_{p\times p}^{\pm}$ are its subalgebras. For brevity, we shall use the designation $W = W_{1\times 1}$, $W_{\pm} = W_{1\times 1}^{\pm}$.

It is easily seen that there is the following partial multiplicativity of the mapping $a \to \mathbb{T}_a$:

$$\mathbb{T}_{aa_+} = \mathbb{T}_a \mathbb{T}_{a_+}, \qquad \mathbb{T}_{a_- a} = \mathbb{T}_{a_-} \mathbb{T}_a \tag{1.1}$$

for any $a(t) \in W_{p\times q}$, $a_+(t) \in W_{q\times k}^+$, $a_-(t) \in W_{l\times p}^-$. By virtue of this property the basic method in the theory of Toeplitz operators with invertible symbols is a Winer-Hopf factorization of symbols.

Let $a(t)$ be an invertible element of $W_{p\times p}$. The representation of $a(t)$ in the form

$$a(t) = a_-(t) d(t) a_+(t)$$

is called a *right Wiener-Hopf factorization* of $a(t)$ with respect to the unit circle $\mathbb{T}$. Here $a_{\pm}(t)$ are invertible elements of $W_{p\times p}^{\pm}$ and $d(t) = \operatorname{diag}[t^{\rho_1}, \dots, t^{\rho_p}]$. The integers $\rho_1, \dots, \rho_p$ are called the *right factorization indices* of $a(t)$. They are uniquely determined by $a(t)$. It is known that all invertible elements of $W_{p\times p}$ admit a Wiener-Hopf factorization.

We shall also need the following definition (see, e.g., [11]). A linear bounded operator A acting in a Banach space is called *generalized invertible* if there exists a linear bounded operator G (a *generalized inverse* of A) such that $AGA = A$. In matrix theory G is also called a (1)-inverse [7]. We shall say that a generalized invertible operator A is *strictly generalized invertible* if A is not one-sided invertible.

As we shall see in the following sections, it is natural to include the matrix T_a in the family of block Toeplitz matrices

$$\{T_{t^{-k}a}\}_{k=-m}^{n}, \qquad \text{where} \quad T_{t^{-k}a} = \|a_{i-j}\|_{\substack{i=k,k+1,\dots,n \\ j=0,1,\dots,m+k}}.$$

For brevity, we shall use the designation $T_k = T_{t^{-k}a}$.

2. GENERALIZED INVERSION OF BLOCK TOEPLITZ MATRICES AND WIENER-HOPF FACTORIZATION OF BLOCK TRIANGULAR MATRIX FUNCTIONS

In this section we establish a connection between a generalized inversion of the finite block Toeplitz matrix

$$T_a = \|a_{i-j}\|_{\substack{i=0,1,\ldots,n \\ j=0,1,\ldots,m}}$$

and the Wiener-Hopf factorization of the block triangular $(p+q)\times(p+q)$ matrix function

$$A(t) = \begin{pmatrix} t^{-m-1}I_q & 0 \\ \sum_{k=-m}^{n} a_k t^k & t^{n+1}I_p \end{pmatrix}.$$

Let

$$A(t) = A_-(t)D(t)A_+(t) \tag{2.1}$$

be a right Wiener-Hopf factorization of $A(t)$ with respect to the unit circle $\mathbb{T}$. We partition the matrix functions $A_\pm(t)$ and $D(t)$ into blocks:

$$A_\pm(t) = \begin{pmatrix} a_{11}^\pm(t) & a_{12}^\pm(t) \\ a_{21}^\pm(t) & a_{22}^\pm(t) \end{pmatrix}, \qquad D(t) = \begin{pmatrix} d_1(t) & 0 \\ 0 & d_2(t) \end{pmatrix},$$

where $a_{11}^\pm(t)$ and $d_1(t)$ have size $q\times q$. In a similar manner we represent $A_\pm^{-1}(t)$:

$$A_\pm^{-1}(t) = \begin{pmatrix} b_{11}^\pm(t) & b_{12}^\pm(t) \\ b_{21}^\pm(t) & b_{22}^\pm(t) \end{pmatrix}.$$

THEOREM 2.1. *The block Toeplitz matrix T_a is invertible (left invertible, right invertible) if and only if the right factorization indices of $A(t)$ are equal to zero (nonnegative, nonpositive). If $A(t)$ has both positive and negative factorization indices, then T_a is strictly generalized invertible.*

The matrix of the operator

$$G = P_{m+1}\left(\mathbb{T}_{b_{11}^{+}} P_{m+1} \mathbb{T}_{d_1^{-1}} P_{n+1} \mathbb{T}_{b_{12}^{-}} + \mathbb{T}_{b_{12}^{+}} P_{m+1} \mathbb{T}_{d_2^{-1}} P_{n+1} \mathbb{T}_{b_{22}^{-}}\right) P_{n+1} | \operatorname{Im} P_{n+1} \tag{2.2}$$

is a generalized (one-sided, two-sided) inverse of T_a.

Proof. It is easily seen that for any matrix functions $\alpha_{\pm}(t)$ with entries in the algebra $W_{\pm}$ we have

$$P_{m+1}\mathbb{T}_{\alpha_+}P_{m+1} = P_{m+1}\mathbb{T}_{\alpha_+}, \qquad P_{n+1}\mathbb{T}_{\alpha_-}P_{n+1} = \mathbb{T}_{\alpha_-}P_{n+1}.$$

Hence

$$G = P_{m+1}\left(\mathbb{T}_{b_{11}^{+}}\mathbb{T}_{d_1^{-1}}\mathbb{T}_{b_{12}^{-}} + \mathbb{T}_{b_{12}^{+}}\mathbb{T}_{d_2^{-1}}\mathbb{T}_{b_{22}^{-}}\right)P_{n+1}|\operatorname{Im} P_{n+1}.$$

Recall that we consider T_a as the matrix of a finite section of the Toeplitz operator $\mathbb{T}_a$, that is, $T_a = P_{n+1}\mathbb{T}_a P_{m+1}|\operatorname{Im} P_{m+1}$. Let us find the operator $A = P_{n+1}T_a G T_a P_{m+1}$. Taking into account the partial multiplicativity of Toeplitz operators (1.1) and the definition of the operators P_{n+1}, P_{m+1}, we obtain

$$\begin{aligned} A = {} & P_{n+1}\left(\mathbb{T}_{ab_{11}^{+}}\mathbb{T}_{d_1^{-1}}\mathbb{T}_{b_{12}^{-}a} + \mathbb{T}_{ab_{12}^{+}}\mathbb{T}_{d_2^{-1}}\mathbb{T}_{b_{22}^{-}a}\right)P_{m+1} \\ & - P_{n+1}\left(\mathbb{T}_{ab_{11}^{+}}\mathbb{T}_{d_1^{-1}}\mathbb{T}_{t^{n+1}b_{12}^{-}} + \mathbb{T}_{ab_{12}^{+}}\mathbb{T}_{d_2^{-1}}\mathbb{T}_{t^{n+1}b_{22}^{-}}\right)\mathbb{T}_{t^{-n-1}a}P_{m+1} \\ & - P_{n+1}\mathbb{T}_{t^{m+1}a}\left(\mathbb{T}_{t^{-m-1}b_{11}^{+}}\mathbb{T}_{d_1^{-1}}\mathbb{T}_{b_{12}^{-}a} + \mathbb{T}_{t^{-m-1}b_{12}^{+}}\mathbb{T}_{d_2^{-1}}\mathbb{T}_{b_{22}^{-}a}\right)P_{m+1} \\ & + P_{n+1}\mathbb{T}_{t^{m+1}a}\left(\mathbb{T}_{t^{-m-1}b_{11}^{+}}\mathbb{T}_{d_1^{-1}}\mathbb{T}_{t^{n+1}b_{12}^{-}} + \mathbb{T}_{t^{-m-1}b_{12}^{+}}\mathbb{T}_{d_2^{-1}}\mathbb{T}_{t^{n+1}b_{22}^{-}}\right)\mathbb{T}_{t^{-n-1}a}P_{m+1}. \end{aligned}$$

Now we transform the first term A_1. It follows from the factorizations $A(t)A_+^{-1}(t) = A_-(t)D(t)$ and $A^{-1}(t)A(t) = D(t)A_+(t)$ that

$$a(t)b_{11}^{+}(t) = a_{21}^{-}(t)d_1(t) - t^{n+1}b_{21}^{+}(t),$$

$$a(t)b_{12}^{+}(t) = a_{22}^{-}(t)d_2(t) - t^{n+1}b_{22}^{+}(t)$$

and

$$b_{12}^{-}(t)a(t) = d_1(t)a_{11}^{+}(t) - t^{-m-1}b_{11}^{-}(t),$$

$$b_{22}^{-}(t)a(t) = d_2(t)a_{21}^{+}(t) - t^{-m-1}b_{21}^{-}(t).$$

Taking into account the relations $P_{n+1}\mathbb{T}_{t^{n+1}I_p} = 0$, $\mathbb{T}_{t^{-m-1}I_q}P_{m+1} = 0$, $\mathbb{T}_{d_j}\mathbb{T}_{d_j^{-1}}\mathbb{T}_{d_j} = \mathbb{T}_{d_j}$ $(j = 1, 2)$, we have $A_1 = P_{n+1}\mathbb{T}_{a_{21}^{-}d_1a_{11}^{+} + a_{22}^{-}d_2a_{21}^{+}}P_{m+1}$.

But it follows from the factorization $A(t) = A_-(t)D(t)A_+(t)$ that

$$a_{21}^{-}(t)d_1(t)a_{11}^{+}(t) + a_{22}^{-}(t)d_2(t)a_{21}^{+}(t) = a(t).$$

Hence $A_1 = P_{n+1}\mathbb{T}_aP_{m+1}$.

Since $t^{n+1}b_{12}^{-}(t) = d_1(t)a_{12}^{+}(t)$ and $t^{n+1}b_{22}^{-}(t) = d_2(t)a_{22}^{+}(t)$, we have for the second term A_2

$$A_2 = -P_{n+1}\mathbb{T}_{a_{21}^{-}d_1a_{12}^{+} + a_{22}^{-}d_2a_{22}^{+}}\mathbb{T}_{t^{-n-1}a}P_{m+1} = -P_{n+1}\mathbb{T}_{t^{n+1}I_p}\mathbb{T}_{t^{-n-1}a}P_{m+1} = 0.$$

Here we use the equality

$$a_{21}^{-}(t)d_1(t)a_{12}^{+}(t) + a_{22}^{-}(t)d_2(t)a_{22}^{+}(t) = t^{n+1}I_p,$$

which follows at once from the factorization of $A(t)$.

Similarly, we can obtain $A_3 = A_4 = 0$. Thus $A = P_{n+1}\mathbb{T}_aP_{m+1}$, that is, $T_aGT_a = T_a$. This means that G is a generalized inverse of T_a. If all factorization indices of $A(t)$ are nonnegative (nonpositive), then in the same manner one can prove that $GT_a = T_a$ $(T_aG = T_a)$. In particular, if all factorization indices are equal to zero, then G is the inverse of T_a. The theorem is proved. ■

For $p = q = 1$ and zero factorization indices of $A(t)$ we arrive at Theorem 1 of [19]. If we denote

$$\mathscr{R}(t) = \begin{pmatrix} b_{11}^{+}(t) & b_{12}^{+}(t) \end{pmatrix}, \qquad \mathscr{L}(t) = \begin{pmatrix} b_{12}^{-}(t) \\ b_{22}^{-}(t) \end{pmatrix}.$$

then (2.2) can be rewritten in the following form:

$$G = P_{m+1}\mathbb{T}_{\mathscr{R}}P_{m+1}\mathbb{T}_{D^{-1}}P_{n+1}\mathbb{T}_{\mathscr{L}}P_{n+1}|\mathrm{Im}\ P_{n+1}. \tag{2.3}$$

Now in order to obtain the generalized inverse G in an explicit form we require an explicit method for a construction of the Wiener-Hopf factorization of $A(t)$ or an explicit method for the construction of $\mathscr{R}(t)$, $\mathscr{L}(t)$, and $D(t) = \mathrm{diag}[t^{\mu_1}, \dots, t^{\mu_{p+q}}]$ in terms of the sequence $a_{-m}, \dots, a_0, \dots, a_n$.

We shall need the following lemma, which can be proved by standard methods (see, e.g., [10, Chapter VIII]).

LEMMA 2.1. *Let $A(t) = A_-(t)D(t)A_+(t)$ be the Wiener-Hopf factorization of $A(t)$. Then*

(1) $-m-1 \leqslant \mu_j \leqslant n+1$, $j = 1, 2, \dots, p+q$,
(2) $[A_+^{-1}(t)]^j$ *is a vector polynomial in t of degree at most $m + \mu_j + 1$,*
(3) $[A_-^{-1}(t)]_j$ *is a vector polynomial in t^{-1} of degree at most $n - \mu_j + 1$.*

In particular, $R_j(t) = [\mathscr{R}(t)]^j$ ($L_j(t) = [\mathscr{L}(t)]_j$), $j = 1, 2, \dots, p+q$, is a vector polynomial in t (t^{-1}) of degree at most $m + \mu_j + 1$ ($n - \mu_j + 1$).

Let us denote

$$r_-(t) = \begin{pmatrix} a_{21}^-(t) & a_{22}^-(t) \end{pmatrix}, \qquad l_+(t) = \begin{pmatrix} b_{21}^+(t) & b_{22}^+(t) \end{pmatrix}.$$

Then it follows from the factorization $A(t)A_+^{-1}(t) = A_-(t)D(t)$ that

$$a(t)\mathscr{R}(t) = r_-(t)D(t) - t^{n+1}l_+(t),$$

or

$$a(t)R_j(t) = t^{\mu_j}r_j^-(t) - t^{n+1}l_j^+(t), \tag{2.4}$$

where $r_j^-(t) = [r_-(t)]^j$, $l_j^+(t) = [l_+(t)]^j$, $j = 1, 2, \dots, p+q$.

LEMMA 2.2. *Let α (ω) be the multiplicity of $-m-1$ ($n+1$) as the factorization index of $A(t)$. Then*

$$\alpha = \dim \ker_R T_{-m}, \qquad \omega = \dim \ker_L T_n.$$

Proof. It follows from (2.4) that $a_{-m}R_j = \cdots = a_n R_j = 0$ and $r_j^-(t) \equiv l_j^+(t) \equiv 0$, $j = 1, 2, \ldots, \alpha$. Since

$$R_j = \left[\begin{pmatrix} b_{11}^+(t) & b_{12}^+(t) \end{pmatrix}\right]^j \qquad l_j^+(t) = \left[\begin{pmatrix} b_{21}^+(t) & b_{22}^+(t) \end{pmatrix}\right]^j,$$

we have

$$\left[A_+^{-1}(t)\right]^j = \begin{pmatrix} R_j \\ 0 \end{pmatrix}, \qquad j = 1, 2, \ldots, \alpha.$$

Hence $R_1, \ldots, R_\alpha$ are linearly independent vectors in $\ker_R T_{-m}$. Thus the dimension of this space is not less than α.

Conversely, let $R_1, \ldots, R_d$ be a basis of $\ker_R T_{-m}$. We form the matrix $(R_1 \cdots R_d)$ and extend it to an invertible $q \times q$ matrix C_{11}. Let us define

$$C = \begin{pmatrix} C_{11} & 0 \\ 0 & I_p \end{pmatrix}.$$

Then

$$C^{-1}A(t)C = \begin{pmatrix} t^{-m-1}I_q & 0 \\ a(t)C_{11} & t^{n+1}I_p \end{pmatrix}.$$

Since $a(t)C_{11} = (0_{p\times d} \;\; a_1(t))$, the matrix $C^{-1}A(t)C$ has the following structure:

$$\begin{pmatrix} t^{-m-1}I_d & 0 & 0 \\ 0 & t^{-m-1}I_{q-d} & 0 \\ 0 & a_1(t) & t^{n+1}I_p \end{pmatrix}.$$

This means that α is not less than d. Hence $d = \alpha$. In an analogous manner we can obtain the second part of the lemma. ■

Let now $j = 1, 2, \ldots, p + q - \omega$. It follows from the expansion (2.4) that the coefficient of t^k in the vector polynomial $a(t)R_j(t)$ is equal to zero for $k = \mu_j + 1, \mu_j + 2, \ldots, n$, that is, the coefficients of the vector polynomial $R_j(t)$ satisfy the system of equations

$$\sum_{i=0}^{n-\mu_j+1} a_{k-j}R_i^j = 0, \qquad k = \mu_j + 1, \mu_j + 2, \ldots, n.$$

In other words, the column formed from the coefficients of the column polynomial $R_j(t)$ is the element of the space $\ker_R T_{\mu_j+1}$ $(j = 1, 2, \dots, p + q - \omega)$. Similarly, if we denote

$$r_+(t) = \begin{pmatrix} a_{11}^+(t) \\ a_{21}^+(t) \end{pmatrix}, \qquad l_-(t) = \begin{pmatrix} b_{11}^-(t) \\ b_{21}^-(t) \end{pmatrix},$$

then from the factorization $A_-^{-1}(t)A(t) = D(t)A_+(t)$ we have

$$\mathscr{L}(t)a(t) = D(t)r_+(t) - t^{-m-1}l_-(t),$$

or

$$L_j(t)a(t) = t^{\mu_j}r_j^+(t) - t^{-m-1}l_j^-(t), \tag{2.5}$$

where $r_j^+(t) = [r_+(t)]_j$, $l_j^-(t) = [l_-(t)]_j$. From this expansion it follows that the row formed from the coefficients of the row polynomial $L_j(t)$ is the element of the space $\ker_L T_{\mu_j-1}$ $(j = \alpha + 1, \alpha + 2, \dots, p + q)$.

These considerations show that we shall need a detailed study of a structure of the right and left kernels for block Toeplitz matrices of the family $\{T_k\}_{k=-m}^{n}$. This will be done in the following section.

3. DEFINITION OF INDICES AND ESSENTIAL POLYNOMIALS

In the following two sections we develop a technique that we shall use in the sequel. The main results were obtained in 1985 [2] for $p = q$.

Our nearest aim is to describe a structure of the right and left kernels of T_k.

Since it is more convenient to deal not with vectors but with generating vector polynomials, we pass from the spaces $\ker_R T_k$ and $\ker_L T_k$ to the isomorphic spaces of generating vector polynomials in t or in t^{-1}. To do this, we introduce operators σ_R and σ_L. For $p = q = 1$ the operator $\sigma_R = \sigma_L$ is the Stieltjes functional used in the theory of orthogonal polynomials.

We define on the space of rational matrix polynomials of the form $R(t) = \sum_{j=-n}^{m} r_j t^j$, $r_j \in \mathbb{C}^{q\times l}$, the operator σ_R into the space $\mathbb{C}^{p\times l}$ according to the formula

$$\sigma_R\{R(t)\} = \sum_{j=-n}^{m} a_{-j}r_j. \tag{3.1}$$

(We use the notation σ_R for all $l \geqslant 1$ because there will be no possibility of misinterpretation.)

By N_k^R $(-m \leqslant k \leqslant n)$ we denote the space of vector polynomials of the form $R(t) = \sum_{j=0}^{m+k} r_j t^j$, $r_j \in \mathbb{C}^{q \times 1}$, such that

$$\sigma_R\{t^{-i}R(t)\} = 0, \qquad i = k, k+1, \ldots, n. \tag{3.2}$$

It is easily seen that N_k^R is the space of generating polynomials of vectors in $\ker_R T_k$. For convenience, we put $N_{-m-1}^R = 0$ and denote by N_{n+1}^R the $(n+m+2)q$-dimensional space of all vector polynomials in t of formal degree $n+m+1$.

It follows from the definition (3.1) that $\sigma_R\{t^{-i}R(t)\}$ coincides with the coefficient of t^i in the vector polynomial $a(t)R(t)$. Hence $R(t) \in N_{k+1}^R$ $(-m \leqslant k \leqslant n)$ iff

$$a(t)R(t) = t^k R_-(t) + t^{n+1}R_+(t), \tag{3.3}$$

where $R_+(t)$ $[R_-(t)]$ is a vector polynomial in t $[t^{-1}]$ of formal degree $m+k$.

Similarly, we define on the space of rational matrix polynomials of the form $L(t) = \sum_{j=-n}^{m} l_j t^j$, $l_j \in \mathbb{C}^{l \times p}$, the operator σ_L into the space $\mathbb{C}^{l \times q}$:

$$\sigma_L\{L(t)\} = \sum_{j=-n}^{m} l_j a_{-j}.$$

The space $\ker_L T_k$ is naturally isomorphic to the space N_k^L of vector polynomials in t^{-1} of the form $L(t) = \sum_{j=0}^{n-k} l_j t^{-j}$, $l_j \in \mathbb{C}^{1 \times p}$, such that

$$\sigma_L\{t^{-i}L(t)\} = 0, \qquad i = k, k-1, \ldots, -m.$$

We put $N_{n+1}^L = 0$ and denote by N_{-m-1}^L the $(n+m+2)p$-dimensional space of all vector polynomials in t^{-1} of formal degree $n+m+1$. It is easily seen that $L(t) \in N_{k-1}^L$ $(-m \leqslant k \leqslant n)$ iff

$$L(t)a(t) = t^k L_+(t) + t^{-m-1}L_-(t), \tag{3.4}$$

where $L_+(t)$ $[L_-(t)]$ is a vector polynomial in t $[t^{-1}]$ of formal degree $n-k$.

Let $\alpha = \dim N_{-m}^R$ and $\omega = \dim N_n^L$. We shall say that the sequence $a_{-m}, \ldots, a_0, \ldots, a_n$ is *left regular* (*right regular*) if $\alpha = 0$ $(\omega = 0)$. The sequence is said to be *regular* if $\alpha = \omega = 0$. We shall also apply the notion of regularity to the symbol $a(t)$.

By d_k^R (d_k^L) denote the dimension of the space N_k^R (N_k^L). Let $\Delta_k^R = d_k^R - d_{k-1}^R$ $(-m \leqslant k \leqslant n+1)$, $\Delta_k^L = d_k^L - d_{k+1}^L$ $(-m-1 \leqslant k \leqslant n)$.

PROPOSITION 3.1. *For any sequence $a_{-m}, \dots, a_0, \dots, a_n$ of complex $p \times q$ matrices we have*

$$\alpha = \Delta_{-m}^R \leqslant \Delta_{-m+1}^R \leqslant \cdots \leqslant \Delta_n^R \leqslant \Delta_{n+1}^R = p + q - \omega, \tag{3.5}$$

$$p + q - \alpha = \Delta_{-m-1}^L \geqslant \Delta_{-m}^L \geqslant \cdots \geqslant \Delta_{n-1}^L \geqslant \Delta_n^L = \omega. \tag{3.6}$$

Proof. It follows from the definition (3.2) that N_k^R and tN_k^R are subspaces of N_{k+1}^R and $N_k^R \cap tN_k^R = tN_{k-1}^R$ for $-m \leqslant k \leqslant n$. Hence, by the Grassman formula,

$$\dim\left(N_k^R + tN_k^R\right) = 2d_k^R - d_{k-1}^R. \tag{3.7}$$

Let us denote by h_{k+1}^R the dimension of any complement H_{k+1}^R of the subspace $N_k^R + tN_k^R$ in the whole space N_{k+1}^R. From (3.7) we have $h_{k+1}^R = \Delta_{k+1}^R - \Delta_k^R$, that is, $\Delta_{k+1}^R \geqslant \Delta_k^R$. It is easily seen that $\Delta_{-m}^R = \alpha$ and $\Delta_{n+1}^R = p + q - \omega$. In a similar manner we can prove the statement of the proposition on the sequence Δ_k^L. ■

It follows from the inequalities (3.5) that there exist $p + q - \alpha - \omega$ integers $\mu_{\alpha+1} \leqslant \cdots \leqslant \mu_{p+q-\omega}$ such that

$$\begin{aligned}
\Delta_{-m}^R &= \cdots = \Delta_{\mu_{\alpha+1}}^R = \alpha, \\
&\vdots \\
\Delta_{\mu_i+1}^R &= \cdots = \Delta_{\mu_{i+1}}^R = i, \\
&\vdots \\
\Delta_{\mu_{p+q-\omega}+1}^R &= \cdots = \Delta_{n+1}^R = p + q - \omega.
\end{aligned} \tag{3.8}$$

If the ith row in these relations is absent, then we assume that $\mu_i = \mu_{i+1}$. By definition, put $\mu_1 = \cdots = \mu_\alpha = -m - 1$ if $\alpha \neq 0$ and $\mu_{p+q-\omega+1} = \cdots = \mu_{p+q} = n + 1$ if $\omega \neq 0$.

Similarly, from (3.6) we have

$$\begin{aligned} \Delta^L_{-m-1} = \cdots &= \Delta^L_{\nu_{\alpha+1}-1} = p+q-\alpha, \\ &\vdots \\ \Delta^L_{\nu_i} = \cdots &= \Delta^L_{\nu_{i+1}-1} = p+q-i, \\ &\vdots \\ \Delta^L_{\nu_{p+q-\omega}} = \cdots &= \Delta^L_n = \omega \end{aligned} \tag{3.9}$$

for some integers $\nu_{\alpha+1} \leqslant \cdots \leqslant \nu_{p+q-\omega}$. Put $\nu_1 = \cdots = \nu_\alpha = -m-1$ (for $\alpha \neq 0$) and $\nu_{p+q-\omega+1} = \cdots = \nu_{p+q} = n+1$ (for $\omega \neq 0$).

PROPOSITION 3.2. *For any sequence $a_{-m}, \dots, a_0, \dots, a_n$ of complex $p \times q$ matrices the integers $\mu_1, \dots, \mu_{p+q}$ coincide with $\nu_1, \dots, \nu_{p+q}$. Moreover,*

$$\sum_{j=1}^{p+q} \mu_j = -\operatorname{ind} T_a. \tag{3.10}$$

Proof. It is easily seen that $\Delta^L_k = p+q-\Delta^R_{k+1}$. This implies that $\mu_j = \nu_j$, $j = 1, \dots, p+q$. Since $d^R_{n+1} = \sum_{j=-m}^{n+1} \Delta^R_j$, it follows from (3.8) that

$$\sum_{j=1}^{p+q} \mu_j = (n+1)p - (m+1)q = -\operatorname{ind} T_a. \qquad \blacksquare$$

DEFINITION 3.1. The integers $\mu_1, \dots, \mu_{p+q}$ defined in (3.8) will be called the *essential indices* (briefly, indices) of the sequence $a_{-m}, \dots, a_0, \dots, a_n$ and its symbol $a(t)$.

From the relations (3.8) we get at once a way to compute the indices of the sequence in terms of the ranks r_k of the matrices T_k $(-m \leqslant k \leqslant n)$:

$$\mu_j = \operatorname{card}\{k \mid q + r_{k-1} - r_k \leqslant j-1\}_{k=-m}^{n+1} - m - 1, \tag{3.11}$$

$j = 1, 2, \dots, p+q$. Here card A is the cardinality of the set A, and by definition $r_{-m-1} = r_{n+1} = 0$.

Since the dimension h_{k+1}^R of the complement H_{k+1}^R of the subspace $N_k^R + tN_k^R$ in the space N_{k+1}^R is equal to $\Delta_{k+1}^R - \Delta_k^R$, it follows from (3.8) that $h_{k+1}^R \neq 0$ iff $k = \mu_j$ ($j = \alpha + 1, \dots, p + q - \omega$). In this case h_{k+1}^R coincides with the multiplicity k_j of the index μ_j. Hence for $k \neq \mu_j$

$$N_{k+1}^R = N_k^R + tN_k^R, \tag{3.12}$$

and for $k = \mu_j$

$$N_{k+1}^R = \left(N_k^R + tN_k^R\right) \dot{+} H_{k+1}^R. \tag{3.13}$$

DEFINITION 3.2. If $\alpha \neq 0$, then any column polynomials $R_1(t), \dots, R_\alpha(t)$ that form a basis for the space N_{-m}^R will be called *right essential polynomials* of the sequence $a_{-m}, \dots, a_0, \dots, a_n$ [and its symbol $a(t)$] corresponding to the index $\mu_1 = \cdots = \mu_\alpha$.

Any polynomials $R_j(t), \dots, R_{j+k_j-1}(t)$ that form a basis for $H_{\mu_j+1}^R$ will be called *right essential polynomials* of the sequence [and its symbol $a(t)$] corresponding to the index μ_j, $\alpha + 1 \leqslant j \leqslant p + 1 - \omega$.

Similarly, for $k \neq \mu_j$

$$N_{k-1}^L = N_k^L + t^{-1}N_k^L,$$

and for $k = \mu_j$

$$N_{k-1}^L = \left(N_k^L + t^{-1}N_k^L\right) \dot{+} H_{k-1}^L.$$

Choosing bases for the space N_n^L (if $\omega \neq 0$) and for the spaces $H_{\mu_j-1}^L$ ($\alpha + 1 \leqslant j \leqslant p + q - \omega$), we obtain a sequence of vector polynomials $L_{\alpha+1}(t), \dots, L_{p+q}(t)$ that will be called *left essential polynomials* of the sequence $a_{-m}, \dots, a_0, \dots, a_n$ and its symbol $a(t)$.

Therefore, for any sequence $a_{-m}, \dots, a_0, \dots, a_n$ there are $p + q$ indices, $p + q - \omega$ right essential polynomials, and $p + q - \alpha$ left essential polynomials. The remaining essential polynomials we shall define in the sequel.

Now we can describe the structure of the right and left kernels of the matrices T_k in terms of the indices and essential polynomials of the sequence $a_{-m}, \dots, a_0, \dots, a_n$.

THEOREM 3.1. *Let the integers* $\mu_1, \dots, \mu_{p+q}$ *be the indices of the sequence* $a_{-m}, \dots, a_0, \dots, a_n$ *and let* $R_1(t), \dots, R_{p+q-\omega}(t)$; $L_{\alpha+q}(t), \dots, L_{p+q}(t)$ *be the essential polynomials of this sequence. Then the vector polynomials*

$$\left\{R_j(t), tR_j(t), \dots, t^{k-\mu_j-1}R_j(t)\right\}_{j=1}^{i} \tag{3.14}$$

are the generating polynomials for elements of a basis of the space $\ker_R T_k$ *for* $k \in (\mu_i; \mu_{i+1}]$, $1 \leqslant i \leqslant p+q-\omega$. *Here we put* $\mu_{p+q+1} = n$ *if* $\omega = 0$.

Similarly, the vector polynomials

$$\left\{L_j(t), t^{-1}L_j(t), \dots, t^{-(\mu_j-k-1)}L_j(t)\right\}_{j=i}^{p+q} \tag{3.15}$$

are the generating polynomials for elements of a basis of the space $\ker_L T_k$ *for* $k \in [\mu_{i+1}; \mu_i)$, $\alpha + 1 \leqslant i \leqslant p+q$. *Here we put* $\mu_0 = -m$ *if* $\alpha = 0$.

Proof. It follows from (3.12) and (3.13) that the polynomials (3.14) generate the space N_k^R. Since $d_k^R = \sum_{j=-m}^{k} \Delta_j^R$, we have

$$d_k^R = ik - \sum_{j=1}^{i} \mu_j. \tag{3.16}$$

It is easily seen that the number of polynomials (3.14) is equal to d_k^R. Hence they form a basis for the space N_k^R.

The second part of the theorem is proved in a similar manner. ■

In particular, it follows from Theorem 3.1 that the kernel structure of a finite Toeplitz matrix T_a is just like that of a Toeplitz operator with an invertible symbol. This fact was first obtained by G. Heinig (see, e.g., [17]).

4. CRITERION OF ESSENTIALNESS

In this section we solve the following problem. What are the conditions in order that given integers shall be the indices and given polynomials shall be the essential polynomials of the sequence $a_{-m}, \dots, a_0, \dots, a_n$? The following theorem gives a criterion for checking essentialness.

THEOREM 4.1. *Let* $a_{-m}, \dots, a_0, \dots, a_n$ *be an arbitrary sequence of complex* $p \times q$ *matrices and* $\omega = \dim \ker_L T_n$. *Let* $\kappa_1, \dots, \kappa_{p+q-\omega}$ *be integers such that* $-m-1 \leqslant \kappa_1 \leqslant \cdots \leqslant \kappa_{k+q-\omega} \leqslant n$ *and*

$$\sum_{j=1}^{p+q-\omega} \kappa_j = (n+1)(p-\omega) - (m+1)q. \tag{4.1}$$

Let $U_1(t), \dots, U_{p+q-\omega}(t)$ *be column polynomials such that* $U_j(t) \in N^R_{\kappa_j+1}$, $1 \leqslant j \leqslant p+q-\omega$. *If* $\omega \neq 0$, *then we put* $\kappa_{p+q-\omega+1} = \cdots = k_{p+q} = n+1$ *if* $w = 0$.

The integers $\kappa_1, \dots, \kappa_{p+q}$ *are the indices and the polynomials* $U_1(t), \dots, U_{p+q-\omega}(t)$ *are right essential polynomials of the sequence if and only if the* $(p+q) \times (p+q-\omega)$ *matrix*

$$\Lambda_R = \begin{pmatrix} \tilde{\sigma}\{t^{-\kappa_1}U_1(t)\} & \cdots & \tilde{\sigma}_R\{t^{-\kappa_{p-q-\omega}}U_{p+q-\omega}(t)\} \\ U_{1,m+k_1+1} & \cdots & U_{p+q-\omega,m+\kappa_{p+q-\omega}+1} \end{pmatrix}$$

or the $(p+q) \times (p+q-\omega)$ *matrix*

$$\hat{\Lambda}_R = \begin{pmatrix} \hat{\sigma}_R\{t^{-n-1}U_1(t)\} & \cdots & \hat{\sigma}_R\{t^{-n-1}U_{p+q-\omega}(t)\} \\ U_{1,0} & \cdots & U_{p+q-\omega,0} \end{pmatrix}$$

is left invertible.

Similarly, let $\alpha = \dim \ker_R T_{-m}$, *and let* $\kappa_{\alpha+1}, \dots, \kappa_{p+q}$ *be integers such that* $-m \leqslant \kappa_{\alpha+1} \leqslant \cdots \leqslant \kappa_{p+q} \leqslant n+1$ *and*

$$\sum_{j=\alpha+1}^{p+q} \kappa_j = (n+1)p - (m+1)(q-\alpha).$$

Let $V_{\alpha+1}(t), \dots, V_{p+q}(t)$ *be row polynomials such that* $V_j(t) \in N^L_{\kappa_j-1}$, $\alpha+1 \leqslant j \leqslant p+q$. *If* $\alpha \neq 0$, *then we put* $\kappa_1 = \cdots = \kappa_\alpha = -m-1$.

The integers $\kappa_1, \dots, \kappa_{p+q}$ *are the indices and the polynomials* $V_{\alpha+1}(t), \dots, V_{p+q}(t)$ *are left essential polynomials of the sequence if and only if the*

$(p+q-\alpha)\times(p+q)$ matrix

$$\Lambda_L = \begin{pmatrix} V_{\alpha+1,0} & \tilde{\sigma}_L\{t^{m+1}V_{\alpha+1}(t)\} \\ \vdots & \vdots \\ V_{p+q,0} & \tilde{\sigma}_L\{t^{m+1}V_{p+q}(t)\} \end{pmatrix}$$

or the $(p+q-\alpha)\times(p+q)$ matrix

$$\hat{\Lambda}_L = \begin{pmatrix} V_{\alpha+1,n-\kappa_{\alpha+1}+1} & \hat{\sigma}_L\{t^{-\kappa_{\alpha+1}}V_{\alpha+1}(t)\} \\ \vdots & \vdots \\ V_{p+q,n-\kappa_{p+q}+1} & \hat{\sigma}_L\{t^{-\kappa_{p+q}}V_{p+q}(t)\} \end{pmatrix}$$

is right invertible.

Here $\tilde{\sigma}_R, \tilde{\sigma}_L$ are the Stieltjes operators for the extended sequence $a_{-m-1}, a_{-m}, \dots, a_0, \dots, a_n$, where a_{-m-1} is an arbitrary matrix; $U_{j,m+\kappa_j+1}$ is the leading coefficient of the column polynomial $U_j(t)$; and $V_{j,0}$ is the constant term of the row polynomial $V_j(t)$. In the matrices $\hat{\Lambda}_R, \hat{\Lambda}_L$ the operators $\hat{\sigma}_R, \hat{\sigma}_L$ correspond to the extended sequence $a_{-m}, \dots, a_0, \dots, a_n$, a_{n+1}, where a_{n+1} is an arbitrary matrix.

Proof. Necessity: Let $k_1, \dots, k_{p+q}$ be the indices, and let $U_1(t), \dots, U_{p+q-\omega}(t)$ be the right essential polynomials of the sequence. Put $r = p+q-\omega$. Suppose that the rank of the matrix Λ_R is less than r. Then there exist numbers $\alpha_1, \dots, \alpha_r$, not all zero, such that

$$\alpha_1\tilde{\sigma}_R\{t^{-\kappa_1}U_1(t)\} + \cdots + \alpha_r\tilde{\sigma}_R\{t^{-\kappa_r}U_r(t)\} = 0 \tag{4.2}$$

and

$$\alpha_1 U_{1,m+\kappa_1+1} + \cdots + \alpha_r U_{r,m+\kappa_r+1} = 0. \tag{4.3}$$

Let the index κ_r has the multiplicity ν, that is, $\kappa_{r+\nu} < \kappa_{r-\nu+1} = \cdots = \kappa_r < k_{r+1}$. We introduce the polynomial

$$Q(t) = \alpha_1 t^{\kappa_r-\kappa_1}U_1(t) + \cdots + \alpha_{r-\nu}t^{\kappa_r-\kappa_{r-\nu}}U_{r-\nu}(t) + \alpha_{r-\nu+1}U_{r-\nu+1}(t) + \cdots + \alpha_r U_r(t).$$

From (4.3) it follows that the degree of this polynomial is not greater than $m + \kappa_r$. Then (4.2) means that $\sigma_R\{t^{-\kappa_r}Q(t)\} = 0$. Since $Q(t) \in N^R_{\kappa_r+1}$, we have $Q(t) \in N^R_{\kappa_r}$ and

$$\begin{aligned}\alpha_{r-\nu+1}U_{r-\nu+1}(t) &+ \cdots + \alpha_r U_r(t) \\ &= Q(t) - t\left[\alpha_1 t^{\kappa_r-\kappa_1-1}U_1(t) + \cdots + \alpha_{r-\nu}t^{\kappa_r-\kappa_{r-\nu}-1}U_{r-\nu}(t)\right] \\ &\in N^R_{\kappa_r} + tN^R_{\kappa_r}. \end{aligned} \tag{4.4}$$

However, $U_{r-\nu+1}(t), \dots, U_r(t)$ are the right essential polynomials corresponding to the index κ_r. Therefore the condition (4.4) is fulfilled iff $\alpha_{r-\nu+1} = \cdots = \alpha_r = 0$. By repeating these arguments for the indices $\kappa_{r-\nu}, \dots, \kappa_1$, we obtain $\alpha_1 = \cdots = \alpha_r = 0$. The contradiction shows that the rank of the matrix Λ_R is equal to r. In an analogous manner we obtain the proofs of the statements about the matrices $\hat{\Lambda}_R, \Lambda_L, \hat{\Lambda}_L$.

The proof of sufficiency is just like that of Theorem 3.1 from [5] and is omitted. ∎

We shall call $\Lambda_R, \hat{\Lambda}_R$ ($\Lambda_L, \hat{\Lambda}_L$) *test matrices* for right (left) essential polynomials.

5. CONSTRUCTION OF THE GENERALIZED INVERSE IN TERMS OF ESSENTIAL POLYNOMIALS

Now we consider a connection between the indices and essential polynomials of $a(t)$ and the Wiener-Hopf factorization of $A(t)$.

THEOREM 5.1. *The factorization indices of $A(t)$ coincide with the essential indices of $a(t)$. Moreover, the polynomials*

$$R_j(t) = \left[\begin{pmatrix} b^+_{11}(t) & b^+_{12}(t)\end{pmatrix}\right]^j, \qquad j = 1, 2, \dots, p+q-\omega,$$

are right essential polynomials of $a(t)$, and

$$L_j(t) = \left[\begin{pmatrix} b^-_{12}(t) \\ b^-_{22}(t)\end{pmatrix}\right]_j, \qquad j = \alpha+1, \alpha+2, \dots, p+q,$$

are left essential polynomials of $a(t)$.

Proof. Let ω be the multiplicity of $n+1$ as the factorization index of $A(t)$. Recall that $\omega = \dim \ker_L T_n$ (Lemma 2.2). If $\rho_1, \dots, \rho_{p+q}$ are the factorization indices of $A(t)$, then

$$\sum_{j=1}^{p+q-\omega} \rho_j = (n+1)(p-\omega) - (m+1)q.$$

Moreover, in Section 2 we showed that $R_j(t) \in N^R_{\rho_j+1}$, $j = 1, 2, \dots, p+q-\omega$. Let us compose the test matrix Λ_R for this system of polynomials. Put $a_{-m-1} = 0$ and find $\tilde{\sigma}_R\{t^{-\rho_j} R_j(t)\}$. It follows from Equation (2.4) that

$$\tilde{\sigma}_R\{t^{-\rho_j} R_j(t)\} = r_j^-(\infty) = \left[\left(a_{21}^-(\infty) a_{22}^-(\infty)\right)\right]^j.$$

We denote $\mathscr{R}_-(t) = t^{-m-1}\mathscr{R}(t) D^{-1}(t)$. It is evident that the leading coefficient of the polynomial $R_j(t)$ coincides with $[\mathscr{R}_-(\infty)]^j$. From the factorization $A(t) A_+^{-1}(t) D^{-1}(t) = A_-(t)$ we have

$$\mathscr{R}_-(t) = \left(a_{11}^-(t)\ a_{12}^-(t)\right)$$

Thus the matrix Λ_R is obtained from the invertible matrix

$$\begin{pmatrix} 0 & I_p \\ I_q & 0 \end{pmatrix} A_-(\infty)$$

by deleting the last ω columns. Therefore Λ_R is a matrix of full rank, and, by Theorem 4.1, $\rho_1, \dots, \rho_{p+q}$ are the essential indices and the polynomials $R_j(t)$, $1 \leqslant j \leqslant p+q-\omega$, are the right essential polynomials of $a(t)$.

The second part of the theorem is proved similarly. ■

This theorem gives a way to compute the factorization indices of $A(t)$ in terms of the essential indices of the sequence $a_{-m}, \dots, a_0, \dots, a_n$. Hence the factorization indices can be explicitly found by (3.11).

Now we show that the factors $A_\pm(t)$ can be explicitly found in terms of the right essential polynomials $R_1(t), \dots, R_{p+q-\omega}(t)$ (for $p \leqslant q$) or in terms of the left essential polynomials $L_{\alpha+1}(t), \dots, L_{p+q}(t)$ (for $p \geqslant q$).

First we extend the system $R_1(t), \dots, R_{p+q-\omega}(t)$ (for $\omega \neq 0$ and $p \leqslant q$) or the system $L_{\alpha+1}(t), \dots, L_{p+q}(t)$ (for $\alpha \neq 0$ and $p \geqslant q$) to a full system consisting of $p+q$ polynomials.

Let $\omega \neq 0$ and $p \leqslant q$. Let us define essential polynomials $R_{p+q-\omega}(t), \ldots, R_{p+q}(t)$ corresponding to the index $n+1$ of multiplicity ω. To do this, we extend the left invertible matrix Λ_R to an invertible matrix Λ_R^e and partition the additional columns $[\Lambda_R^e]^j$ of the matrix Λ_R^e into blocks $\sigma_j^R \in \mathbb{C}^{p\times 1}$ and $r_j \in \mathbb{C}^{q\times 1}$:

$$[\Lambda_R^e]^j = \begin{pmatrix} \sigma_j^R \\ r_j \end{pmatrix}, \qquad p+q-\omega+1 \leqslant j \leqslant p+q.$$

Moreover, we extend the sequence $a_{-m-1}, a_{-m}, \ldots, a_0, \ldots, a_n$ by an arbitrary right invertible matrix a_{n+1}. Then the matrix $(a_{n+1}, a_n, \ldots, a_{-m}, a_{-m-1})$ is also right invertible. Hence the equation

$$\tilde{\sigma}\left\{t^{-(n+1)} \sum_{i=0}^{n+m+2} x_i t^i\right\}$$
$$\equiv a_{n+1}x_0 + a_n x_1 + \cdots + a_{-m}x_{n+m+1} + a_{-m-1}x_{n+m+2} = y$$

($x_i \in \mathbb{C}^{q\times 1}$, $y \in \mathbb{C}^{p\times 1}$) is solvable for any y.

DEFINITION 5.1. Let $\omega \neq 0$, $p \leqslant q$. Arbitrary column polynomials $R_{p+q-\omega+1}(t), \ldots, R_{p+q}(t)$ of formal degree $n+m+2$ such that

$$\tilde{\sigma}_R\left\{t^{-(n+1)}R_j(t)\right\} = \sigma_j^R, \qquad R_{j,\,n+m+2} = r_j,$$

$j = p+q-\omega+1, \ldots, p+1$, are called *right essential polynomials* of the sequence $a_{-m}, \ldots, a_0, \ldots, a_n$ corresponding to the index $n+1$.

In a similar manner we define deficient left essential polynomials $L_1(t), \ldots, L_\alpha(t)$ is $\alpha \neq 0$ and $p \geqslant q$.

DEFINITION 5.2. Let $\alpha \neq 0$ and $p \geqslant q$. We extend the right invertible matrix Λ_L to an invertible matrix Λ_L^e by the rows

$$[\Lambda_L^e]_j = \left(l_j, \sigma_j^L\right), \qquad l_j \in \mathbb{C}^{1\times p}, \qquad \sigma_j^L \in \mathbb{C}^{1\times q},$$

$j = 1, \ldots, \alpha$. The sequence $a_{-m-1}, a_{-m}, \ldots, a_0, \ldots, a_n$ is extended by an arbitrary left invertible matrix a_{n+1}. Arbitrary row polynomials $L_1(t), \ldots,$

$L_\alpha(t)$ in t^{-1} of formal degree $n+m+2$ such that

$$\tilde{\sigma}_L\{t^{m+1}L_j(t)\} = \sigma_j^L, \qquad L_{j,0} = l_j,$$

$j = 1, \dots, \alpha$, are called *left essential polynomials* of the sequence $a_{-m}, \dots, a_0, \dots, a_n$ corresponding to the index $-m-1$.

Note that the equations $\tilde{\sigma}_L\{t^{m+1}L_j(t)\} = \sigma_j^L$ are solvable because a_{n+1} is left invertible.

Thus for any sequence of matrices there are $p+q$ right essential polynomials or $p+q$ left essential polynomials.

THEOREM 5.2. *Let $a(t) = \sum_{j=-m}^{n} a_j t^j$ be a rational $p \times q$ matrix polynomial. Suppose that $a(t)$ is right regular or $p \leqslant q$. Let $\mu_1, \dots, \mu_{p+q}$ be the essential indices, and let*

$$\mathscr{R}(t) = \begin{pmatrix} R_1(t) & \cdots & R_{p+q}(t) \end{pmatrix}$$

be the matrix of the right essential polynomials of $a(t)$.

Then the right Wiener-Hopf factorization of $A(t)$ with respect to $\mathbb{T}$ can be constructed by the formula

$$A(t) = A_-(t)D(t)B_+^{-1}(t), \tag{5.1}$$

where

$$A_-(t) = \begin{pmatrix} t^{-m-1}\mathscr{R}(t)D^{-1}(t) \\ r_-(t) \end{pmatrix}, \qquad B_+(t) = \begin{pmatrix} \mathscr{R}(t) \\ l_+(t) \end{pmatrix},$$

and the matrix polynomials $r_-(t), l_+(t)$ are uniquely determined by the expansion

$$a(t)\mathscr{R}(t) = r_-(t)D(t) - t^{n+1}l_+(t). \tag{5.2}$$

Similarly, if $a(t)$ is left regular or $p \geqslant q$, then

$$A(t) = B_-^{-1}(t)D(t)A_+(t), \tag{5.3}$$

is the right Wiener-Hopf factorization of $A(t)$. *Here* $A_+(t) = (r_+(t)\; t^{n+1}D^{-1}(t)\mathscr{L}(t))$, $B_-(t) = (l_-(t)\; \mathscr{L}(t))$,

$$\mathscr{L}(t) = \begin{pmatrix} L_1(t) \\ \vdots \\ L_{p+q}(t) \end{pmatrix}$$

is the matrix of the left essential polynomials of $a(t)$, *and* $l_-(t), r_+(t)$ *are uniquely determined by the expansion*

$$\mathscr{L}(t)a(t) = D(t)r_+(t) - t^{-m-1}l_-(t). \tag{5.4}$$

Proof. For the construction of the factorization we shall use the full system of right or left essential polynomials. Hence we must consider the two cases.

Suppose that $a(t)$ is a right regular of $p \leqslant q$. Let $\mu_1, \dots, \mu_{p+q}$ be the essential indices, and let $R_1(t), \dots, R_{p+q}(t)$ be right essential polynomials of $a(t)$. Recall that $\mu_{p+q-\omega+1} = \cdots = \mu_{p+q} = n+1$ if $\omega = \dim \ker_L T_n \neq 0$. The polynomials $R_{p+q-\omega+1}(t), \dots, R_{p+q}(t)$ corresponding to the index $n+1$ are constructed by the matrix Λ_R^e (see Definition 5.1).

If $\mu_j \leqslant n$, then the condition $R_j(t) \in N^R_{\mu_j+1}$ is equivalent to the following relation:

$$a(t)R_j(t) = t^{\mu_j}r_j^-(t) - t^{n+1}l_j^+(t) \tag{5.5}$$

[see Equation (3.3)]. Here $r_j^-(t)$ $[l_j^+(t)]$ is a column polynomial in t^{-1} $[t]$ of degree at most $m + \mu_j$ if $\mu_j \geqslant -m$, and $r_j^-(t) = l_j^+(t) \equiv 0$ if $\mu_j = -m-1$. The polynomials $r_j^-(t), l_j^+(t)$ are uniquely determined by the above expansion. Let us compare the coefficients of t^{μ_j} in (5.5) for $\mu_j \geqslant -m$:

$$a_{\mu_j}R_{j,0} + a_{\mu_j-1}R_{j,1} + \cdots a_{-m}R_{j,m+\mu_j} = r_j^-(\infty).$$

In the matrix Λ_R we put $a_{-m-1} = 0$. Then the previous equation can be rewritten as follows:

$$\tilde{\sigma}_R\{t^{-\mu_j}R_j(t)\} = r_j^-(\infty). \tag{5.6}$$

It is easily seen that this equation is valid for $\mu_j = -m-1$ too.

Now let $\mu_j = n+1$, and let $R_j(t)$ be a right essential polynomial corresponding to the index $n+1$. The expansion (5.5) is also valid in this case, and all coefficients of the polynomials $r_j^-(t), l_j^+(t)$ except the constant terms are uniquely determined. Let us compare the coefficients of t^{n+1} in (5.5):

$$a_n R_{j,1} + a_{n-1} R_{j,2} + \cdots + a_{-m} R_{j,n+m+1} = r_j^-(\infty) - l_j^+(0).$$

Let a_{n+1} be the right invertible matrix from Λ_R^e (see Definition 5.1). The constant terms $r_j^-(\infty)$ and $l_j^+(0)$ are related by the equation

$$\tilde{\sigma}_R\{t^{-(n+1)}R_j(t)\} = r_j^-(\infty) - l_j^+(0) + a_{n+1}R_{j,0}.$$

In (5.5) we put

$$l_j^+(0) = a_{n+1}R_{j,0} \tag{5.7}$$

for $\mu_j = n+1$. Then $r_j^-(\infty)$ is uniquely determined by the equation

$$\tilde{\sigma}_R\{t^{-(n+1)}R_j(t)\} = r_j^-(\infty).$$

Now the relations (5.5)–(5.6) are fulfilled for all right essential polynomials. We rewrite these equations in the matrix form

$$\begin{pmatrix} a(t) & t^{n+1}I_p \end{pmatrix}\begin{pmatrix} \mathscr{R}(t) \\ l_+(t) \end{pmatrix} = r_-(t)D(t), \tag{5.8}$$

$$\begin{pmatrix} \tilde{\sigma}_R\{t^{-\mu_1}R_1(t)\} & \cdots & \tilde{\sigma}_R\{t^{-\mu_{p+q}}R_{p+q}(t)\} \end{pmatrix} = r_-(\infty). \tag{5.9}$$

Here

$$\mathscr{R}(t) = \begin{pmatrix} R_1(t) & \cdots & R_{p+q}(t) \end{pmatrix}, \qquad r_-(t) = \begin{pmatrix} r_1^-(t) & \cdots & r_{p+q}^-(t) \end{pmatrix}$$

$$l_+(t) = \begin{pmatrix} l_1^+(t) & \cdots & l_{p+q}^+(t) \end{pmatrix}, \qquad D(t) = \operatorname{diag}[t^{\mu_1}, \dots, t^{\mu_{p+q}}].$$

Define

$$\mathscr{R}_-(t) = t^{-m-1}\mathscr{R}(t)D^{-1}(t). \tag{5.10}$$

Since the column $[\mathscr{R}(t)]^j = R_j(t)$ is a polynomial in t of formal degree $m + \mu_j + 1$, the column $[\mathscr{R}_-(t)]^j$ is a polynomial in t^{-1} of the same formal degree. We rewrite (5.10) as follows:

$$t^{-m-1}\mathscr{R}(t) = \mathscr{R}_-(t)D(t). \tag{5.11}$$

Now from (5.8), (5.11) we obtain

$$\begin{pmatrix} t^{-m-1}I_q & 0 \\ a(t) & t^{n+1}I_p \end{pmatrix}\begin{pmatrix} \mathscr{R}(t) \\ l_+(t) \end{pmatrix} = \begin{pmatrix} \mathscr{R}_-(t) \\ r_-(t) \end{pmatrix} D(t).$$

Let us introduce $(p+q) \times (p+q)$ matrix functions

$$B_+(t) = \begin{pmatrix} \mathscr{R}(t) \\ l_+(t) \end{pmatrix}, \qquad A_-(t) = \begin{pmatrix} \mathscr{R}_-(t) \\ r_-(t) \end{pmatrix}.$$

$B_+(t)$ $[A_-(t)]$ is a matrix polynomial in t $[t^{-1}]$. Hence $B_+(t)$ $[A_-(t)]$ is analytic in the inner domain D_+ [the outer domain D_-] bounded by the contour $\mathbb{T}$. Thus we get

$$A(t)B_+(t) = A_-(t)D(t).$$

Since the sum of the essential indices of $a(t)$ is $(n+1)p - (m+1)q$, we obtain det $B_+(t) =$ det $A_-(t) =$ const. Let us find $A_-(\infty)$. From (5.9), (5.10) we have

$$A_-(\infty) = \begin{pmatrix} \mathscr{R}_-(\infty) \\ r_-(\infty) \end{pmatrix} = \begin{pmatrix} R_{1,m+\mu_1+1} & \cdots & R_{p+q,m+\mu_{p+q}+1} \\ \tilde{\sigma}_R\{t^{-\mu_1}R_1(t)\} & \cdots & \tilde{\sigma}_R\{t^{-\mu_{p+q}}R_{p+q}(t)\} \end{pmatrix}.$$

It follows from this that

$$A_-(\infty) = \begin{pmatrix} 0 & I_p \\ I_q & 0 \end{pmatrix} \Lambda_R^e. \tag{5.12}$$

Hence $\det B_+(t) = \det A_-(t) \neq 0$, and $B_+^{-1}(t)$ $[A_-^{-1}(t)]$ is a matrix polynomial in t $[t^{-1}]$. Thus

$$A(t) = A_-(t) D(t) B_+^{-1}(t)$$

is a Wiener-Hopf factorization of $A(t)$ with respect to $\mathbb{T}$.

The case when $a(t)$ is left regular or $p \geqslant q$ can be analyzed in a similar manner. ■

Using Theorems 5.2 and 5.1, now we can recover left (right) essential polynomials if we know $p + q$ right (left) ones. We can do this by the following procedure. Let $a(t)$ be a right regular rational matrix polynomial or $p \leqslant q$. Let $R_1(t), \dots, R_{p+q}(t)$ be right essential polynomials of $a(t)$. The matrix $a(t)\mathscr{R}(t)$ can uniquely be expanded in the form

$$a(t)\mathscr{R}(t) = r_-(t) D(t) - t^{n+1} l_+(t).$$

Let us form the matrix

$$A_-(t) = \begin{pmatrix} t^{-m-1}\mathscr{R}(t) D^{-1}(t) \\ r_-(t) \end{pmatrix}.$$

By Theorem 5.2, this matrix is the factor of the right Wiener-Hopf factorization of $A(t)$. Then, by Theorem 5.1, the row polynomials

$$L_j(t) = \begin{bmatrix} b_{12}^-(t) \\ b_{22}^-(t) \end{bmatrix}_j, \qquad j = \alpha + 1, \alpha + 2, \dots, p + q,$$

are left essential polynomials of $a(t)$. Here

$$A_-^{-1}(t) = \begin{pmatrix} b_{11}^-(t) & b_{12}^-(t) \\ b_{21}^-(t) & b_{22}^-(t) \end{pmatrix}.$$

If $\alpha \neq 0$, then the system of these left essential polynomials can be extended by the polynomials $L_j(t)$, $1 \leqslant j \leqslant \alpha$. We shall call them the *left essential polynomials corresponding to the index* $-m-1$.

DEFINITION 5.3. Let $L_1(t), \dots, L_{p+q}(t)$ be the left essential polynomials that are constructed with the help of the right essential polynomials $R_1(t), \dots, R_{p+q}(t)$ according to the above-mentioned procedure. The essential polynomials $L_1(t), \dots, L_{p+q}(t)$ and $R_1(t), \dots, R_{p+q}(t)$ are called the *conforming essential polynomials* of $a(t)$.

Similarly, if we know $p+q$ left essential polynomial [$a(t)$ is left regular or $p \geqslant q$], then we can recover the right essential polynomials $R_1(t), \dots, R_{p+q}(t)$ and constructed the conforming polynomials.

REMARK 5.1. Let $R_1(t)$, $R_2(t)$ be right essential polynomials of a scalar sequence. It is easily seen that if

$$L_1(t) = \frac{1}{\sigma_0} t^{-(m+\mu_2+1)} R_2(t), \qquad L_2(t) = \frac{1}{\sigma_0} t^{-(m+\mu_1+1)} R_1(t),$$

then $R_1(t)$, $R_2(t)$, $L_1(t)$, $L_2(t)$ are the conforming essential polynomials of this sequence. Here $\sigma_0 = \sigma\{t^{-\mu_2} R_{1,\, m+\mu_1+1} R_2(t) - t^{-\mu_1} R_{2,\, m+\mu_2+1} R_1(t)\}$ and, by the essentialness criterion, $\sigma_0 \neq 0$.

Now we can formulate our results (Theorem 2.1, Theorem 5.1, Theorem 5.2) on the generalized inversion of block Toeplitz matrix T_a without the use of the Wiener-Hopf factorization of the auxiliary matrix function $A(t)$.

THEOREM 5.3. *Let* $a(t) = \sum_{j=-m}^{n} a_j t^j$ *be a rational* $p \times q$ *matrix polynomial. Let* $\mu_1, \dots, \mu_{p+1}$ *be the essential indices, and let*

$$\mathscr{R}(t) = \begin{pmatrix} R_1(t) & \cdots & R_{p+q}(t) \end{pmatrix}, \qquad \mathscr{L}(t) = \begin{pmatrix} L_1(t) \\ \vdots \\ L_{p+q}(t) \end{pmatrix}$$

be the matrices of the conforming right and left essential polynomials of $a(t)$.

Then the matrix of the operator

$$G = P_{m+1} \mathbb{T}_{\mathscr{R}} P_{m+1} \mathbb{T}_{D^{-1}} P_{n+1} \mathbb{T}_{\mathscr{L}} P_{n+1} | \operatorname{Im} P_{n+1},$$

where $D(t) = \operatorname{diag}[t^{\mu_1}, \ldots, t^{\mu_{p+q}}]$, *is a generalized (one-sided, two-sided) inverse of* T_a.

Let us find the formula for G in terms of the coefficients of the matrix polynomials $\mathscr{R}(t), \mathscr{L}(t)$. Let $\lambda_1 < \cdots < \lambda_r$ be the distinct essential indices of $a(t)$, and let $\nu_1, \ldots, \nu_r$ be their multiplicities ($\nu_1 + \cdots + \nu_r = p + q$). Then

$$D^{-1} = t^{-\lambda_1}\Pi_{-\lambda_1} + \cdots + t^{-\lambda_r}\Pi_{-\lambda_r}.$$

Here $\Pi_{-\lambda_j} = \|\varepsilon_i^j \delta_{ik}\|_{i,k=1}^{p+q}$, where

$$\varepsilon_i^j = \begin{cases} 1, & i = \nu_1 + \cdots + \nu_{j-1} + 1, \ldots, \nu_1 + \cdots + \nu_j, \\ 0 & \text{otherwise}. \end{cases}$$

Put $\Pi_k = 0$ for $-n \leqslant k \leqslant m$, $k \neq -\lambda_1, \ldots, \lambda_r$. Then the matrix Π of the operator $P_{m+1}\mathbb{T}_{D^{-1}}P_{n+1}|\operatorname{Im} P_{n+1}$ has the following form:

$$\Pi = \begin{pmatrix} \Pi_0 & \Pi_{-1} & \cdots & \Pi_{-n} \\ \Pi_1 & \Pi_0 & \cdots & \Pi_{-n+1} \\ \vdots & \vdots & & \vdots \\ \Pi_m & \Pi_{m-1} & \cdots & \Pi_{m-n} \end{pmatrix}.$$

It is easily seen that Π is a subpermutation matrix.

From Theorem 5.3 we have

$$G = \begin{pmatrix} \mathscr{R}_0 & 0 & \cdots & 0 \\ \mathscr{R}_1 & \mathscr{R}_0 & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathscr{R}_m & \mathscr{R}_{m-1} & \cdots & \mathscr{R}_0 \end{pmatrix} \Pi \begin{pmatrix} \mathscr{L}_0 & \mathscr{L}_{-1} & \cdots & \mathscr{L}_{-n} \\ 0 & \mathscr{L}_0 & \cdots & \mathscr{L}_{-n+1} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \mathscr{L}_0 \end{pmatrix}. \quad (5.13)$$

Here $\mathscr{R}_j \in \mathbb{C}^{p\times(p+q)}$ [$\mathscr{L}_j \in \mathbb{C}^{(p+q)\times p}$] are the coefficients of $\mathscr{R}(t)$ [$\mathscr{L}(t)$].

We note that Theorem 5.2 can be applied to the problem of the explicit construction of a Wiener-Hopf factorization for block triangular matrix functions of the form

$$G(t) = \begin{pmatrix} G_{11}(t) & 0 \\ G_{21}(t) & G_{22}(t) \end{pmatrix},$$

where $G_{11}(t)$ $[G_{22}(t)]$ is a $q \times q$ $[p \times p]$ matrix function admiting a right Wiener-Hopf factorization of the form

$$G_{11}(t) = G_{11}^{-}(t)t^{\nu_1}G_{11}^{+}(t) \qquad [G_{22}(t) = G_{22}^{-}(t)t^{\nu_2}G_{22}^{+}(t)].$$

In particular, we can obtain an explicit solution of the factorization problem for an arbitrary 2×2 triangular matrix function. This was done in the paper [1].

6. GENERATING MATRIX POLYNOMIALS FOR THE GENERALIZED INVERSES

In this section we obtain a formula for the generating matrix polynomial

$$G(t,s) = \sum_{i=0}^{m} \sum_{j=0}^{n} g_{ij} t^i s^{-j}$$

for the generalized inverse

$$G = \|g_{ij}\|_{\substack{i=0,\dots,m \\ j=0,\dots,n}}.$$

Let $\mathscr{P}(\alpha,\beta)$ $(-m \leqslant \alpha \leqslant \beta \leqslant n)$ be the projector acting by the formula

$$\mathscr{P}(\alpha,\beta) \sum_{i=-m}^{n} r_i t^i = \sum_{i=\alpha}^{\beta} r_i t^i.$$

If the operator $\mathscr{P}(\alpha,\beta)$ acts on a polynomial in t and s, then the notation $\mathscr{P}_t(\alpha,\beta)$ means that the operator acts on the variable t.

PROPOSITION 6.1. *The generating matrix polynomial of the generalized inverse G from Theorem 5.3 is found by the formula*

$$G(t,s) = \mathscr{P}_t(0,m)\mathscr{P}_s(-n,0)\frac{\mathscr{R}(t)D_\sigma^{-1}(t,s)\mathscr{L}(s)}{1-ts^{-1}}. \qquad (6.1)$$

Here $\mathscr{R}(t)$, $\mathscr{L}(s)$ are the matrices of the conforming essential polynomials,

$$D_\sigma(t,s) = \operatorname{diag}[t^{\mu_1},\dots,t^{\mu_\sigma}, s^{\mu_{\sigma+1}},\dots,s^{\mu_{p+q}}],$$

and the integer σ is found from the condition

$$\mu_1 \leqslant \cdots \leqslant \mu_\sigma \leqslant 0 < \mu_{\sigma+1} \leqslant \cdots \leqslant \mu_{p+q}.$$

Proof. Consider the matrix function

$$\mathscr{B}(t, s) = \frac{\mathscr{R}(t) D_\sigma^{-1}(t, s) \mathscr{L}(s)}{1 - ts^{-1}}.$$

Since for the conforming essential polynomials the condition $\mathscr{R}(t)D^{-1}(t)\mathscr{L}(t) = 0$ is fulfilled, $\mathscr{B}(t, s)$ is a polynomial in t of order at most $\max(m + \mu_{p+q}, m)$, and s^{-1} of order at most $\max(n - \mu_1, n)$.

Let $B = \|b_{ij}\|_{i,j=0}^{\infty}$ ($b_{ij} \in \mathbb{C}^{q \times p}$) be the matrix of the operator $\mathbb{B} = \mathbb{T}_R \mathbb{T}_{D^{-1}} \mathbb{T}_{\mathscr{L}}$. Recall that $\mathbb{B}$ is an operator from $l^1_{p\times 1}$ into $l^1_{q \times 1}$. We shall show that $\mathscr{B}(t, s)$ is the generating polynomial of $\mathbb{B}$. The proof is similar to the proof of a formula for the generating function of the inverse of a Toeplitz operator [18].

Apply the operator $\mathbb{B}$ to the sequence $E = (I_p, s^{-1}I_p, s^{-2}I_p, \ldots)$. For $|s| > 1$ the sequence belongs to $l^1_{p\times 1}$ and has the symbol (the Fourier transform)

$$\sum_{j=0}^{\infty} t^j s^{-j} I_p = \frac{1}{1 - ts^{-1}} I_p, \qquad |t| = 1.$$

The symbol of the sequence $\mathbb{B}E$ is the function $\sum_{i,j=0}^{\infty} b_{ij} t^i s^{-j}$, that is, the generating function of B.

On the other hand, the sequence $\mathbb{T}_{\mathscr{L}} E = (\mathscr{L}(s), s^{-1}\mathscr{L}(s), s^{-2}\mathscr{L}(s), \ldots)$ has the symbol $\mathscr{L}(s)/(1 - ts^{-1})$. Hence the symbol of the sequence $\mathbb{T}_{D^{-1}} \mathbb{T}_{\mathscr{L}} E$ is

$$P_+ \frac{D^{-1}(t) \mathscr{L}(s)}{1 - ts^{-1}},$$

where the projector P_+ acts by the formula

$$P_+\left(\sum_{j=-\infty}^{\infty} r_j t^j = \sum_{j=0}^{\infty} r_j t^j \right).$$

Since

$$P_+ \frac{t^{-\mu}}{1 - ts^{-1}} = \begin{cases} \dfrac{t^{-\mu}}{1 - ts^{-1}}, & \mu \leqslant 0, \\ \dfrac{s^{-\mu}}{1 - ts^{-1}}, & \mu \geqslant 0, \end{cases}$$

we have

$$P_+ \frac{D^{-1}(t)\mathscr{L}(s)}{1 - ts^{-1}} = \frac{D_\sigma^{-1}(t, s)\mathscr{L}(s)}{1 - ts^{-1}}.$$

Thus the symbol of the sequence $\mathbb{T}_{\mathscr{R}}\mathbb{T}_{D^{-1}}\mathbb{T}_{\mathscr{L}}E$ is the function

$$P_+ \frac{\mathscr{R}(t) D_\sigma^{-1}(t, s)\mathscr{L}(s)}{1 - ts^{-1}} = \frac{\mathscr{R}(t) D_\sigma^{-1}(t, s)\mathscr{L}(s)}{1 - ts^{-1}} = \mathscr{B}(t, s).$$

Hence, $\mathscr{B}(t, s)$ is the generating function of the operator $\mathbb{B}$:

$$\mathscr{B}(t, s) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b_{ij} t^i s^{-j}, \qquad |t| = 1, \quad |s| > 1.$$

Since $\mathscr{B}(t, s)$ is a polynomial in t, s^{-1}, we can omit the conditions $|t| = 1$, $|s| > 1$.

It is evident that the generating polynomial of the matrix of the operator $G = P_{m+1}\mathbb{T}_{\mathscr{R}}\mathbb{T}_{D^{-1}}\mathbb{T}_{\mathscr{L}}P_{n+1}|\operatorname{Im} P_{n+1}$ coincides with the polynomial $\mathscr{P}_t(0, m)\mathscr{P}_s(-n, 0)\mathscr{B}(t, s)$. The proposition is proved. ■

If in (6.1) we replace $D_\sigma(t, s)$ by

$$D_k(t, s) = \operatorname{diag}[t^{\mu_1}, \dots, t^{\mu_k}, s^{\mu_{k+1}}, \dots, s^{\mu_{p+q}}],$$

$0 \leqslant k \leqslant p + q$, then we obtain the generating polynomial of another generalized inverses of T_a.

PROPOSITION 6.2. *The matrix polynomial*

$$G_k(t,s) = \mathscr{P}_t(0,m)\mathscr{P}_s(-n,0)\frac{\mathscr{R}(t)D_k^{-1}(t,s)\mathscr{L}(s)}{1-ts^{-1}}, \qquad 0 \leqslant k \leqslant p+q, \tag{6.2}$$

is the generating polynomial of some generalized inverse G_k of T_a.

Proof. For $k = \sigma$ the statement is proved in Proposition 6.1. Hence it is sufficient to prove that for all k $(1 \leqslant k \leqslant p+q)$ the matrices G_k and G_{k-1} are generalized inverses of T_a simultaneously, that is, $T_a K T_a = 0$, where $K = G_{k-1} - G_k$. For the generating function $K(t,s) = G_{k-1}(t,s) - G_k(t,s)$ the last condition can be rewritten in the following form:

$$\sigma_R^t \sigma_L^s\{t^{-i}s^j K(t,s)\} = 0, \qquad i = 0,1,\dots,n, \quad j = 0,1,\dots,m.$$

If

$$\mathscr{B}_k(t,s) = \frac{\mathscr{R}(t)D_k^{-1}(t,s)\mathscr{L}(s)}{1-ts^{-1}},$$

then

$$\mathscr{B}_{k-1}(t,s) - \mathscr{B}_k(t,s) = R_k(t)s^{-\mu_k}d_k(t,s)L_k(s),$$

where $d_k(t,s) = (1-(ts^{-1})^{-\mu_k})/(1-ts^{-1})$. Hence

$$K(t,s) = \mathscr{P}_t(0,m)\mathscr{P}_s(-n,0)R_k(t)s^{-\mu_k}d_k(t,s)L_k(s).$$

Let $\mu_k < 0$. Then

$$d_k(t,s) = \sum_{j=0}^{|\mu_k|-1} t^j s^{-j}$$

and

$$K(t,s) = \sum_{j=0}^{|\mu_k|-1} t^j R_k(t)\left[\mathscr{P}_s(-n,0)s^{-j-u_k}L_k(s)\right].$$

Therefore,

$$\sigma_R^t\{t^{-i}K(t,s)\} = \sum_{j=0}^{|\mu_k|-1} \sigma_R\{t^{-i+j}R_k(t)\}\mathscr{P}_s(-n,0)s^{-j-\mu_k}L_k(s) = 0$$

for $i = 0, 1, \dots, n$. Here we use the inequality $\mu_k + 1 \leqslant i - j \leqslant n$ and the definition of the right essential polynomial $R_k(t)$. Thus $T_a K = 0$.

In a similar manner we can prove that $KT_a = 0$ for $\mu_k > 0$. If $\mu_k = 0$, the $d_k(t, s) = 0$ and $K = 0$. Thus, we always have $T_a KT_a = 0$. The proposition is proved. ∎

7. SOME SPECIAL CASES OF THE GENERALIZED INVERSION FORMULAS

Now we consider some special cases of (5.13), (6.1), and (6.2).

7.1

If all indices of $a(t)$ are equal to zero, then the sequence is regular and the matrix T_a is invertible. Let

$$\mathscr{R}(t) = \begin{pmatrix} R_1(t) & \cdots & R_{p+q}(t) \end{pmatrix}$$

be the matrix of arbitrary right essential polynomials, and let

$$\mathscr{L}_c(t) = \begin{pmatrix} L_1^c(t) \\ \vdots \\ L_{p+q}^c(t) \end{pmatrix}$$

be the matrix of conforming left essential polynomials. If $L_1(t), \dots, L_{p+q}(t)$ are arbitrary left essential polynomials, then there exists an invertible matrix C such that

$$\mathscr{L}_c(t) = C\mathscr{L}(t), \qquad \text{where} \quad \mathscr{L}(t) = \begin{pmatrix} L_1(t) \\ \vdots \\ L_{p+1}(t) \end{pmatrix}.$$

It follows from this that $\Lambda_L^c = C\Lambda_L$, where Λ_L^c $[\Lambda_L]$ is the test matrix for $\mathscr{L}_c(t)$ $[\mathscr{L}(t)]$. From the definition of conforming left essential polynomials we have

$$\Lambda_L^c = B_-(\infty)\begin{pmatrix} 0 & I_p \\ -I_q & 0 \end{pmatrix}^{-1},$$

where

$$B_-(\infty) = A_-^{-1}(\infty) = \Lambda_R^{-1}\begin{pmatrix} 0 & I_p \\ I_q & 0 \end{pmatrix}^{-1}.$$

Here Λ_R is the test matrix for $\mathscr{R}(t)$. Thus $C = \Lambda^{-1}$, where

$$\Lambda = \Lambda_L\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}\Lambda_R.$$

Applying Theorem 5.3 and Proposition 6.1, we arrive at the following result.

COROLLARY 7.1. *Let $R_1(t), \dots, R_{p+q}(t)$ be any linearly independent polynomials in the space N_1^R, and $L_1(t), \dots, L_{p+q}(t)$ be any linearly independent polynomials in N_{-1}^L. (The dimension of these spaces is not less than $p + q$.)*

The block Toeplitz matrix T_a is invertible if and only if the matrix

$$\Lambda_R = \begin{pmatrix} \tilde{\sigma}_R\{R_1(t)\} & \cdots & \tilde{\sigma}_R\{R_{p+q}(t)\} \\ R_{1,m+1} & \cdots & R_{p+q,m+1} \end{pmatrix}$$

or the matrix

$$\Lambda_L = \begin{pmatrix} L_{1,0} & \tilde{\sigma}_L\{t^{m+1}L_1(t)\} \\ \vdots & \vdots \\ L_{p+q,0} & \tilde{\sigma}_L\{t^{m+1}L_{p+q}(t)\} \end{pmatrix}$$

is invertible. Here $R_{j,m+1}$ is the leading coefficient of the polynomial $R_j(t)$; $L_{j,0}$ is the constant term of $L_j(t)$; and $\tilde{\sigma}_R$, $\tilde{\sigma}_L$ are the Stieltjes operators for the sequence $a_{-m-1}, a_{-m}, \dots, a_0, \dots, a_n$, where a_{-m-1} is an arbitrary matrix.

If the matrix Λ_R is invertible, Λ_L is also invertible, and vice versa. Moreover, in this case the polynomials $R_1(t), \dots, R_{p+q}(t)$; $L_1(t), \dots, L_{p+q}(t)$ are the essential polynomials of the sequence $a_{-m}, \dots, a_0, \dots, a_n$, and the generating polynomial for the inverse of T_a is constructed by the formula

$$\mathscr{B}(t,s) = \frac{\begin{pmatrix} R_1(t) & \cdots & R_{p+q}(t) \end{pmatrix} \Lambda^{-1} \begin{pmatrix} L_1(s) \\ \vdots \\ L_{p+q}(s) \end{pmatrix}}{1 - ts^{-1}}, \tag{7.1}$$

where $\Lambda = \Lambda_L \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \Lambda_R$.

This result was first established in 1985 [2] for $p = q$ (scalar case in [4]). Since the coefficients of essential polynomials are solutions of systems of homogeneous linear equations, they are nonuniquely determined parameters. Therefore (7.1) contains a family of inversion formulas. Choosing special bases for the spaces N_1^R, N_{-1}^L (the spaces of essential polynomials), we may obtain some special cases of the inversion formula.

For example, if we normalize the essential polynomials by the conditions $\Lambda_R = \Lambda_L = I_{p+q}$, then we obtain

COROLLARY 7.2. *The block Toeplitz matrix T_a is invertible if and only if there exist solution of the systems of matrix equations*

$$\begin{aligned} &\sum_{j=0}^{m} a_{i-j}\alpha_j = \delta_{i0} I_p, \quad i = 0, 1, \dots, n, \\ &\sum_{j=0}^{m} a_{i-j}\beta_j = -a_{i-m-1}, \quad i = 0, 1, \dots, n, \end{aligned} \tag{7.2}$$

or the systems

$$\sum_{j=0}^{n} \delta_{j+1} a_{j-1} = -a_{-i-1}, \quad i = 0, 1, \ldots, m,$$

$$\sum_{j=0}^{n} \gamma_{j+1} a_{j-i} = \delta_{im} I_q, \quad i = 0, 1, \ldots, m, \tag{7.3}$$

where a_{-m-1} is an arbitrary matrix. If the systems (7.2) are solvable, then the systems (7.3) are also solvable and vice versa. The inversion formula for T_a has the following form:

$$B = L(\alpha_0, \ldots, \alpha_m) U(I_p, \delta_1, \ldots, \delta_n) - L(\beta_0, \ldots, \beta_m) U(0, \gamma_1, \ldots, \gamma_n).$$

Here

$$L(x_0, \ldots, x_m) = \begin{pmatrix} x_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ x_m & \cdots & x_0 \end{pmatrix},$$

$$U(y_0, \ldots, y_n) = \begin{pmatrix} y_0 & \cdots & y_m & \cdots & y_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & y_0 & \cdots & y_{n-m} \end{pmatrix}$$

for $m \leqslant n$, and

$$L(x_0, \ldots, x_m) = \begin{pmatrix} x_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ x_n & \cdots & x_0 \\ \vdots & & \vdots \\ x_m & \cdots & x_{m-n} \end{pmatrix},$$

$$U(y_0, \dots, y_n) = \begin{pmatrix} y_0 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ 0 & \cdots & y_0 \end{pmatrix}$$

for $m > n$.

The parameters α_j, β_j, δ_j, γ_j are the coefficients of the matrix normalized essential polynomials. In the scalar case an analogous result was first obtained by Li-Gun-Y [20]. In the block case the invertibility of T_a was proved in [13]. In that article the inversion formula, which use only the solutions of systems (7.2) or only the solutions of systems (7.3), was found.

In an similar manner we can obtain from (7.1) other well-known inversion formulas (the Sakhnovich formula, the Gohberg-Heinig formula, and the Gohberg-Krupnik formula).

7.2

Let $p = q = 1$. Applying Theorem 5.3, Proposition 6.2 for $k = 0$, and Remark 5.1, we obtain

COROLLARY 7.3. *Let μ_1, μ_2 be the indices and let $R_1(t)$, $R_2(t)$ be the right essential polynomials of a scalar sequence $a_{-m}, \dots, a_0, \dots, a_n$. Then the polynomial*

$$G(t,s) = \frac{1}{\sigma_0}\mathscr{P}_t(0,m)\mathscr{P}_s(-n,0)s^{-(n+1)}\frac{R_1(s)R_2(t) - R_1(t)R_2(s)}{1 - ts^{-1}}$$

is the generating polynomial of a generalized (one-sided, two-sided) inverse of T_a.

This result was establish by a different method in [3]. For Hankel matrices a similar formula was found in [17].

We note that Theorem 3 of [3] about a recovery of the initial sequence by indices and essential polynomials can be generalized to the block case.

7.3

In conclusion we note that the results of this paper can be formulated in the same form as the results of the theory of Toeplitz operators. This enables us to state that the proposed technique of indices and essential polynomial is an analog of the Wiener-Hopf factorization method.

From Equations (5.2), (5.4) it is easily seen that an arbitrary rational $p \times q$ matrix polynomial $a(t) = \sum_{j=-m}^{n} a_j t^j$ can be represented in the form

$$a(t) = r_-(t) D(T) r_+(t). \tag{7.4}$$

Here the matrix polynomials $r_\pm(t)$ in $t^{\pm 1}$ satisfy the following conditions:

(1) there exists a matrix polynomial $r_+^{(-1)}(t)$ $[r_-^{(-1)}(t)]$ in t $[t^{-1}]$ such that $r_+^{(-1)}(t) r_+(t) = I_q$ $[r_-(t) r_-^{(-1)}(t) = I_p]$;

(2) $r_+^{(-1)}(t) D^{-1}(t) r_-^{(-1)}(t) = 0$;

(3) $\mathscr{R}_-(t) = t^{-m-1} r_+^{(-1)}(t) D^{-1}(t)$ $[\mathscr{L}_+(t) = t^{n+1} D^{-1}(t) r_-^{(-1)}(t)]$ is a matrix polynomial in t^{-1} $[t]$;

(4) $\det(r_+(t)\ \ \mathscr{L}_+(t))$ and $\det\begin{pmatrix} \mathscr{R}_-(t) \\ r_-(t) \end{pmatrix}$ are constants.

It turns out that any $r_+^{(-1)}(t)$, $r_-^{(-1)}(t)$ are matrices of conforming essential polynomials of $a(t)$. The representation (7.4) of a rational matrix polynomial $a(t)$ we shall call an *essential factorization* of $a(t)$.

The following theorem shows that in the finite-dimensional case the essential factorization plays a role of a Wiener-Hopf factorization.

THEOREM 7.1. *Let*

$$T_a = \|a_{i-j}\|_{\substack{i=0,\dots,n \\ j=0,\dots,m}}$$

be an arbitrary block Toeplitz matrix. T_a is strictly generalized invertible if and only if its symbol $a(t) = \sum_{j=-m}^{n} a_j t^j$ has both positive and negative essential indices. T_a is left (right) invertible if and only if all essential indices of $a(t)$ are nonnegative (nonpositive). Thus, T_a is invertible if and only if all indices are equal to zero. Moreover,

$$\operatorname{ind} T_a = -\sum_{j=1}^{p+q} \mu_j,$$

$$\dim\ker T_a = -\sum_{\mu_j<0} \mu_j, \qquad \dim\operatorname{coker} T_a = \sum_{\mu_j>0} \mu_j. \tag{7.5}$$

If

$$a(t) = r_-(t) D(t) r_+(t)$$

is a essential factorization of $a(t)$, then the matrix of the operator

$$G = P_{m+1}\mathbb{T}_{r_+^{(-1)}} P_{m+1} \mathbb{T}_{D^{-1}} P_{n+1} \mathbb{T}_{r_-^{(-1)}} P_{n+1} |\mathrm{Im}\, P_{n+1} \tag{7.6}$$

is a generalized (one-sided, two-sided) inverse of T_a. ■

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