On the Inverses of Toeplitz-plus-Hankel Matrices

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ABSTRACT

It is well known that the inverses of Hankel and Toeplitz matrices can be represented as Bezoutians of polynomials. In the present note a Bezoutian-type formula for the inverses of Toeplitz-plus-Hankel matrices and a complete characterization of Toeplitz-plus-Hankel matrix inverses are given.

INTRODUCTION

Utilizing some earlier results concerning Toeplitz matrix inversion presented in [1], F. I. Lander [5] remarked that the inverse of a regular Hankel matrix, i.e. a matrix of the form $[s_{i+j}]_0^{n-1}$, can be represented as a Bezoutian of two polynomials and, vice versa, any regular Bezoutian is the inverse of a Hankel matrix. A similar result holds for Toeplitz matrices, i.e. matrices of the form $[t_{i-j}]_0^{n-1}$.

The main aim of the present note is to show that a Bezoutian-type formula also exists for the inverse of a matrix of the form $A = T + H$, where T is Toeplitz and H is Hankel. We shall call matrices of this kind T + *H-matrices*. A second aim will be a complete characterization of the class of T + H-matrix inverses.

In the first section we shall introduce Bezoutian concepts and quote some known results. Furthermore, we formulate our main theorem. It turns out that Hankel, Toeplitz, and $T+H$-matrices are special types of a class of matrices which we shall call "ω-structured matrices." This concept will be introduced in the present paper (Section 2) for the first time. In Section 3 we

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shall deduce an inversion formula for $T+H$-matrices. Let us note that formulas of this kind are important for constructing fast inversion algorithms for $T+H$-matrices. Our paper [2] is dedicated to the investigation of such algorithms. In Section 4 the sufficiency of the condition of the main theorem will be proved, which is an analogue of Theorem I, 2.1 in [3].

1. BEZOUTIANS

It is convenient to define the Bezoutian concepts in the language of generating functions. The generating function of an $m\times n$ matrix $A=[a_{ij}]_{0,\ 0}^{m-1,n-1}$ is, by definition, the polynomial in two variables

$$A(\lambda,\mu)=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}a_{ij}\lambda^i\mu^j.$$

Identifying vectors $a=(a_i)_0^{n-1}\in\mathbf{C}^n$ with the corresponding $n\times 1$ matrices, this notation will also be used for vectors of $\mathbf{C}^n$.

DEFINITION 1.1. A matrix B is called an *H-Bezoutian* ("H" refers to Hankel) iff there are polynomials $g_i(\lambda)$, $f_i(\lambda)$ $(i=1,2)$ such that

$$(\lambda-\mu)B(\lambda,\mu)=\sum_{i=1}^{2}g_i(\lambda)f_i(\mu). \tag{1.1}$$

In case

$$g_2(\lambda)=-f_1(\lambda),\qquad g_1(\lambda)=f_2(\lambda), \tag{1.2}$$

B is said to be a *classical H-Bezoutian.*

This Bezoutian concept was introduced in [6]. Concerning the classical Bezoutian concept we refer to [4].

THEOREM 1.1. *A regular matrix B is an H-Bezoutian iff B^{-1} is Hankel. Moreover, any regular H-Bezoutian is classical.*

The proof of the fact that B^{-1} is Hankel iff B is a classical Bezoutian was already given in [5]. The stronger version of Theorem 1.1 was shown in [3, I.2.3].

DEFINITION 1.2. *A matrix B is called a T-Bezoutian* ("T" refers to Toeplitz) iff there exist polynomials $g_i(\lambda)$, $f_i(\lambda)$ $(i=1,2)$ such that

$$(1-\lambda\mu)B(\lambda,\mu)=\sum_{i=1}^{2} g_i(\lambda)f_i(\mu).$$

In case

$$g_2(\lambda)=-f_1(\lambda^{-1})\lambda^n, \qquad g_1(\lambda)=f_2(\lambda^{-1})\lambda^n,$$

with n the degree of $f_1(\lambda)$ and $f_2(\lambda)$, B is said to be a *classical T-Bezoutian.*

THEOREM 1.2. *A regular matrix B is a T-Bezoutian iff B^{-1} is Toeplitz. Moreover, any regular T-Bezoutian is classical.*

The proof of this theorem is, in principle, the same as that of Theorem 1.1.

DEFINITION 1.3. A matrix B will be called a *$T+H$-Bezoutian* iff there are polynomials $g_i(\lambda)$, $f_i(\lambda)$ $(i=1,2,3,4)$ such that

$$(\lambda-\mu)(1-\lambda\mu)B(\lambda,\mu)=\sum_{i=1}^{4} g_i(\lambda)f_i(\mu). \tag{1.3}$$

Clearly, any H-Bezoutian as well as any T-Bezoutian is a $T+H$-Bezoutian. On the other hand, the sum of a T-Bezoutian and an H-Bezoutian is not necessarily a $T+H$-Bezoutian, as simple examples show.

The main result of our note is the following one.

THEOREM 1.3. *A regular matrix B is a $T+H$-Bezoutian iff B^{-1} is a $T+H$-matrix.*

The two directions of the proof will be given in Sections 3 and 4.

2. MATRICES WITH ω-STRUCTURE

In this section we introduce the concept of ω-structured matrices. This concept seems to be fruitful also in other situations and will be developed in a further publication of the authors.

Let L_n denote the class of all $n \times n$ matrices with complex entries. With any given matrix $\omega = [\omega_{st}]_0^l \in L_{l+1}$ we associate two transformations ∇_ω: $L_n \to L_{n+l}$ and $\nabla_\omega^0 : L_n \to L_{n-l}$ defined by

$$\nabla_\omega A := \left[\sum_{s,t=0}^{l} a_{i-s,j-t}\omega_{st} \right]_0^{n-1+l}, \tag{2.1}$$

$$\nabla_\omega^0 A := \left[\sum_{s,t=0}^{l} a_{i-s,j-t}\omega_{st} \right]_l^{n-1}, \tag{2.2}$$

where $A = [a_{ij}]_0^{n-1}$ and $a_{ij} := 0$ if $i, j \notin \{0, 1, \ldots, n-1\}$. Clearly, ∇_ω and ∇_ω^0 are linear operators. Furthermore, the transformation ∇_ω can be represented in terms of generating functions.

PROPOSITION 2.1.

$$(\nabla_\omega A)(\lambda, \mu) = \omega(\lambda, \mu) A(\lambda, \mu).$$

The proof is an elementary calculation.

From Proposition 2.1 it becomes clear that ∇_ω $(\omega \neq 0)$ is left invertible.

DEFINITION 2.1. A matrix $A \in L_n$ is said to possess an ω-structure iff $\nabla_\omega^0 A = 0$.

Let us give some examples. Put

$$\omega_H := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

then, obviously, A has an ω_H-structure iff A is Hankel. Defining

$$\omega_T := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

the matrices with an ω_T-structure are just the Toeplitz matrices. Note that

$$\omega_H(\lambda, \mu) = \lambda - \mu \quad \text{and} \quad \omega_T(\lambda, \mu) = 1 - \lambda\mu.$$

Hence, by Proposition 2.1,

$$(\nabla_{\omega_H} A)(\lambda, \mu) = (\lambda - \mu) A(\lambda, \mu)$$

and

$$(\nabla_{\omega_T} A)(\lambda, \mu) = (1 - \lambda\mu) A(\lambda, \mu).$$

Therefore, H- and T-Bezoutians can be characterized by means of transformations ∇_ω. In that manner B is an H-Bezoutian (T-Bezoutian) iff rank $\nabla_{\omega_H} B \leqslant 2$ (rank $\nabla_{\omega_T} B \leqslant 2$). Moreover, the equality holds if $B \neq 0$. Now let us characterize $T + H$-matrices.

PROPOSITION 2.2. *Suppose*

$$\omega = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \tag{2.3}$$

Then A has an ω-structure iff A is Toeplitz-plus-Hankel.

Proof. We compare the linear space $\mathscr{A}_n$ of all $n \times n$ $T + H$-matrices with the kernel of ∇_ω^0. It is easily verified that $\nabla_\omega^0 A = 0$ if $A \in \mathscr{A}_n$; that means

$$\mathscr{A}_n \subseteq \ker \nabla_\omega^0. \tag{2.4}$$

Thus, it remains to prove that the dimensions of $\mathscr{A}_n$ and $\ker \nabla_\omega^0$ coincide. First we compute $\dim \mathscr{A}_n$. Let $\mathscr{T}_n$ denote the space of $n \times n$ Toeplitz and $\mathscr{H}_n$ the space of $n \times n$ Hankel matrices. Obviously, $\dim \mathscr{T}_n = \dim \mathscr{H}_n = 2n - 1$. Because $\mathscr{A}_n$ is the algebraic sum of $\mathscr{T}_n$ and $\mathscr{H}_n$, we obtain

$$\dim \mathscr{A}_n = \dim \mathscr{T}_n + \dim \mathscr{H}_n - \dim(\mathscr{T}_n \cap \mathscr{H}_n).$$

The intersection $\mathscr{T}_n \cap \mathscr{H}_n$ consists of all "checkered" matrices

$$\begin{bmatrix} a & b & a & \cdots \\ b & a & b & \cdots \\ a & b & a & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and is therefore two-dimensional. This implies

$$\dim \mathscr{A}_n = 2(2n - 1) - 2 = 4n - 4. \tag{2.5}$$

Next we observe that any matrix $A \in \ker \nabla_\omega^0$ is uniquely determined by its first two rows and its first and last columns, which means

$$\dim\ker \nabla_\omega^0 \leqslant 2n + 2(n-2) = 4n - 4.$$

Taking into account (2.4) and (2.5), we obtain $\dim\ker\nabla_\omega^0 = \dim \mathscr{A}_n$. Consequently, $\ker \nabla_\omega^0 = \mathscr{A}_n$. ∎

We observe that, for ω defined by (2.3),

$$\begin{aligned}\omega(\lambda,\mu) &= \lambda - \mu + \lambda\mu^2 - \lambda^2\mu = (\lambda-\mu)(1-\lambda\mu)\\ &= \omega_H(\lambda,\mu)\omega_T(\lambda,\mu).\end{aligned}$$

Therefore, by Proposition 2.1,

$$(\nabla_\omega A)(\lambda,\mu) = (\lambda-\mu)(1-\lambda\mu)A(\lambda,\mu).$$

Consequently, according to Definition 1.3, B is a $T+H$-Bezoutian iff $\operatorname{rank}\nabla_\omega B \leqslant 4$.

3. INVERSION FORMULA

Throughout this and the next section let ∇ denote the transformation ∇_ω for ω defined by (2.3). In order to obtain an inversion formula for $T+H$-matrices we study the action of the transformation ∇. Let S_n denote the matrix $[\delta_{i-1,j}]_0^{n-1}$ of the forward shift in $\mathbb{C}^n$, and W_n the sum of S_n and its transpose. An elementary computation yields the following fact.

PROPOSITION 3.1. *For $A = [a_{ij}]_0^{n-1} \in L_n$, the matrix ∇A has the form*

$$\nabla A = \begin{bmatrix} 0 & -c_1^T & 0 \\ b_1 & AW_n - W_nA & b_2 \\ 0 & -c_2^T & 0 \end{bmatrix}, \tag{3.1}$$

where

$$b_1 = \begin{bmatrix} a_{00} & \cdots & a_{n-1,0}\end{bmatrix}^T, \qquad b_2 = \begin{bmatrix} a_{0,n-1} & \cdots & a_{n-1,n-1}\end{bmatrix}^T,$$

$$c_1 = \begin{bmatrix} a_{00} & \cdots & a_{0,n-1}\end{bmatrix}^T, \qquad c_2 = \begin{bmatrix} a_{n-1,0} & \cdots & a_{n-1,n-1}\end{bmatrix}^T.$$

Now we assume that A is a $T+H$-matrix

$$A = \left[t_{i-j} + s_{i+j}\right]_0^{n-1}.$$

Then we have

$$AW_n - W_nA = -g_1e_0^T - g_2e_{n-1}^T + e_0f_1^T + e_{n-1}f_2^T, \tag{3.2}$$

where

$$g_1 = [t_1 + s_{-1} \quad \cdots \quad t_n + s_{n-2}]^T,$$

$$g_2 = [t_{-n} + s_n \quad \cdots \quad t_{-1} + s_{2n-1}]^T,$$

$$f_1 = [t_{-1} + s_{-1} \quad \cdots \quad t_{-n} + s_{n-2}]^T,$$

$$f_2 = [t_n + s_n \quad \cdots \quad t_1 + s_{2n-1}]^T,$$

$$e_0 = [1 \quad 0 \quad \cdots \quad 0]^T, \qquad e_{n-1} = [0 \quad \cdots \quad 0 \quad 1]^T,$$

and s_{-1}, s_{2n-1}, t_n, t_{-n} are arbitrary numbers.

Equation (3.2) represents a W_nW_n-reduction of the $T+H$-matrix A in the sense of [3]. According to the theory developed in this monograph one has to consider now the following "fundamental" equations

$$Ax_1 = g_1, \qquad Ax_2 = g_2, \qquad Ax_3 = e_0, \qquad Ax_4 = e_{n-1}, \tag{3.3}$$

$$A^Ty_1 = e_0, \qquad A^Ty_2 = e_{n-1}, \qquad A^Ty_3 = f_1, \qquad A^Ty_4 = f_2. \tag{3.4}$$

THEOREM 3.1. *Suppose A is an $n \times n$ $T+H$-matrix and the equations* (3.3) *or* (3.4) *are solvable. Then A is regular, and its inverse is completely determined from the solutions of* (3.3) *and* (3.4) *by the following formula:*

$$A^{-1}(\lambda, \mu) = \frac{1}{(\lambda - \mu)(1 - \lambda\mu)} \sum_{i=1}^{4} u_i(\lambda) v_i(\mu), \tag{3.5}$$

where

$$\begin{aligned} u_1(\lambda) &= -1 + \lambda x_1(\lambda), & v_1(\lambda) &= \lambda y_1(\lambda), \\ u_2(\lambda) &= \lambda x_2(\lambda) - \lambda^{n+1}, & v_2(\lambda) &= \lambda y_2(\lambda), \\ u_3(\lambda) &= \lambda x_3(\lambda), & v_3(\lambda) &= 1 - \lambda y_3(\lambda), \\ u_4(\lambda) &= \lambda x_4(\lambda), & v_4(\lambda) &= -\lambda y_4(\lambda) + \lambda^{n+1}. \end{aligned} \tag{3.6}$$

Proof. First we prove the regularity of A. Without loss of generality we may assume that the equations (3.4) are solvable. [In the case that the equations (3.3) are solvable we consider A^T, which is a $T+H$-matrix again, instead of A.] Let u belong to the kernel of A, i.e. $Au=0$. Then, according to (3.2),

$$\begin{aligned} AW_n u &= -g_1 e_0^T u - g_2 e_{n-1}^T u + e_0 f_1^T u + e_{n-1} f_2^T u \\ &= -g_1 y_1^T Au - g_2 y_2^T Au + e_0 y_3^T Au + e_{n-1} y_4^T Au, \end{aligned}$$

and consequently

$$e_0^T u = e_{n-1}^T u = 0$$

and

$$AW_n u = 0.$$

With the same arguments we conclude $AW_n^{k+1}u=0$ and $e_0^T W_n^k u = e_{n-1}^T W_n^k u = 0$ for $k=1,2,\dots$. This implies $u=0$, and the regularity of A is proved.

Next we prove the inversion formula (3.5). From (3.2) we obtain

$$A^{-1}W_n - W_n A^{-1} = x_1 y_1^T + x_2 y_2^T - x_3 y_3^T - x_4 y_4^T. \tag{3.7}$$

According to Proposition 3.1, we have

$$\nabla A^{-1} = \begin{bmatrix} 0 & -y_1^T & 0 \\ x_3 & A^{-1}W_n - W_n A^{-1} & x_4 \\ 0 & -y_2^T & 0 \end{bmatrix}. \tag{3.8}$$

Taking (3.7) into account, we conclude

$$\nabla A^{-1} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ x_1 & x_2 & x_3 & x_4 \\ 0 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & y_1^T & 0 \\ 0 & y_2^T & 0 \\ 1 & -y_3^T & 0 \\ 0 & -y_4^T & 1 \end{bmatrix}.$$

Using generating functions, this can be written as

$$\nabla A^{-1}(\lambda,\mu)=\left[-1+\lambda x_1(\lambda)\right]\mu y_1(\mu)+\left[\lambda x_2(\lambda)-\lambda^{n+1}\right]\mu y_2(\mu)$$
$$+\lambda x_3(\lambda)\left[1-\mu y_3(\mu)\right]+\lambda x_4(\lambda)\left[-\mu y_4(\mu)+\mu^{n+1}\right].$$

Together with Proposition 2.1, this leads just to the formula (3.5), and the theorem is proved. ■

We proceed with some additional remarks.

3.1

Besides the matrix $A=[t_{i-j}+s_{i+j}]_0^{n-1}$, we consider the following well-defined $(n-2)\times(n+2)$ matrix:

$$A_1:=\left[t_{i-j}+s_{i+j}\right]_{i=1,\,j=-1}^{n-2,\,n},$$

which has full rank in case A is regular. Therefore, A_1 has a four-dimensional kernel. We shall show that there is a close relation between the kernel of A_1 and the solutions of the fundamental equations (3.3). Suppose x_i $(i=1,2,3,4)$ to be the solutions of (3.3). Then the vectors u_i defined by (3.6) are linearly independent and

$$A_1u_i=0\qquad(i=1,2,3,4).\tag{3.9}$$

On the other hand, assume A is regular and $\{z_i\}_{i=1}^4$ is a basis of the kernel of A_1. Then on account of (3.9) there exists a regular 4×4 matrix C such that

$$[u_1\quad u_2\quad u_3\quad u_4]=[z_1\quad z_2\quad z_3\quad z_4]C.$$

In other words, the vectors u_i $(i=1,2,3,4)$ are linear combinations of the vectors z_1, z_2, z_3, and z_4. Thus we obtain the following assertions.

PROPOSITION 3.2. *Let the equations* (3.3) *be solvable. Then the vectors* u_i *defined by* (3.6) *form a basis of the kernel of* A_1.

3.2

Let us now prove that the entries of a $T+H$-matrix inverse can be evaluated recurrently.

Proposition 3.3. *Suppose A is a regular $n \times n$ $T+H$-matrix. Then the entries c_{jk} $(j, k = 0, \dots, n-1)$ of A^{-1} can be determined recurrently as follows*:

$$c_{j,-1} := 0, \qquad c_{j,0} = e_j^T x_1,$$

$$c_{j,k+1} = c_{j+1,k} + c_{j-1,k} - c_{j,k-1} + e_j^T\left(x_1 y_1^T + x_2 y_2^T - x_3 y_3^T - x_4 y_4^T\right) e_k, \tag{3.10}$$

where $e_j = (\delta_{ij})_{i=0}^{n-1}$ and x_i, y_i $(i = 1,2,3,4)$ are the solutions of the equations (3.3).

Proof. Let c_k denote the $(k+1)$th column of A^{-1}. Then from (3.7) one concludes

$$A^{-1} W_n e_k = W_n c_k + x_1 y_1^T e_k + x_2 y_2^T e_k - x_3 y_3^T e_k - x_4 y_4^T e_k.$$

Since $W_n e_k = e_{k-1} + e_{k+1}$, this implies (3.10). ■

3.3

For the construction of A^{-1} it suffices to know the solutions x_i.

Proposition 3.4. *The entries c_{jk} of a $T+H$-matrix inverse A^{-1} can be evaluated via*

$$c_{j,-1} := 0, \qquad c_{j,0} = e_j^T x_1,$$

$$c_{j,k+1} = c_{j+1,k} + c_{j-1,k} - c_{j,k-1} + e_j^T\left(x_1 e_0^T + x_2 e_{n-1}^T - x_3 f_1^T - x_4 f_2^T\right) c_k, \tag{3.11}$$

where $c_k := A^{-1} e_k$ and f_1, f_2 defined by (3.2).

Proof. From (3.2) and (3.7) it follows immediately that

$$W_n c_k - A^{-1} W_n e_k = \left(-x_1 e_0^T - x_2 e_{n-1}^T + x_3 f_1^T + x_4 f_2^T\right) c_k,$$

which leads to (3.11). ■

3.4

It is easy to verify that a $T+H$-matrix is symmetric iff the Toeplitz part has this property, and in this case the inversion formulas (3.5) and (3.10) can be simplified using the following relations between the fundamental solutions of (3.3) and (3.4):

$$y_1 = x_3, \qquad y_2 = x_4, \qquad y_3 = x_1, \qquad y_4 = x_2. \tag{3.12}$$

3.5

The definition of the concept of classical H- and T-Bezoutians includes the fact that there is a close relation between the fundamental solutions x_i and y_i. Section 3.3 above shows that there also exist such relations for $T+H$-Bezoutians. However, these relations are not so transparent as in the Toeplitz and Hankel cases. For this reason we could not find, hitherto, a natural definition of the concept "classical $T+H$-Bezoutian."

4. CHARACTERIZATION OF $T+H$-MATRIX INVERSES

In this section we prove the converse part of Theorem 1.3.

THEOREM 4.1. *Suppose that B is a regular matrix such that*

$$\operatorname{rank} \nabla B = 4.$$

Then B is the inverse of a $T+H$-matrix.

Proof. According to Proposition 3.1 the matrix ∇B admits a representation

$$\nabla B = \tilde{b}_1 e_0^T + \tilde{b}_2 e_n^T - e_0 \tilde{c}_1^T - e_n \tilde{c}_2^T + \sum_{i=1}^{4} \tilde{u}_i \tilde{v}_i^T, \tag{4.1}$$

where

$$\tilde{b}_1 = \begin{bmatrix} 0 \\ Be_0 \\ 0 \end{bmatrix}, \qquad \tilde{b}_2 = \begin{bmatrix} 0 \\ Be_{n-1} \\ 0 \end{bmatrix}, \qquad \tilde{c}_1 = \begin{bmatrix} 0 \\ B^T e_0 \\ 0 \end{bmatrix}, \qquad \tilde{c}_2 = \begin{bmatrix} 0 \\ B^T e_{n-1} \\ 0 \end{bmatrix},$$

$$\tilde{u}_i = \begin{bmatrix} 0 \\ u_i \\ 0 \end{bmatrix}, \qquad \tilde{v}_i = \begin{bmatrix} 0 \\ v_i \\ 0 \end{bmatrix},$$

and

$$BW_n - W_n B = \sum_{i=1}^{4} u_i v_i^T.$$

We intend to show that ∇B can be represented in the form

$$\nabla B = \tilde{b}_1 [1 \quad * \quad 0] + \tilde{b}_2 [0 \quad * \quad 1] - \begin{bmatrix} 1 \\ * \\ 0 \end{bmatrix} \tilde{c}_1^T - \begin{bmatrix} 0 \\ * \\ 1 \end{bmatrix} \tilde{c}_2^T, \qquad (4.2)$$

where $*$ stands for some vector of $\mathbf{C}^n$. In view of (4.1) we have a representation

$$\nabla B = R + \sum_{i=1}^{m} \tilde{u}_i \tilde{v}_i^T \qquad (4.3)$$

for $m = 4$, where R denotes a matrix possessing the form of the right-hand side of (4.2). It remains to show that there is a representation (4.3) for $m-1$, too. For this we utilize the elementary fact that if $\operatorname{rank} \sum_{i=1}^{m} g_i f_i^T < m$ then the vectors g_i or f_i are linearly dependent. Since $\operatorname{rank} \nabla B = 4$, the vectors

$$\tilde{b}_1, \quad \tilde{b}_2, \quad \begin{bmatrix} 1 \\ * \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ * \\ 1 \end{bmatrix}, \quad \tilde{u}_i \ (i = 1, \dots, m) \qquad (4.4)$$

or the corresponding row vectors are linearly dependent. Assume the vectors (4.4) are linearly dependent.[1] In view of the special form of these vectors, we have linear dependence already for $\tilde{b}_1$, $\tilde{b}_2$, $\tilde{u}_i$ $(i = 1, \dots, m)$. Taking the regularity of B into account, we obtain that one of the vectors $\tilde{u}_i$, say $\tilde{u}_m$, is a linear combination of the others:

$$\tilde{u}_m = \alpha_1 \tilde{b}_1 + \alpha_2 \tilde{b}_2 + \sum_{i=1}^{m-1} \beta_i \tilde{u}_i.$$

[1] In the other case we can proceed analogously.

Substituting this into (4.3), we obtain

$$\begin{aligned}\nabla B = \tilde{b}_1([1 \quad * \quad 0] + [0 \quad \alpha_1 v_m \quad 0]) \\ + \tilde{b}_2([0 \quad * \quad 1] + [0 \quad \alpha_2 v_m \quad 0]) \\ - \begin{bmatrix} 1 \\ * \\ 0 \end{bmatrix} \tilde{c}_1^T - \begin{bmatrix} 0 \\ * \\ 1 \end{bmatrix} \tilde{c}_2^T + \sum_{i=1}^{m-1} \tilde{u}_i(\tilde{v}_i + \beta_i \tilde{v}_m),\end{aligned}$$

which is indeed a representation of the form (4.3) for $m-1$. Finally (4.2) is obtained. Using the notation of Proposition 3.1, this means in particular that there are vectors $z_i \in \mathbf{C}^n$ such that

$$BW_n - W_n B = Be_0 z_1^T + Be_{n-1} z_2^T + z_3 e_0^T B + z_4 e_{n-1}^T B.$$

Applying B^{-1} from both sides, the latter leads to

$$-\left(B^{-1}W_n - W_n B^{-1}\right) = e_0 f_1^T + e_{n-1} f_2^T + f_3 e_0^T + f_4 e_{n-1}^T, \tag{4.5}$$

where $f_i = (B^{-1})^T z_i$ $(i = 1,2)$, $f_i = B^{-1} z_i$ $(i = 3,4)$. The relation (4.5) shows in particular that for the corresponding transformation ∇^0 defined by (2.2)

$$\nabla^0 B^{-1} = 0$$

holds. Consequently, by Proposition 2.2, B^{-1} is a $T+H$-matrix, and the theorem is proved. ∎

For completeness let us remark the following fact.

THEOREM 4.2. *Let B be an $n \times n$ matrix, $n \geqslant 2$. If* $\operatorname{rank} \nabla B < 4$, *then the first and last column or the first and last row of B are linearly dependent, which means, in particular, that B is singular.*

Proof. Obviously, the relation (4.2) holds if $\operatorname{rank} \nabla B \leqslant 4$ and the first and the last columns as well as the first and the last rows are linearly independent. Thus, both the system of vectors

$$\tilde{b}_1, \quad \tilde{b}_2, \quad \begin{bmatrix} 1 \\ * \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ * \\ 1 \end{bmatrix},$$

and the corresponding system of row vectors

$$\tilde{c}_1^T, \quad \tilde{c}_2^T, \quad [1 \quad * \quad 0], \quad [0 \quad * \quad 1]$$

are linearly independent. This implies rank $\nabla B = 4$. ■

Finally, we note that, in contrast with H- and T-Bezoutians, there are nontrivial $T + H$-Bezoutians B with rank $\nabla B < 4$. For example, if B is a checkered matrix of odd order, then rank $\nabla B \leqslant 2$.

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