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Transformation Techniques for Toeplitz and Toeplitz-plus-Hankel Matrices. I. Transformations

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ABSTRACT

Transformations of the form $A \to \mathscr{C}_1^T A \mathscr{C}_2$ are investigated that transform Toeplitz and Toeplitz-plus-Hankel matrices into generalized Cauchy matrices. $\mathscr{C}_1$ and $\mathscr{C}_2$ are matrices related to the discrete Fourier transformation or to various real trigonometric transformations. Combining these results with pivoting techniques, in paper II algorithms for Toeplitz and Toeplitz-plus-Hankel systems will be presented that are more stable than classical algorithms. © Elsevier Science Inc., 1997

1. INTRODUCTION

Transformations of the form $\Phi : A \to \mathscr{C}_1^T A \mathscr{C}_2$ mapping one class of matrices with displacement structure into another class with displacement structure appear in quite a number of papers in different contexts. A classical

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example is the Frobenius-Fischer transformation (see [19, 16]) transforming Hermitian Toeplitz into real Hankel matrices and so the trigonometric moment problem into the algebraic one. The general form of such transformations is described in [16]. Another result concerning transformations of this kind is Fiedler's theorem [8], which claims that if $\mathscr{C}_1$ and $\mathscr{C}_2$ are *any* inverse Vandermonde matrices then Φ maps Hankel matrices into Löwner matrices. Recall that a Löwner matrix is a matrix of the form $[(a_i - b_j)/(c_i - d_j)]$ (see [7]). As a particular case of this theorem one obtains Lander's result (see [16]), which claims that for a given Hankel matrix there exist inverse Vandermonde matrices such that the transformed matrix is block diagonal. This result is related to the one on Vandermode reduction of Bezoutians (see [1]).

In this paper we study transformations mapping Toeplitz and Toeplitz-plus-Hankel matrices into generalized Cauchy matrices. Recall that a matrix $C = [a_{ij}]$ is said to be a *generalized Cauchy matrix* if for certain n-tuples of complex numbers $c = (c_i)_1^n$ and $d = (d_j)_1^n$ the matrix

$$\nabla(c,d)C = \left[(c_i - d_j)a_{ij}\right]_1^n$$

has a rank r which is "small" compared with the order of C. The integer r will be called the *Cauchy rank* of C (with respect to c and d). Cauchy matrices in the classical sense are matrices for which $(c_i - d_j)a_{ij} = 1$. Since we always consider *generalized* Cauchy matrices, we will omit this attribute. Löwner matrices are matrices with Cauchy rank 2. We will also deal with matrices of Cauchy rank 4. In our paper two cases of Cauchy matrices will appear: (1) $c_i \neq d_j$ for all i and j, and (2) $c = d$.

There are quite a few theoretical motivations to study transformations between different classes of structured matrices, since the algebraic theory of one class can be transferred to the other class. But the main motivation for this paper was a more practical, numerical one. Let us explain this. The classical algorithms of Levinson and Schur types usually work well if the matrix is positive definite. However, if the matrix is indefinite they very often suffer from instability even if the matrix is well conditioned. The reason is that all these algorithms are based on recursions of the nested principal submatrices, which may be ill conditioned.

The first proposal to overcome the problem of ill-conditioned principal sections is to apply a lookahead strategy, i.e. to jump from one well-conditioned section to the next one. This proposal was made first in [5] and [6] and further developed by many authors (see [13] and references therein). However, lookahead strategies have two disadvantages. First, there is the problem of step-size estimation. A "good" step-size estimator will slow down the algorithm significantly. Secondly, even with a good step-size estimator the algorithm may not be fast, i.e. of complexity lower than $O(n^3)$. As an example we consider a Toeplitz matrix

$$T = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + E,$$

where E is a Toeplitz matrix with a small norm and I denotes the identity matrix. The matrix T is well conditioned, but there is no fast lookahead Toeplitz algorithm for it.

For general unstructured matrices, pivoting is the main tool to avoid instability. This cannot be applied to Toeplitz and related matrices, since permutations of columns or rows destroys the structure of the matrix. In contrast with Toeplitz-like matrices, Cauchy matrices do not have this disadvantage: Permutations of rows and columns do no destroy their structure. On the other hand, for Cauchy matrices there exist fast algorithms for inversion and factorization with essentially the same complexity as the classical algorithms for Toeplitz and Hankel matrices. Concerning literature on this topic we refer to [16, 10, 11, 14, 20, 12]. We will discuss this topic in more detail in the second paper of this pair.

Thus, it remains to find suitable transformations from Toeplitz-like into Cauchy matrices. To our knowledge, it was first noticed in [14] that discrete Fourier transformations do this job in an efficient and stable way. In [15] it was remarked that the DFT is also convenient for transforming Toeplitz-plus-Hankel into Cauchy matrices. This idea was further developed in [12]. In the later paper also a mixed sine-I-cosine-III transformation was used to transform real Toeplitz-plus-Hankel matrices into Cauchy matrices. Some transformation results for symmetric Toeplitz matrices appear implicitly in papers on optimal preconditioners (see [26] and [17] for DFT and [18] for the sine-I transformation).

The present paper continues the investigation in this direction. Our main aim is to give a systematic account of transformations from Toeplitz and Toeplitz-plus-Hankel matrices into generalized Cauchy matrices. Special attention is paid to transformations that preserve certain properties like symmetry and realness.

In Section 2 we consider transformations of Toeplitz matrices by DFT into matrices with Cauchy rank 2, and in Section 3 transformations of Toeplitz-plus-Hankel matrices by DFT into matrices with Cauchy rank 4. Section 4 is dedicated to the transformation of real Toeplitz-plus-Hankel matrices into real Cauchy matrices. It turns out that many common real trigonometric transformations, such as sine-I-IV, cosine-I-IV,[1] the Hartley, and the real DFT, transform real Toeplitz-plus-Hankel matrices into matrices with Cauchy rank 4. No special advantage can be gained in the case of a nonsymmetric Toeplitz matrix. But in the case of a real symmetric Toeplitz matrix the sine-I, sine-II, cosine-I, and cosine-II transformations map them into the direct sum of two matrices of about half the size with Cauchy rank 2.[2]

Since all transformations listed above are almost unitary, the condition of the matrix remains essentially unchanged. Furthermore, for all of these transformations fast and stable algorithms do exist (see [21, 23, 24, 25, 28]).

The method used in Section 2–4 is mainly straightforward computation. An alternative approach via displacement structure is presented in Section 5. The advantage of the displacement approach is that is can be generalized to Toeplitz-like matrices, i.e. to matrices T for which $T - S^T TS$ has a small rank, where S denotes the operator of forward shift. For the classical Toeplitz and Toeplitz-plus-Hankel matrices, however, we found the direct approach simpler and more instructive.

In paper II we will present algorithms for the solution of the Cauchy systems emerging from the transformation of Toeplitz and Toeplitz-plus-Hankel systems. These will include the LU factorization of the corresponding Cauchy matrices and their inverses together with partial pivoting techniques.

Let us finally note two other possible applications of the results concerning transformations from Toeplitz into Cauchy matrices. The first one concerns preconditioners for Toeplitz matrices (see also [27] and references therein). Let U be a unitary matrix such that for a Toeplitz matrix T, $C = U^{-1}TU$ is a Cauchy matrix. Consider preconditioners of the form $U^{-1}DU$ where D is diagonal. The optimal (in the Frobenius norm) diagonal preconditioner of C is the diagonal of C, and hence the optimal preconditioner for T is $U^{-1}\,\mathrm{diag}(C)\,U$. If one takes the DFT for U, then one obtains in this way the T. Chan preconditioner. If U is the sine-I transformation, then one obtains the preconditioners proposed in [2]. For the other trigonometric transformations new types of preconditioners are obtained. The importance of Cauchy matrices for iterative methods for Toeplitz methods are recognized in [18].

The second application concerns representations of Toeplitz-like matrices with the help of trigonometric transformations. These representations are based on the representation of the corresponding Cauchy matrices. Related results were obtained using other methods, for example, in [9], [18], and [3].

[1] The numbering of the sine and cosine transforms is not unique in the literature. We mainly follow the standard source [22].

[2] After writing this paper we observed that this can be achieved also for the other transformations.

The representations give rise to fast matrix-vector multiplication algorithms which can be then used, for example, in iterative solvers. This will be discussed in more detail elsewhere.

2. TRANSFORMATIONS OF TOEPLITZ MATRICES BY DFT

In this section we show how Toeplitz matrices can be transformed into generalized Cauchy matrices with the help of complex DFT. In contrast with the approach in [14, 12], we do not make explicit use of their displacement structure but give direct proofs instead.

For $\lambda \in \mathbf{C}$, let $l(\lambda)$ denote the column $l(\lambda) = [1 \;\; \lambda \;\cdots\; \lambda^{n-1}]^T$ and S the matrix of the forward shift operator,

$$S = \begin{bmatrix} 0 & & & 0 \\ 1 & & & \\ & \ddots & & \\ 0 & & 1 & 0 \end{bmatrix}.$$

We use the fact that matrices S^k and $(S^k)^T$ $(k = 0, \dots, n-1)$ form a basis in the space of all $n \times n$ Toeplitz matrices.

For two complex numbers λ and μ with $\lambda\mu \neq 1$ we have for $k = 0, 1, \dots, n-1$

$$l(\lambda)^T S^k l(\mu) = \frac{\lambda^n \mu^{n-k} - \lambda_k}{\lambda\mu - 1}, \qquad l(\lambda)^T S^{kT} l(\mu) = \frac{\lambda^{n-k}\mu^n - \mu^k}{\lambda\mu - 1}. \tag{2.1}$$

Moreover,

$$l(\lambda)^T S^k l(\lambda^{-1}) = \lambda^k (n-k) \quad \text{and} \quad l(\lambda)^T S^{kT} l(\lambda^{-1}) = \lambda^{-k}(n-k). \tag{2.2}$$

Let now $T = [a_{i-j}]_1^n$ be a Toeplitz matrix. Then

$$T = \sum_{k=0}^{n-1}{}' a_k S^k + \sum_{k=0}^{n-1}{}' a_{-k} S^{kT}. \tag{2.3}$$

The prime on the sum sign is according to the following convention:

$$\sum_{k=0}^{m}{}' a_k := \frac{a_0}{2} + \sum_{k=1}^{m} a_k.$$

We introduce the functions

$$a_{+}(t) = \sum_{k=0}^{n-1}{}' a_k t^k, \qquad a_{-}(t) = \sum_{k=0}^{n-1}{}' a_{-k} t^{-k}, \qquad a(t) = a_{-}(t) + a_{+}(t).$$

Furthermore we fix two complex numbers ξ and η with $|\xi| = |\eta| = 1$. Let c_i denote the nth roots of ξ, and d_j the nth roots of η. From (2.1) we get for $c_i \neq d_j$

$$l(c_i)^T Tl\left(d_j^{-1}\right) = \frac{\tilde{f}(c_i) - f(d_j)}{c_i - d_j} d_j, \tag{2.4}$$

where

$$\tilde{f}(t) = \xi\eta^{-1} a_{-}(t) - a_{+}(t), \qquad f(t) = a_{-}(t) - \xi\eta^{-1} a_{+}(t).$$

Furthermore, (2.2) leads to

$$l(c_i)^T Tl\left(c_i^{-1}\right) = na(c_i) - \{a'_{+}(c_i) - a'_{-}(c_i)\}c_i, \tag{2.5}$$

where the prime indicates the derivative. For a given n-tuple $c = (c_k)_1^n$, we denote by $V(c)$ the Vandermonde matrix

$$V(c) = \begin{bmatrix} 1 & c_1 & \cdots & c_1^{n-1} \\ 1 & c_2 & \cdots & c_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & c_n & \cdots & c_n^{n-1} \end{bmatrix}.$$

If c is the n-tuple of the nth roots of ξ (in a certain unspecified order), then we set

$$\mathscr{F}(\xi) = \frac{1}{\sqrt{n}} V(c).$$

Note that $\mathscr{F}(1)$ is the DFT in the usual sense and

$$\mathscr{F}(\xi) = \mathscr{F}(1) \operatorname{diag}\left(1, c_1, \ldots, c_1^{n-1}\right).$$

As an immediate consequence of (2.4) and (2.5) we obtain the following.

THEOREM 1. *Let ξ, η be two complex numbers with $|\xi| = |\eta| = 1$, c_k the nth roots of ξ, and d_k the nth roots of η $(k = 1, \dots, n)$. Then for a Toeplitz matrix $T = [a_{i-j}]_1^n$ the matrix $C := \mathscr{F}(\xi)T\mathscr{F}(\eta^{-1})^T$ has Cauchy rank $\leqslant 2$. The entries c_{ij} of the matrix C are given by*

$$c_{ij} = \begin{cases} \dfrac{d_j}{n} \dfrac{\tilde{f}(c_i) - f(d_j)}{c_i - d_j}, & c_i \neq d_j, \\[2ex] a(c_i) - \dfrac{1}{n}\{a'_+(c_i) - a'_-(c_i)\}c_i, & c_i = d_j. \end{cases}$$

REMARK 2. For arbitrary Vandermonde matrices $V(c)$ and $V(d^{-1})$, where $d^{-1} := (d_1^{-1}, \dots, d_n^{-1})$, the matrix $C = V(c)TV(d^{-1})^T$ has Cauchy rank $\leqslant 4$ with respect to c and d. This is also true for confluent Vandermonde matrices.

REMARK 3. One gets a Cauchy matrix with Cauchy rank 2 if T is multiplied by the inverses of Vandermonde matrices. This was first observed by M. Fiedler [8]. Since this fact seems to be not relevant for the construction of fast stable algorithms, we do not discuss it in detail.

We now consider some special cases.

2.1. *Nonsymmetric Standard Choice*

In [14] it was proposed to choose $\zeta = 1$ and $\eta = -1$. In this case we have $f(t) = -\tilde{f}(t) = a(t)$. The entries of $C = \mathscr{F}(1)T\mathscr{F}(-1)^T$ are given by

$$c_{ij} = -\frac{\sigma\omega_j}{n} \frac{a(\omega_i) + a(\sigma\omega_j)}{\omega_i - \sigma\omega_j},$$

where ω_i are the nth unit roots and $\sigma = \exp(\pi \mathrm{i}/n)$. Here and in the sequel i is the imaginary unit $\sqrt{-1}$.

2.2. *Hermitian Toeplitz Matrices*

If the Toeplitz matrix T is Hermitian, i.e. $a_{-i} = \bar{a}_i$, it is desirable to have also a Hermitian matrix after the transformation. Therefore, it is convenient to choose $\xi = \eta = 1$. In this case we have $f(t) = \tilde{f}(t) = a_-(t) - a_+(t)$.

Furthermore, since in the Hermitian case $a_+(t) = \overline{a_-(t)}$, we have $f(t) = 2i \operatorname{Im} a_+(t)$. The relation (2.5) goes over into

$$l(\omega_i)^T T l(\overline{\omega}_i) = -2 \operatorname{Re}\{n a_+(\omega_i) + \omega_i a'_+(\omega_i)\}.$$

We arrive at the following.

THEOREM 4. *Let T be a Hermitian Toeplitz matrix. Then $C = \mathscr{F}(1) T \mathscr{F}(1)^*$ is a Hermitian matrix with Cauchy rank $\leqslant 2$ (with respect to $c = d = \omega$) given by $C = [c_{ij}]_1^n$,*

$$c_{ij} = \begin{cases} \dfrac{2}{ni} \dfrac{\operatorname{Im} a_+(\omega_i) - \operatorname{Im} a_+(\omega_j)}{1 - \omega_i \overline{\omega}_j}, & i \neq j, \\ -2 \operatorname{Re}\left(a_+(\omega_i) + \dfrac{1}{n} \omega_i a'_+(\omega_i)\right), & i \neq j. \end{cases}$$

The following fact concerning the matrix C is still more important for our construction of fast algorithms in paper II.

COROLLARY 5. *If T is a Hermitian Toeplitz matrix and $D(\omega) = \operatorname{diag}(\omega_j)_1^N$, then $\hat{C} = (ni/2) \mathscr{F}(1) T \mathscr{F}(1)^* D$ is a complex symmetric matrix satisfying*

$$D(\omega)\hat{C} - \hat{C} D(\omega) = Z K Z^T,$$

where[3]

$$Z = \operatorname{col}\begin{bmatrix} 1 & \operatorname{Im} a_+(\omega_i) \end{bmatrix}_1^n, \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Hermitian Toeplitz matrices can be transformed even into real Cauchy matrices. For this we use a linear fractional transformation mapping the unit circle into the real line. We fix a complex number ζ with absolute value 1

[3] $\operatorname{col}(a_j)_1^n$ means the column vector with the components a_j.

different from the nth unit roots ω_k. We introduce numbers x_j by

$$x_j = \frac{\zeta + \omega_j}{\zeta - \omega_j}\,\mathrm{i} = \frac{2\,\mathrm{Im}\,\overline{\omega}_j\zeta}{|\zeta - \omega_j|^2}.$$

Clearly, the x_j are real and $\omega_j = \dfrac{x_j - \mathrm{i}}{x_j + \mathrm{i}}\,\zeta$. Hence

$$1 - \omega_i\overline{\omega}_j = 1 - \frac{x_i - \mathrm{i}}{x_i + \mathrm{i}}\,\frac{x_j + \mathrm{i}}{x_j - \mathrm{i}}$$

$$= -2\mathrm{i}(x_i + \mathrm{i})^{-1}(x_i - x_j)(x_j - \mathrm{i})^{-1}.$$

This leads to the following.

THEOREM 6. *Let* T *be a Hermitian Toeplitz matrix and* $D_{\pm} = \mathrm{diag}((x_j + \mathrm{i})^{-1})_1^n$. *Then* $\tilde{C} = D_+\mathscr{F}(1)T\mathscr{F}(1)^*D_-$ *is a real symmetric Cauchy matrix given by* $\tilde{C} = [\tilde{c}_{ij}]_1^n$,

$$\tilde{c}_{ij} = \begin{cases} \dfrac{1}{n}\,\dfrac{\mathrm{Im}\,a_+(\omega_i) - \mathrm{Im}\,a_+(\omega_j)}{x_i - x_j}, & i \neq j, \\[2ex] -2(x_i^2 + 1)\,\mathrm{Re}\left(a_+(\omega_i) + \dfrac{1}{n}\,\omega_i a'_+(\omega_i)\right), & i = j. \end{cases}$$

If we have a real symmetric Toeplitz matrix, then Theorem 6 describes a complex transformation into a real symmetric Cauchy matrix. In Section 4 we show that such matrices can be transformed into two real symmetric Cauchy matrices of about half the size with the help of real transformations.

2.3. *Symmetric Toeplitz Matrices*

We discuss now a transformation that transforms complex symmetric Toeplitz matrices into symmetric Cauchy matrices. In Theorem 1 let c_i be the roots of i and $d_j = c_j^{-1}$. Then the d_j run over all nth roots of $-\mathrm{i}$, and $f(t) = -\tilde{f}(t) = a(t)$. If now T is symmetric, then we have $a(t^{-1}) = a(t)$. Thus Theorem 1 goes over into the following.

THEOREM 7. *Let T be a complex symmetric Toeplitz matrix. Then $C = \mathcal{F}(\mathrm{i})T\mathcal{F}(\mathrm{i})^T$ is a symmetric matrix with Cauchy rank $\leqslant 2$ given by $C = [c_{ij}]_1^n$,*

$$c_{ij} = \frac{a(c_i) + a(c_j)}{1 - c_i c_j},$$

where the c_i are the nth roots of i.

In order to get the matrix C in a form which is more convenient for the application of the algorithms described in paper II, we use the same linear fractional substitution as in the previous subsection. Here, however, it is possible to choose $\zeta = 1$. That means we get

$$x_j = \frac{1 + c_j}{1 - c_j}\,\mathrm{i}.$$

Then the x_j are real and the entries of C can be represented in the form

$$c_{ij} = (x_i + \mathrm{i})\,\frac{a(c_i) + a(c_j)}{2\mathrm{i}(x_i + x_j)}\,(x_j + \mathrm{i}).$$

Thus we get the following.

COROLLARY 8. *If T is a complex symmetric Toeplitz matrix, $D(x) = \mathrm{diag}(x_j)_1^n$, $D_\pm$ is as in Theorem 2.5, and $\tilde{C} = 2\mathrm{i}\,D_+\mathcal{F}(\mathrm{i})T\mathcal{F}(\mathrm{i})^T D_-$, then*

$$D(x)\tilde{C} + \tilde{C}D(x) = ZKZ^T,$$

where

$$Z = \mathrm{col}\begin{bmatrix} 1 & a(c_1) \end{bmatrix}_1^n, \qquad K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

3. TRANSFORMATION OF COMPLEX TOEPLITZ-PLUS-HANKEL MATRICES BY DFT

In this section we show that complex Toeplitz-plus-Hankel matrices can be transformed into matrices with Cauchy rank $\leqslant 4$ with the help of DFT. Suppose that $\lambda = e^{i\phi}$ and $\mu = e^{i\psi}$ ($\phi, \psi \in \mathbf{R}$). Then

$$\lambda^{-1}\mu^{-1}(\lambda - \mu)(\lambda\mu - 1) = 2(\cos\phi - \cos\psi).$$

We apply this relation for $\lambda = c_k = \exp \phi_k i$ and $\mu = d_j = \exp \psi_j i$, where c_k are the nth roots of ξ and d_j the nth roots of η, $|\xi| = |\eta| = 1$. Then we obtain from (2.4), for a Toeplitz matrix T defined by (2.3),

$$2c_i d_j(\cos\phi_i - \cos\psi_j)l(c_i)^T Tl(d_j) = (c_i - d_j)\left\{\tilde{g}(c_i) - g\left(d_j^{-1}\right)\right\}, \quad (3.1)$$

where

$$\tilde{g}(t) = \xi\eta a_-(t) - a_+(t), \qquad g(t) = a_-(t) - \xi\eta a_+(t),$$

and $a_-(t)$ and $a_+(t)$ are defined as in Section 2.

We consider now Hankel matrices

$$H = \left[b_{i+j}\right]_0^{n-1} = \sum_{k=0}^{n-1}{}' \left(b_{n-1-k}JS^k + b_{n-1+k}S^kJ\right), \quad (3.2)$$

where J denotes the counteridentity,

$$J = \begin{bmatrix} & & & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & & & \end{bmatrix}.$$

Then

$$(c_i - d_j)l(c_i)^T Hl(d_j) = \tilde{h}(c_i) - h(d_j),$$

where

$$\tilde{h}(t) = \sum_{k=0}^{n-1}{}' \left(b_{n-1-k}t^{n-k} - \eta b_{n-1+k}t^{k}\right),$$

$$h(t) = \sum_{k=0}^{n-1}{}' \left(b_{n-1-k}t^{n-k} - \xi b_{n-1+k}t^{k}\right).$$

Hence

$$2c_i d_j(\cos\phi_i - \cos\psi_j) l(c_i)^T H l(d_j) = (c_i d_j - 1)\left\{\tilde{h}(c_i) - h(d_j)\right\}. \quad (3.3)$$

For a Toeplitz-plus-Hankel matrix $A = T + H$ we have now

$$\begin{aligned} &2c_i d_j(\cos\phi_i - \cos\psi_j) l(c_i)^T A l(d_j) \\ &\quad = \left\{c_i\tilde{g}(c_i) - \tilde{h}(c_i)\right\} - \left\{\tilde{g}(c_i) - c_i\tilde{h}(c_i)\right\}d_j \\ &\quad\quad - c_i\left\{g(d_j^{-1}) + d_j h(d_j)\right\} + \left\{d_j g(d_j^{-1}) + h(d_j)\right\}. \end{aligned}$$

Thus we have proved the following.

THEOREM 9. *Let ξ, η be two given complex numbers with $|\xi| = |\eta| = 1$; c_i, d_j the nth roots of ξ and η, respectively; and $\cos\phi_i = \operatorname{Re} c_i$, $\cos\psi_i = \operatorname{Re} d_i$. Then for a Toeplitz-plus-Hankel matrix $A = T + H$, the matrix $C := \mathcal{F}(\xi)A\mathcal{F}(\eta)^T$ has Cauchy rank $\leqslant 4$ with respect to $(\cos\phi_i)_1^n$ and $(\cos\psi_j)_1^n$. If $\xi \neq \eta$ and $\xi\eta \neq 1$, then $\cos\phi_i \neq \cos\psi_j$ and the entries c_{ij} of C are given by*

$$c_{ij} = \frac{\tilde{p}(c_i) - \tilde{q}(c_i)d_j - c_i q(d_j) + p(d_j)}{2nc_i(\cos\phi_i - \cos\psi_j)d_j},$$

where

$$\tilde{p}(t) = t\tilde{g}(t) - h(t), \qquad \tilde{q}(t) = \tilde{g}(t) - t\tilde{h}(t),$$

$$p(t) = tg(t^{-1}) + h(t), \qquad q(t) = g(t^{-1}) + th(t).$$

REMARK 10.

1. The entries of C can also be described in the cases $\xi = \eta$ and $\xi\eta = 1$ using the relations

$$l(c_i)^T H l(c_i) = \xi \sum_{k=0}^{n-1}{}' (n-k)\left(b_{n-1-k} c_i^{n-k-1} + b_{n-1+k} c_i^{n+k-1}\right)$$

and (2.5).

2. For a simple implementation it is desirable to have also $\cos \phi_i \neq \cos \phi_j$ for $i \neq j$. This can be guaranteed if ξ and η are chosen normal. One possibility is $\xi = -\eta = (1/\sqrt{2})(1 + \mathrm{i})$.

3. In the case of a Hermitian Toeplitz-plus-Hankel matrix the transformed matrix will be Hermitian again if $\xi = \bar{\eta}$. We suggest choosing $\xi = -\eta = \mathrm{i}$ (rather than $\xi = \eta = 1$). With this choice we have $\cos \phi_i = \cos \psi_i$ and $\cos \phi_i \neq \cos \phi_j$ for $i \neq j$.

4. REAL TRIGONOMETRIC TRANSFORMATIONS

The disadvantages of the transformation with the help of DFT is that complex arithmetic is required even if the matrices are real. In this section we discuss some real trigonometric transformations. These transformations, however, transform Toeplitz matrices into matrices with Cauchy rank $\leqslant 4$ rather than 2. This value can also be achieved for Toeplitz-plus-Hankel matrices. Therefore in this section we derive transformation formulas first for Toeplitz-plus-Hankel matrices. Later in the section we show that these formulas can be much simplified in the case of real symmetric Toeplitz matrices.

4.1. *Transformation with Chebyshev-Vandermonde Matrices*

As the DFT is a special Vandermonde matrix, the real trigonometric transformations are special Chebyshev-Vandermonde matrices, up to diagonal factors.

Polynomials $u_k(\lambda)$ $(k = 0, 1, \ldots)$ satisfying the recursion

$$u_{k+1}(\lambda) = 2\lambda u_k(\lambda) - u_{k-1}(\lambda) \qquad (k = 1, 2, \ldots) \tag{4.1}$$

will be called *polynomials of Chebyshev type*. The Chebyshev polynomials of the first kind $T_k(\lambda)$,

$$T_k(\cos\theta) = \cos k\theta,$$

have this property and satisfy the initial conditions $u_0 = 1$, $u_1(\lambda) = \lambda$, and the Chebyshev polynomials of the second kind $U_k(\lambda)$,

$$U_k(\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta},$$

also satisfy this recursion with the initial conditions $u_0 = 1$, $u_1(\lambda) = 2\lambda$. If $u_0(\lambda)$ and $u_1(\lambda)$ are fixed, then (4.1) defines $u_k(\lambda)$ also for negative k. In particular, $U_{-1} = 0$ and $T_{-1}(\lambda) = \lambda$.

For two sequences of polynomials $u_k(\lambda)$ and $\tilde{u}_k(\lambda)$ satisfying (4.1), we introduce the vectors $u(\lambda) = (u_k(\lambda))_0^{n-1}$ and $\tilde{u}(\lambda) = (\tilde{u}_k(\lambda))_0^{n-1}$. The following lemma is crucial for the further investigation.

LEMMA 11. *If S denotes the matrix of the forward shift, then*

$$2(\lambda-\mu)\tilde{u}(\lambda)^T S^k u(\mu) = \tilde{u}_n u_{n-k-1} - \tilde{u}_k u_{-1} + \tilde{u}_{k-1}u_0 - \tilde{u}_{n-1}u_{n-k},$$

$$2(\lambda-\mu)\tilde{u}(\lambda)^T S^{kT} u(\mu) = \tilde{u}_{n-k} u_{n-1} - \tilde{u}_0 u_{k-1} + \tilde{u}_{-1}u_k - \tilde{u}_{n-k-1}u_n,$$

$$\begin{aligned}2(\lambda-\mu)\tilde{u}(\lambda)^T J S^k u(\mu) &= \tilde{u}_{n-k} u_0 - \tilde{u}_0 u_{n-k}\\ &\quad + \tilde{u}_{-1}u_{n-1-k} - \tilde{u}_{n-1-k}u_{-1},\end{aligned}$$

$$2(\lambda-\mu)\tilde{u}(\lambda)^T S^k J u(\mu) = \tilde{u}_n u_k - \tilde{u}_k u_n + \tilde{u}_{k-1}u_{n-1} - \tilde{u}_{n-1}u_{k-1}.$$

Proof. We have

$$\tilde{\mu}(\lambda)^T S^k u(\mu) = \sum_{i=0}^{n-k-1} \tilde{u}_{i+k}(\lambda)u_i(\mu).$$

According to the recursion (4.1) we get

$$\begin{aligned}&2(\lambda-\mu)\tilde{u}(\lambda)^T S^k u(\mu)\\ &\quad = \sum_{i=0}^{n-k-1}\left\{(\tilde{u}_{i+k+1} + \tilde{u}_{u+k-1})u_i - \tilde{u}_{i+k}(u_{i+1}+u_{i-1})\right\}.\end{aligned}$$

Telescoping the latter sums, we obtain the first equality. Analogously, the other relations are verified. ∎

A matrix of the form

$$\mathscr{U}(x) = \left[u_{j-1}(x_i)\right]_{i,j=1}^{n}, \tag{4.2}$$

where $x = (x_i)_1^n \in \mathbf{R}^n$, is called a Chebyshev-Vandermonde matrix. The following is an immediate consequence of Lemma 11.

PROPOSITION 12. *If $\tilde{\mathscr{U}}(\tilde{x})$ and $\mathscr{U}(x)$ are Chebyshev-Vandermonde matrices, then for any Toeplitz-plus-Hankel matrix A the matrix $\tilde{\mathscr{U}}(\tilde{x})A\mathscr{U}(x)^T$ has Cauchy rank $\leqslant 8$.*

We are looking now for special choices of $u_j(\lambda)$ and x_i for which the transformed matrix has Cauchy rank $\leqslant 4$. There are many possibilities. We restrict ourselves to those which lead to the classical sine and cosine transformations, because for them fast and stable algorithms are well known and furthermore they have some additional symmetry properties that simplify the computation. In particular we will get matrices with a 2×2 block structure $[C_{ij}]_1^2$ such that the C_{ij} have Cauchy rank $\leqslant 2$. In the case of a symmetric Toeplitz matrix we even have $C_{12} = C_{21} = 0$.

4.2. *Sine-I Transformation*

Let us deal first with the case of Chebyshev polynomials of the second kind, $\tilde{u}(\lambda) = u(\lambda) = U(\lambda) = (U_k(\lambda))_0^{n-1}$. We introduce

$$x_i := \cos\frac{i\pi}{n+1} \quad \text{and} \quad y_i := \sin\frac{i\pi}{n+1} \qquad (i = 1,\dots,n).$$

The x_i $(i = 1,\dots,n)$ are just the roots of the polynomial $U_n(\lambda)$. Furthermore we denote

$$s_{ij} := \sin\frac{ij\pi}{n+1} = y_i U_{j-1}(x_i), \qquad c_{ij} := \sin\frac{ij\pi}{n+1} = T_j(x_i).$$

The matrix $\mathscr{U}(x)$ is related to the sine-I transform, which is the matrix-vector multiplication, by

$$\mathscr{S}_n^{\mathrm{I}} := \sqrt{\frac{2}{n+1}}\left[\sin\frac{ij\pi}{n+1}\right]_{i,j=1}^{n}.$$

From Lemma 11 we get

$$2(\lambda-\mu)U(\lambda)^T U(\mu) = U_n(\lambda)U_{n-1}(\mu) - U_{n-1}(\lambda)U_n(\mu),$$

which implies $U(x_i)^T U(x_j) = 0$ for $i \neq j$, and $2U(x_i)^T U(x_i) = U_n'(x_i)U_{n-1}(x_i)$. Taking into account that $U_{n-1}(x_i) = (-1)^{i+1}$ and

$$y_i^2 U_n'(x_i) = (-1)^{i+1}(n+1),$$

we obtain the well-known fact that $\mathscr{S}_n^I = (\mathscr{S}_n^I)^{-1}$.

It is important to observe the symmetry relations

$$s_{ik} = (-1)^{i+1} s_{i,n-k+1}, \qquad c_{ik} = (-1)^i c_{i,n-k+1}.$$

In particular, $s_{in} = y_i U_{n-1}(x_i) = (-1)^{i+1} y_i$. Using these relations, we obtain from Lemma 11

$$\begin{aligned}
2(x_i - x_j) y_i y_j U(x_i)^T S^k U(x_j) &= s_{ik} y_j - (-1)^{i+j} y_i s_{jk},\\
2(x_i - x_j) y_i y_j U(x_i)^T S^{kT} U(x_j) &= (-1)^{i+j} s_{ik} y_j - y_i s_{jk},\\
2(x_i - x_j) y_i y_j U(x_i)^T J S^k U(x_j) &= (-1)^{i+1} s_{ik} y_j - (-1)^{j+1} y_i s_{jk},\\
2(x_i - x_j) y_i y_j U(x_i)^T S^k J U(x_j) &= (-1)^{j+1} s_{ik} y_j - (-1)^{i+1} y_i s_{jk}.
\end{aligned} \tag{4.3}$$

From the relations (4.3) we may conclude how Toeplitz-plus-Hankel matrices are transformed by the sine transform except for the main diagonal. In order to evaluate the main diagonal we differentiate the first relation in Lemma 11 with respect to μ and obtain

$$2U(x_i)^T S^k U(x_i) = U_{n-1}(x_i) U_{n-k}'(x_i).$$

Since

$$U_k'(\lambda) = \frac{1}{1-\lambda^2}\{\lambda U_k(\lambda) - (k+1)T_{k+1}(\lambda)\},$$

we conclude

$$y_i^2 U_{n-k}'(x_i) = (-1)^{i+1} t_{ik},$$

where

$$t_{ik} := \frac{1}{y_i} x_i s_{ik} + (n-k+1)c_{ik}.$$

Hence

$$2y_i^2 U(x_i)^T S^k U(x_i) = t_{ik}. \tag{4.4}$$

We get the same expression for $2y_i^2 U(x_i)^T S^{kT} U(x_i)$.

Differentiating the third relation in Lemma 11 with respect to λ, we obtain

$$2y_i^2 U(x_i)^T J S^k U(x_i) = y_i^2 U'_{n-k}(x_i) = (-1)^{i+1} t_{ik}. \tag{4.5}$$

Due to symmetry or skew symmetry of the vectors $U(x_i)$, we get the same expression for $2y_i^2 U(x_i)^T S^k J U(x_i)$.

We consider a Toeplitz-plus-Hankel matrix $A = [a_{i-j} + b_{i+j}]_0^{n-1}$. This matrix can be represented in the form

$$A = \sum_{k=0}^{m-1}{}' \left(a_k S^k + a_{-k} S^{kT} + b_{n-1-k} J S^k + b_{n-1+k} S^k J \right). \tag{4.6}$$

We introduce the numbers

$$f_i^{\pm} = \frac{1}{n+1} \sum_{k=0}^{n-1}{}' s_{ik} a_{\pm k}, \qquad g_i^{\pm} = \frac{1}{n+1} \sum_{k=0}^{n-1}{}' s_{ik} b_{n-1\pm k}, \tag{4.7}$$

$$\begin{aligned} h_i &= \frac{1}{n+1} \sum_{k=0}^{n-1}{}' t_{ik}(a_k + a_{-k}), \\ l_i &= \frac{1}{n+1} \sum_{k=0}^{n-1}{}' t_{ik}(b_{n-1-k} + b_{n-1+k}), \end{aligned} \tag{4.8}$$

and $f_i = f_i^+ + f_i^-$, $g_i = g_i^+ + g_i^-$. We arrive at the following.

THEOREM 13. *Let A be given by* (4.6). *Then the matrix* $\mathcal{S}_n^I A \mathcal{S}_n^I = [\gamma_{ij}]_1^n$ *has Cauchy rank* $\leqslant 4$ *with respect to* $(x_i)_1^n$ *and the entries are given by*

$$\gamma_{ij} = \begin{cases} \dfrac{\alpha_i^{(j)} y_j - y_i \beta_j^{(i)}}{x_i - x_j} & i \neq j, \\ h_i + (-1)^{i+1} l_i, & i = j, \end{cases}$$

where

$$\alpha_i^{(j)} = f_i^+ + (-1)^{i+j} f_i^- + (-1)^{i+1} g_i^- + (-1)^{j+1} g_i^+,$$

$$\beta_j^{(i)} = (-1)^{i+j} f_j^+ + f_j^- + (-1)^{j+1} g_j^- + (-1)^{i+1} g_j^+.$$

Let Π denote the even-odd shuffle matrix, $\Pi(x_i)_1^n = (x_2, x_4,, \dots, x_1, x_3, \dots)$. From Theorem 9 we get now the following.

COROLLARY 14. *If A is a Toeplitz-plus-Hankel matrix, then $\Pi^T \mathcal{S}_n^I A \mathcal{S}_n^I \Pi$ has a 2×2 block structure $[C_{ik}]_1^2$ where C_{ik} have Cauchy rank $\leqslant 2$.*

In particular, for a real symmetric Toeplitz matrix

$$T = [a_{|i-j|}]_1^n = \sum_{k=0}^{n-1}{}' a_k \left\{ S^k + (S^k)^T \right\} \tag{4.9}$$

we get the following.

THEOREM 15. *Let T be given by* (4.9). *Then*

$$\Pi^T \mathcal{S}_n^I T \mathcal{S}_n^I \Pi = \begin{bmatrix} C_{\text{even}} & 0 \\ 0 & C_{\text{odd}} \end{bmatrix},$$

where $C_{\text{even}} = [c_{pq}^{\text{even}}]_1^{m_1}$ and $C_{\text{odd}} = [c_{pq}^{\text{odd}}]_1^{m_2}$, with $m_1 = [(n+1)/2]$, $m_2 = [n/2]$, are given by

$$c_{pq}^{\text{even}} = \begin{cases} \dfrac{f_{2p} y_{2q} - y_{2p} f_{2q}}{x_{2p} - x_{2q}}, & p \neq q, \\ h_{2p}, & p = q, \end{cases}$$

$$c_{pq}^{\text{odd}} = \begin{cases} \dfrac{f_{2p-1} y_{2q-1} - y_{2p-1} f_{2q-1}}{x_{2p-1} - x_{2q-1}}, & p \neq q, \\ h_{2p+1}, & p = q. \end{cases}$$

4.3. Cosine-I Transformation

We assume now that in Lemma 11 $\tilde{u}_k(\lambda) = u_k(\lambda) = T_k(\lambda)$. We introduce the vector polynomials

$$T(\lambda) = (T_k(\lambda))_0^{n-1} \quad \text{and} \quad \tilde{T}(\lambda) = (\epsilon_k T_k(\lambda))_0^{n-1},$$

where

$$\epsilon_k = \begin{cases} \frac{1}{2}, & k = 0, n-1, \\ 1, & k = 1, \ldots, n-2. \end{cases}$$

We consider these vector polynomials at the points

$$x_j := \cos \frac{j\pi}{n-1} \qquad (i = 0, \ldots, n-1).$$

Furthermore we introduce

$$y_j := \sin \frac{j\pi}{n-1}$$

and

$$c_{ij} := \cos \frac{ij\pi}{n-1} = T_i(x_j), \qquad s_{ij} = \sin \frac{ij\pi}{n-1} = y_j U_{i-1}(x_j).$$

Let us point out that the quantities x_i, y_i, c_{ij}, and s_{ij} are different to those in the previous subsection.

The vectors $\tilde{T}(x_j)$ are related to the cosine-I transformation, which is the matrix-vector multiplication, by

$$\mathscr{C}_n^{\mathrm{I}} = \sqrt{\frac{2}{n-1}} \left[\epsilon_j \cos \frac{ij\pi}{n-1} \right]_{i,j=0}^{n-1} = \sqrt{\frac{2}{n-1}} \left[\tilde{T}_j(x_i) \right]_{i,j=0}^{n-1}.$$

In contrast with the sine-I transformation, $\mathscr{C}_n^{\mathrm{I}}$ is not symmetric and not unitary. But, as for the sine-I transform, the relation $(\mathscr{C}_n^{\mathrm{I}})^{-1} = \mathscr{C}_n^{\mathrm{I}}$ holds.

We have the following symmetry relations for the c_{ik} and s_{ik}:

$$c_{ik} = (-1)^i c_{i,n-1-k}, \qquad s_{ik} = (-1)^{i+1} s_{i,n-1-k}.$$

In particular,

$$T_{n-1}(x_i) = c_{i,n-1} = (-1)^i, \qquad T_n(x_i) = c_{in} = (-1)^i x_i.$$

In order to study the action of the cosine-I transformation on Toeplitz and Toeplitz-plus-Hankel matrices, we study their action on the powers of the shift S^k. We have to distinguish the cases (a) $k \neq 0, n-1$, (b) $k = n-1$, and (c) $k = 0$.

Case (a): $k \neq 0, n-1$. Applying Lemma 11 we get,

$$\begin{aligned} &2(\lambda - \mu)T(\lambda)^T S^k T(\mu) \\ &\quad = T_n(\lambda)T_{n-k-1}(\mu) - T_k(\lambda)\mu T_{k-1}(\lambda) - T_{n-1}(\lambda)T_{n-k}(\mu). \end{aligned} \tag{4.10}$$

This implies

$$\begin{aligned} &2(\lambda - \mu)T(\lambda)^T S^k \tilde{T}(\mu) \\ &\quad = T_n(\lambda)T_{n-k-1}(\mu) - \lambda T_k(\lambda) + T_{k-1}(\lambda) - T_{n-1}(\lambda)T_{n-k}(\mu) \end{aligned} \tag{4.11}$$

for $k = 1, \dots, n-2$.

From (4.11) we obtain now

$$\begin{aligned} &2(x_i - x_j)\tilde{T}^T(x_i)S^k\tilde{T}(x_j) \\ &\qquad = (-1)^{i+j}\{x_j T_k(x_j) - T_{k-1}(x_j)\} - x_i T_k(x_i) + T_{k-1}(x_i). \end{aligned}$$

In view of

$$T_{k-1}(x_i) - x_i T_k(x_i) = c_{i,k-1} - x_i c_{ik} = y_i s_{ik},$$

we conclude

$$2(x_i - x_j)\tilde{T}(x_i)^T S^k \tilde{T}(x_j) = y_i s_{ik} - (-1)^{i+j} y_j s_{jk} \tag{4.12}$$

for $k = 1, \dots, n-2$.

Analogously, for $k \neq 0, n-1$,

$$2(x_i - x_j)\tilde{T}(x_i)^T S^{kT}\tilde{T}(x_j) = (-1)^{i+j} y_i s_{ik} - y_j s_{jk},$$

$$2(x_i - x_j)\tilde{T}(x_i)^T J S^{k}\tilde{T}(x_j) = (-1)^{i} y_i s_{ik} - (-1)^j y_j s_{jk},$$

$$2(x_i - x_j)\tilde{T}(x_i)^T S^{k} J\tilde{T}(x_j) = (-1)^{j} y_i s_{ik} - (-1)^i y_j s_{jk}.$$

In order to compute the diagonal of the cosine transformed matrices, we differentiate (4.11) with respect to μ and obtain

$$\begin{aligned} 2\tilde{T}(x_i)^T S^k \tilde{T}(x_i) &= -T_{n-1}(x_i)T_{n-k-1}(x_i) - T_n(x_i)T'_{n-k-1}(x_i) \\ &\quad + T_{n-1}(x_i)T'_{n-k}(x_i) \end{aligned} \tag{4.13}$$

for $k = 1, \ldots, n-1$. Since $T'_k(\lambda) = kU_{k-1}(\lambda)$, we conclude, as long as $y_i \neq 0$,

$$2\tilde{T}(x_i)^T S^k \tilde{T}(x_i) = -c_{ik} + (n-k-1)x_i \frac{s_{i,n-k-1}}{y_i} - (n-k)\frac{s_{i,n-k}}{y_i}.$$

Due to $s_{i,k-1} = x_i s_{ik} - y_i c_{ik}$, this implies

$$t_{ik} := 2\tilde{T}(x_i)^T S^k \tilde{T}(x_i) = (n-k-1)c_{ik} - \frac{x_i s_{ik}}{y_i}.$$

For $i = 0$ or $i = n-1$ we have $y_i = 0$. In this case t_{ik} can be calculated directly from (4.13), giving

$$\tilde{T}(x_i)^T S^k \tilde{T}(x_i) = \begin{cases} 2(n-k-1) & \text{for} \quad i = 0, \\ (-1)^k(n-k-1) & \text{for} \quad i = n-1. \end{cases}$$

Furthermore,

$$\begin{aligned} \tilde{T}(x_i)^T S^{kT} \tilde{T}(x_i) &= (-1)^i \tilde{T}(x_i)^T J S^k \tilde{T}(x_i) \\ &= (-1)^i \tilde{T}(x_i)^T S^k J \tilde{T}(x_i) = t_{ik}/2. \end{aligned}$$

Case (b): $k = n - 1$. In this case we obtain via direct calculation the relations

$$\tilde{T}(x_i)^T S^{n-1} \tilde{T}(x_j) = (-1)^i/4, \qquad \tilde{T}(x_i)^T S^{T(n-1)} \tilde{T}(x_j) = (-1)^j/4,$$

$$\tilde{T}(x_i)^T S^{n-1} J \tilde{T}(x_j) = \tfrac{1}{4}, \qquad \tilde{T}(x_i)^T J S^{n-1} \tilde{T}(x_j) = (-1)^{i+j}/4/.$$

Case (c): $k = 0$. For this case we use the fact that $(\mathscr{C}_n^I)^2 = I_n$. From this we obtain for $i \neq 0, n-1$

$$\tilde{T}(x_i)^T \tilde{T}(x_j) = \frac{n-1}{2} \delta_{ij} - \tfrac{1}{4}\{1 + (-1)^{i+j}\}. \tag{4.14}$$

Furthermore,

$$\tilde{T}(x_0)^T \tilde{T}(x_0) = \tilde{T}(x_{n-1})^T \tilde{T}(x_{n-1}) = \frac{2n-3}{2}$$

and

$$\tilde{T}(x_0)^T \tilde{T}(x_{n-1}) = \tilde{T}(x_{n-1})^T \tilde{T}(x_0) = \begin{cases} 0, & n \text{ even} \\ -\frac{1}{2}, & n \text{ odd}. \end{cases}$$

The last relation shows that (4.14) is valid for all $i \neq j$.

In order to calculate $\tilde{T}(x_i)^T J \tilde{T}(x_j)$ one has only to multiply the previous expressions by $(-1)^j$.

Now we have a complete collection of transformation formulas and a theorem can be formulated which is completely analogous to Theorem 3.1. As a consequence we obtain the following.

COROLLARY 16. *If A is a Toeplitz-plus-Hankel matrix, then the matrix $\Pi^T \mathscr{C}_n^I A (\mathscr{C}_n^I)^T \Pi$ has a 2×2 block structure $[C_{ij}]_1^2$ such that the C_{ij} have Cauchy rank $\leqslant 2$. If A is a symmetric Toeplitz matrix, then moreover $C_{12} = C_{21} = 0$, i.e., the transformed matrix is the direct sum of two matrices with Cauchy rank $\leqslant 2$.*

4.4. *Cosine-III and Sine-III Transformations*

We study now the transformation of Toeplitz-plus-Hankel matrices with the cosine-III transformation. Due to the weaker symmetry properties of this transformation, we will not get an essential simplification for the case of a symmetric Toeplitz matrix.

With the Chebyshev polynomials of first kind $T_k(\lambda)$ we form the vector

$$\hat{T}(\lambda)_0^{n-1} = (\eta_k T_k(\lambda))_0^{n-1},$$

where

$$\eta_k = \begin{cases} \frac{1}{2}, & k = 0, \\ 1, & k = 1, \dots, n-1, \end{cases}$$

and consider $\hat{T}(\lambda)$ at the Chebyshev nodes

$$x_j = \cos \frac{(2j+1)\pi}{2n} \qquad (j = 0, \dots, n-1),$$

which are the roots of $T_n(\lambda)$. Furthermore we define

$$y_j = \sin \frac{(2j+1)\pi}{2n}$$

and

$$c_{ij} = T_i(x_j) = \cos \frac{i(2j+1)\pi}{2n}, \qquad s_{ij} = y_j U_i(x_j) = \sin \frac{i(2j+1)\pi}{2n}.$$

Note again again the quantities x_j, y_j, c_{ij}, and s_{ij} are different to those in the previous two subsections.

The vectors $\hat{T}(x_j)$ are related to the cosine-III transformation, which is defined as the matrix-vector multiplication by

$$\mathscr{C}_n^{\mathrm{III}} = \sqrt{\frac{2}{n}} \left[\eta_j \cos \frac{j(2i+1)\pi}{2n} \right]_{i,j=0}^{n-1} = \sqrt{\frac{2}{n}} \left[T_j(x_i) \right]_{ij=0}^{n-1}.$$

The inverse of $\mathscr{C}_n^{\mathrm{III}}$ is the matrix of the cosine-II transformation

$$\mathscr{C}_n^{\mathrm{II}} = \sqrt{\frac{2}{n}} \left[\cos \frac{i(2j+1)\pi}{2n} \right]_{i,j=0}^{n-1}.$$

This follows from the equality

$$2(\lambda-\mu)T(\lambda)^T\hat{T}(\mu)=T_n(\lambda)T_{n-1}(\mu)-T_{n-1}(\lambda)T_n(\mu),$$

which is a consequence of (4.10). From (4.10) we obtain also the equalities

$$\begin{aligned}&2(\lambda-\mu)\hat{T}(\lambda)^T S^k\hat{T}(\mu)\\&\quad=T_n(\lambda)T_{n-k-1}(\mu)+T_{k-1}(\lambda)-\lambda T_k(\lambda)-T_{n-1}(\lambda)T_{n-k}(\mu)\end{aligned}\tag{4.15}$$

for $k=1,\dots,n-1$, and

$$2(\lambda-\mu)\hat{T}(\lambda)^T\hat{T}(\mu)=T_n(\lambda)T_{n-1}(\mu)-T_{n-1}(\lambda)T_n(\mu)-\tfrac{1}{2}(\lambda-\mu).\tag{4.16}$$

We have the symmetry relation

$$c_{ik}=(-1)^i s_{i,n-k}.$$

In particular, $c_{i,n-1}=(-1)^i y_i$. With these relations and $c_{i,k-1}=x_i c_{ik}+y_i s_{ik}$ we conclude from (4.15)

$$2(x_i-x_j)\hat{T}(x_i)^T S^k\hat{T}(x_j)=y_i\left\{s_{ik}-(-1)^{i+j}s_{jk}\right\}\tag{4.17}$$

for $k=1,\dots,n-1$, and

$$2\hat{T}(x_i)^T\hat{T}(x_j)=-\tfrac{1}{2}.\tag{4.18}$$

Analogously we obtain the following relations, using the equality $s_{i,k+1}=x_i s_{ik}+y_i c_{ik}$:

$$2(x_i-x_j)\hat{T}(x_i)^T S^{kT}\hat{T}(x_j)=\left\{(-1)^{i+j}s_{ik}-s_{jk}\right\}y_j,\tag{4.19}$$

$$2(x_i-x_j)\hat{T}(x_i)^T JS^k\hat{T}(x_j)=(-1)^i y_i c_{ik}-(-1)^j y_j c_{jk},\tag{4.20}$$

$$2(x_i-x_j)\hat{T}(x_i)^T S^k J\hat{T}(x_j)=(-1)^j c_{i,k-1}y_j-(-1)^i y_i c_{j,k-1}.\tag{4.21}$$

The relations (4.19) and (4.21) hold for $k=1,\dots,n-1$, whereas (4.20) holds for $k=0,\dots,n-1$.

Differentiating (4.15) with respect to μ and putting $\lambda = \mu = x_i$ we obtain

$$2\hat{T}(x_i)^T S^k \hat{T}(x_i) = 2\hat{T}(x_i)^T S^{kT} \hat{T}(x_i) = (n-k)c_{ik}. \qquad (4.22)$$

Furthermore, after some elementary calculations one gets

$$2\hat{T}(x_i)^T J S^k \hat{T}(x_i) = (-1)^i \left((n-k)s_{i,k-1} - \frac{c_{ik}}{y_i} - s_{ik} \right), \qquad (4.23)$$

$$2\hat{T}(x_i)^T S^k J \hat{T}(x_i) = (-1)^i \left(\frac{nc_{ik}}{y_i} + (k-1)s_{i,k-1} \right). \qquad (4.24)$$

Now with the help of (4.17)–(4.24) one can show how Toeplitz-plus-Hankel matrices transform with the cosine-III transformation. In particular, we obtain the following.

COROLLARY 17. *If A is a Toeplitz-plus-Hankel matrix, then the matrix* $\Pi^T \mathscr{C}_n^{\mathrm{III}} A (\mathscr{C}_n^{\mathrm{III}})^T \Pi$ *has a* 2×2 *block structure* $[C_{ij}]_1^2$ *such that the* C_{ij} *have Cauchy rank* $\leqslant 2$.

Note that similar formulas hold for the sine-III transformation which is defined by

$$\mathscr{S}^{\mathrm{III}} = \sqrt{\frac{2}{n}} \left[\eta_j \sin \frac{j(2i-1)\pi}{2n} \right]_{i,j=1}^n.$$

4.5. *Cosine-II and Sine-II Transformations*

We show now that also the cosine-II and sine-II transformations are also suitable for the transformation of Toeplitz and Toeplitz-plus-Hankel matrices into Cauchy matrices. Because of their symmetry properties they are convenient for symmetric Toeplitz matrices. For this we consider the polynomials $V_k(\lambda)$ of Chebyshev type defined by

$$V_k^{\pm}(\lambda) := U_k(\lambda) - \{\pm U_{k-1}(\lambda)\}. \qquad (4.25)$$

Then $V_{-1}^{\pm}(\lambda) = \pm 1$, $V_0(\lambda) = 1$. Furthermore, it is easily checked that

$$V_k^+(\cos\theta) = \frac{\cos\frac{2k+1}{2}\theta}{\cos\frac{\theta}{2}} \quad \text{and} \quad V_k^-(\cos\theta) = \frac{\sin\frac{2k+1}{2}\theta}{\sin\frac{\theta}{2}}.$$

We define

$$x_i := \cos\frac{i\pi}{n}, \qquad \xi_i := \cos\frac{i\pi}{2n}, \qquad \zeta_i := \sin\frac{i\pi}{2n}$$

and

$$c_{ik} := \cos\frac{i(2k+1)\pi}{2n} = \xi_i V_k^+(x_i), \quad s_{ik} := \sin\frac{i(2k+1)\pi}{2n} = \zeta_i V_k^-(x_i).$$

In particular, $c_{i,-1} = c_{i0} = \xi_i$ and $s_{i,-1} = -s_{i0} = -\zeta_i$. We have the following symmetry relations:

$$c_{i,n-k-1} = (-1)^i c_{ik}, \qquad s_{i,n-k-1} = (-1)^{i+1} s_{ik}.$$

From Lemma 11 we obtain now the following relations for $V(\lambda) = (V_k(\lambda))_0^{n-1}$:

$$\begin{aligned}
&2(x_i - x_j)\xi_i\xi_j V^+(x_i)^T S^k V^+(x_j)\\
&\qquad = (c_{i,k-1} - c_{i,k})\xi_j - (-1)^{i+j}\xi_i(c_{j,k-1} - c_{j,k}),\\
&2(x_i - x_j)\xi_i\xi_j V^+(x_i)^T S^{kT} V^+(x_j)\\
&\qquad = (-1)^{i+j}(c_{i,k-1} - c_{i,k})\xi_j - \xi_i(c_{j,k-1} - c_{j,k}),\\
&2(x_i - x_j)\xi_i\xi_j V^+(x_i)^T J S^k V^+(x_j)\\
&\qquad = (-1)^i(c_{i,k-1} - c_{i,k})\xi_j - (-1)^j\xi_i(c_{j,k-1} - c_{j,k}),\\
&2(x_i - x_j)\xi_i\xi_j V^+(x_i)^T S^k J V^+(x_j)\\
&\qquad = (-1)^j(c_{i,k-1} - c_{i,k})\xi_j - (-1)^i\xi_i(c_{j,k-1} - c_{j,k})
\end{aligned}$$

and

$$2(x_i - x_j)\zeta_i\zeta_j V^-(x_i)^T S^k V^-(x_j)$$

$$= (s_{i,k-1} + s_{ik})\zeta_j - (-1)^{i+j}\zeta_i(s_{j,k-1} + s_{jk}),$$

$$2(x_i - x_j)\zeta_i\zeta_i V^-(x_i)^T S^{kT} V^-(x_j)$$

$$= (-1)^{i+j}(s_{i,k-1} + s_{ik})\zeta_j - \zeta_i(s_{j,k-1} + s_{jk}),$$

$$2(x_i - x_j)\zeta_i\zeta_j V^-(x_i)^T J S^k V^-(x_j)$$

$$= (-1)^i(s_{i,k-1}s_{ik})\zeta_j - (-1)^j\zeta_i(s_{j,k-1} + s_{jk}),$$

$$2(x_i - x_j)\zeta_i\zeta_j V^-(x_i)^T S^k J V^-(x_j)$$

$$= (-1)^j(s_{i,k-1} + s_{ik})\zeta_i - (-1)^i\zeta_i(s_{j,k-1} + s_{jk}).$$

This leads to the following.

COROLLARY 18. *If A is a Toeplitz-plus-Hankel matrix, then the matrices $\Pi^T\mathscr{C}_n^{II}A(\mathscr{C}_n^{II})^T\Pi$ and $\Pi^T\mathscr{S}_n^{II}A(S^{II})^T\Pi$ have a 2×2 block structure $[C_{ij}]_1^2$ such that the C_{ij} have Cauchy rank $\leqslant 2$. If A is a symmetric Toeplitz matrix, then moreover $C_{12} = C_{21} = 0$, i.e., the transformed matrix is the direct sum of two matrices with Cauchy rank $\leqslant 2$.*

4.6. *Mixed Transformations*

Of course, it is possible to combine different transformations. We show this for the combination of the sine-I and cosine-I transformation. The advantage of this combination is that a symmetric Toeplitz matrix will be transformed into the direct sum of two matrices with Cauchy rank 2. However these two matrices are clearly not symmetric. A potential advantage of this kind of transformation is that in the case of even order the nodes of the corresponding Cauchy matrices are pairwise different. This leads to simpler recursions in Cauchy solvers discussed in paper II.

Let $\tilde{T}(\lambda)$ be defined as in Section 4.3, and $U(\lambda)$ as in 4.2. According to Lemma 11 and (4.11) we have

$$2(\lambda - \mu)U(\lambda)^T S^k T(\mu)$$
$$= U_n(\lambda)T_{n-k-1}(\mu) - \lambda U_k(\lambda) + U_{k-1}(\lambda) - U_{n-1}(\lambda)T_{n-k}(\mu). \quad (4.26)$$

Let x_i, y_i, c_{ij}, s_{ij} $(i, j = 1, \dots, n)$ be defined as in Section 4.2, and let

$$c'_{ij} = \cos \frac{ij\pi}{n-1}, \quad x'_j = c'_{1j} \qquad (i, j = 0, \dots, n-1).$$

Then we get from (4.26)

$$2(x_i - x'_j)U(x_i)^T S^k \tilde{T}(x'_j) = -x_i s_{i,k+1} + s_{ik} + (-1)^{i+j} c'_{j,k-1}.$$

Taking into account that $s_{ik} = x_i s_{i,k+1} - y_i c_{i,k+1}$, we conclude that

$$2(x_i - x'_j)U(x_i)^T S^k \tilde{T}(x'_j) = -c_{i,k+1} + (-1)^{i+j} c'_{j,k-1} \quad (4.27)$$

for $i = 1, \dots, n$ and $j = 0, \dots, n-1$.

Analogously,

$$2(x_i - x'_j)U(x_i)^T S^{kT} \tilde{T}(x'_j) = -(-1)^{i+j} c_{i,k+1} + c'_{j,k-1}, \quad (4.28)$$

$$2(x_i - x'_j)U(x_i)^T J S^k \tilde{T}(x'_j) = (-1)^i c_{i,k+1} - (-1)^j c'_{j,k-1}, \quad (4.29)$$

$$2(x_i - x'_j)U(x_i)^T S^k J \tilde{T}(x'_j) = -(-1)^j c_{i,k+1} + (-1)^i c'_{j,k-1} \quad (4.30)$$

for $i = 1, \dots, n$ and $j = 0, \dots, n-1$.

Let us assume that the order n is even. Then $x_i \neq x'_j$ for all i and j. For a given Toeplitz-plus-Hankel matrix A defined by (4.6), we introduce the numbers

$$f_i^{\pm} = \frac{1}{\sqrt{n^2-1}} \sum_{k=0}^{n-1}{}' c_{i,k+1} a_{\pm k}, \qquad f_j'^{\pm} = \frac{1}{\sqrt{n^2-1}} \sum_{k=0}^{n-1}{}' c'_{j,k-1} a_{\pm k},$$

$$g_i^{\pm} = \frac{1}{\sqrt{n^2-1}} \sum_{k=0}^{n-1}{}' c_{i,k+1} b_{n-1\pm k}, \qquad g_j'^{\pm} = \frac{1}{\sqrt{n^2-1}} \sum_{k=0}^{n-1}{}' c'_{j,k-1} b_{n-1\pm k}.$$

From (4.27)–(4.30) we get the following.

THEOREM 19. *Let A be given by (4.6). Then the matrix $\mathscr{S}_n^I A \mathscr{C}_n^I = [\gamma_{ij}]_1^n$ has Cauchy rank $\leqslant 4$, and the entries are given by*

$$\gamma_{ij} = \frac{\alpha_i^{(j)} y_j - y_i \beta_j^{(i)}}{x_i - x_j'},$$

where

$$\alpha_i^{(j)} = -f_i^+ - (-1)^{i+j} f_i^- + (-1)^i g_i^- - (-1)^j g_i^+,$$

$$\beta_j^{(i)} = -(-1)^{i+j} f_j'^+ - f_j'^- + (-1)^j g_j'^- - (-1)^i g_j'^+.$$

THEOREM 20. *Let T be given by (2.3). Then*

$$\Pi^T \mathscr{S}_n^I T \mathscr{C}_n^I \Pi = \begin{bmatrix} C_{\text{even}} & 0 \\ 0 & C_{\text{odd}} \end{bmatrix},$$

where $C_{\text{even}} = [c_{pq}^{\text{even}}]_1^{m_1}$ and $C_{\text{odd}} = [c_{pq}^{\text{odd}}]_1^{m_2}$, with $m_1 = [(n+1)/2]$, $m_2 = [n/2]$, are given by

$$c_{pq}^{\text{even}} = \begin{cases} \dfrac{f_{2p} y_{2q} - y_{2p} f_{2q}}{x_{2p} - x_{2q}}, & p \neq q, \\ h_{2p}, & p = q, \end{cases}$$

$$c_{pq}^{\text{odd}} = \begin{cases} \dfrac{f_{2p-1} y_{2q-1} - y_{2p-1} f_{2q-1}}{x_{2p-1} - x_{2q-1}}, & p \neq q, \\ h_{2p+1}, & p = q. \end{cases}$$

4.7. *Real Modifications of DFT and the Hartley Transformation*

There are some real modifications of the complex DFT which can also be used to transform Toeplitz-plus-Hankel into Cauchy matrices. Among them is the Hartley transformations.

Let c_i $(i = 1, \dots, n)$ denote the nth roots of 1 or -1 ordered in such a way that $c_{2k} = \bar{c}_{2k-1}$ $[0 < k < (n-1)/2]$, and let $\alpha_i \in \mathbf{C}$ be given such that $\alpha_{2k}\alpha_{2k-1}$ is nonreal for all k. We introduce vectors $u_i = (u_{ij})_{j=0}^{n-1}$ by $u_{ij} = \alpha_i c_i^j + \bar{\alpha}_i c_i^j$ and the matrix $\mathscr{R}_n$ by

$$\mathscr{R}_n = [u_{i,j-1}]_1^n.$$

The matrix $\mathscr{R}_n$ is obtained from the DFT $\mathscr{F}_n(1)$ or $\mathscr{F}_n(-1)$ after multiplication from the left by a permutation matrix and a block diagonal matrix with blocks

$$\begin{bmatrix} \alpha_i & \alpha_{i+1} \\ \bar{\alpha}_i & \alpha_{i+1} \end{bmatrix}.$$

Clearly $\mathscr{R}_n$ is nonsingular if $\alpha_{2k}\alpha_{2k-1}$ is nonreal.

We consider two special cases. First we choose $\alpha_{2k-1} = \frac{1}{2}$ and $\alpha_{2k} = \mathrm{i}/2$. Then we obtain the real DFT $\mathscr{F}\mathbf{R}_n$ with entries $\cos(2ij\pi/n)$ and $\sin(2ij\pi/n)$.

Secondly, we choose $\alpha = (1 - \mathrm{i})/2$. In this case we obtain a row permutation of the discrete Hartley transformation, which is, by definition, the matrix-vector multiplication by

$$\mathscr{H}_n = \left[\cos \frac{2ij\pi}{n} + \sin \frac{2ij\pi}{n}\right]_1^n.$$

As can be checked, both the real DFT and the Hartley transformation transform Toeplitz-plus-Hankel matrices into matrices with Cauchy rank $\leqslant 4$. Due to the lack of symmetry properties, these transformations do not appear to offer any advantage for transforming symmetric Toeplitz matrices. Therefore we refrain from presenting the explicit formulas.

4.8. More Transformations

There are more Chebyshev Vandermonde transformations transforming Toeplitz-plus-Hankel matrices into matrices with Cauchy rank $\leqslant 4$ which we did not include in this paper. However most of them do not have the nice symmetry properties of the sine-I and cosine-I transformations.

For example, the cosine-IV and sine-IV transformations

$$\mathscr{C}_n^{\mathrm{IV}} = \sqrt{\frac{2}{n}}\left[\cos \frac{(2i+1)(2j+1)\pi}{4n}\right]_0^{n-1}$$

and

$$\mathscr{S}_n^{\mathrm{IV}} = \sqrt{\frac{2}{n}}\left[\sin \frac{(2i+1)(2j+1)\pi}{4n}\right]_0^{n-1}$$

have similar properties to the cosine-III and sine-III transformations. To get the corresponding formulas one has to take, as in Section 4.5, the polynomials $V_k^{\pm}(\lambda) = U_k(\lambda) - \{\pm U_{k-1}(\lambda)\}$ and to consider them at the points $x_i = \{(2i+1)\pi\}/2n$.

Furthermore, one can consider the vectors $U(\lambda)$ at the roots of $U_n(\lambda) - \eta$ for $\eta = \pm 1$, which are $\cos(2i\pi/n)$ and $\cos\{(2j+1)\pi/n+2\}$ for $\eta = 1$, and $\cos\{2i\pi/(n+2)\}$ and $\cos\{(2i-1)\pi/n\}$ for $\eta = -1$. For general Toeplitz-plus-Hankel matrices it is recommended to combine the cases $\eta = 1$ and $\eta = -1$. Similarly one can consider the vector $U(\lambda)$ at the roots of $U_n(\lambda) \pm U_{n-1}(\lambda)$. In all cases one gets transformations transforming Toeplitz-plus-Hankel matrices into Cauchy rank $\leqslant 4$.

5. DISPLACEMENT APPROACH

We discuss now a different approach to obtaining the transformation results in the previous sections. This approach is based on a quite general but very simple idea. This idea was used in [14] and also in [12]. The approach utilizes the concept of displacement structure.

Let U, V be two fixed matrices. The UV displacement rank of a matrix A is by definition the rank r of $\nabla(A) := AU - VA$. If r is small compared with the order of A, then A is said to possess a UV displacement structure. Assume that U and V admit diagonalizations

$$U = Q_1 D(c) Q_1^{-1}, \qquad V = Q_2 D(d) Q_2^{-1},$$

$D(c) = \operatorname{diag}(c_i)_1^n$, $D(d) = \operatorname{diag}(d_j)_1^n$. Then the following is obvious.

PROPOSITION 21. *If A has UV displacement rank r, then $C = Q_2^{-1} A Q_1$ has Cauchy rank r.*

We present now a survey of the displacement operators corresponding to the transformations discussed in Sections 2–4. In Section 2 we considered the complex DFT transformation $\mathscr{F}_n(\xi)$. The displacement operator corresponding to this transformation is the ξ-cyclic shift operator

$$U(\xi) = \begin{bmatrix} 0 & & & \xi \\ 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix},$$

for which

$$\mathscr{F}_n(\xi) D(c) \mathscr{F}_n^{-1}(\xi) = U(\xi),$$

where c is the n-tuple of the nth roots of ξ, and ξ is chosen in one of the several ways described in Section 2.

In Sections 3 and 4 we considered real trigonometric transformations. The displacement operator U for these transformations has the eigenvectors which are the columns of the transpose of the matrix of trigonometric transformations. The corresponding displacement operators are listed below.

All rows with the possible exception of the first and last two are of the form $[0 \ \cdots \ 0 \ 1 \ 0 \ 1 \ 0 \ \cdots \ 0]$. The entries with differ from the displacement operator for the sine-I transformation are written in boldface:

$$\overset{\text{sin-I}}{\begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & \ddots & & \ddots & & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}} \qquad \overset{\text{cos-I}}{\begin{bmatrix} 0 & 1 & & & & \\ \mathbf{2} & 0 & 1 & & & \\ & \ddots & & \ddots & & \\ & & & 1 & 0 & \mathbf{2} \\ & & & & 1 & 0 \end{bmatrix}}$$

$$\overset{\text{sin-II}}{\begin{bmatrix} \mathbf{-1} & 1 & & & & \\ 1 & 0 & 1 & & & \\ & \ddots & & \ddots & & \\ & & & 1 & 0 & 1 \\ & & & & 1 & \mathbf{-1} \end{bmatrix}} \qquad \overset{\text{cos-II}}{\begin{bmatrix} \mathbf{1} & 1 & & & & \\ 1 & 0 & 1 & & & \\ & \ddots & & \ddots & & \\ & & & 1 & 0 & 1 \\ & & & & 1 & \mathbf{1} \end{bmatrix}}$$

$$\overset{\text{sin-III}}{\begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & \ddots & & \ddots & & \\ & & & \mathbf{2} & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}} \qquad \overset{\text{cos-III}}{\begin{bmatrix} 0 & 1 & & & & \\ \mathbf{2} & 0 & 1 & & & \\ & \ddots & & \ddots & & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}}$$

$$\overset{\text{sin-IV}}{\begin{bmatrix} \mathbf{-1} & 1 & & & & \\ 1 & 0 & 1 & & & \\ & \ddots & & \ddots & & \\ & & & 1 & 0 & 1 \\ & & & & 1 & \mathbf{1} \end{bmatrix}} \qquad \overset{\text{cos-IV}}{\begin{bmatrix} \mathbf{1} & 1 & & & & \\ 1 & 0 & 1 & & & \\ & \ddots & & \ddots & & \\ & & & 1 & 0 & 1 \\ & & & & 1 & \mathbf{-1} \end{bmatrix}}$$

Hartley and Real DFT

$$U = \begin{bmatrix} 0 & 1 & & & & \mathbf{1} \\ 1 & 0 & 1 & & & \\ & \ddots & & \ddots & & \\ & & & 1 & 0 & 1 \\ \mathbf{1} & & & & 1 & 0 \end{bmatrix}.$$

In the last case the transformation is not uniquely determined by the displacement operator U, since U has double eigenvalues.

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