SPECTRAL PROPERTIES OF TOEPLITZ-PLUS-HANKEL MATRICES

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Abstract - We study asymptotic and uniform properties of eigenvalues of a large class of real symmetric matrices that can be decomposed into the sum of a Toeplitz matrix and a Hankel matrix. In particular, we show that their properties are essentially driven by those of the Toeplitz part. A special subclass of structured matrices arising in an approximation problem is analyzed in detail.

1. Introduction

The class of *Toeplitz-plus-Hankel matrices* (T+H matrices, for brevity) is the set of those matrices that can be decomposed into the sum of a Toeplitz matrix and a Hankel matrix. This class is one of the best known classes of matrices with a *displacement structure* [12]. A class of matrices is said to have a displacement structure if there exist suitable matrices A and B such that the linear operator

$$\mathcal{L}(M) = AM - MB,$$

transforms all matrices in that class into matrices whose rank does not depend on the order n. This fact allows to devise fast $O(n^2)$ algorithms for the solution of linear systems with matrices in a displacement structure class. Indeed, all Schur complements involved in the gaussian elimination process inherit the displacement structure of the initial matrix, hence they can be described by a number of parameters that grows only linearly with n. Fast algorithms for T+H systems can be found in [4, 5].

Inverse T+H matrices are completely characterized in [7, 8] by exploiting their relationship with Bézout matrices. Various formulas have been proposed that transform T+H matrices into Cauchy matrices [4]. These formulas involve discrete trigonometric transforms, and their implementation is numerically stable and very efficient.

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Circulant based preconditioners for T+H matrices arising in inverse scattering problems are found in [13], generalizing a preconditioning technique known for Toeplitz matrices. Moreover, a link between a class of tridiagonal matrices arising from finite difference discretizations of 2-point boundary value problems and some T+H matrices is established in [10] by means of a congruence defined by trigonometric transforms.

Spectral properties of Toeplitz matrices and Hankel matrices are active but separate research fields; for example, the asymptotic distribution of eigenvalues of Toeplitz matrices is well understood [6], and the behavior of their extreme eigenvalues has been the subject of an extensive research [11, 17, 19, 20]. Analogous results concerning eigenvalues of Hankel matrices are obtained in [2, 21, 25]. However, little is known about spectral properties of T+H matrices, as positive definiteness, conditioning and asymptotic behavior of their eigenvalues.

Our aim in this paper is to study the asymptotic distribution of the eigenvalues and other spectral properties of symmetric T+H matrices whose coefficients are Fourier coefficients of a real bounded function. Matrices with such structure arise in various problems of approximation theory and signal processing; in fact, the present work was originally motivated by a problem in applied mathematics [3] where symmetric T+H systems arise in the approximate solution of an inverse problem for the Laplace equation. To start with a motivating example, consider the matrix

$$X \equiv (f_{i+j} + f_{i-j}), \qquad i, j = 0 \dots n,$$

where the numbers f_k are cosine-Fourier coefficients of a real even function (see below); then, if $v_0 \dots v_n$ are the cosine-Fourier coefficients of the trigonometric polynomial $v(x)$, the entries of the vector

$$X(v_0/2, v_1 \dots v_n)^t$$

are the first $n+1$ cosine-Fourier coefficients of the function $f(x)v(x)$.

In the next section we recall some definitions and basic results on generating functions and their relationship with eigenvalues of Toeplitz matrices. Much of the subsequent work is devoted to their generalization to T+H matrices. In section 3 we prove some asymptotic results for uniformly bounded sequences of T+H matrices. In particular, we show that the distribution of their eigenvalues is essentially driven by the spectral properties of the Toeplitz part. In section 4 we provide a more detailed analysis of eigenvalues of matrices with structure similar to the matrix X above. An application of the previous results to the problem that motivated originally this work is shown in section 5.

Some results in this paper are in a preliminary form; indeed, the author is aware that a wider class (in particular, some unbounded sequence) of T+H matrices can be considered, by means of the approach described in [22, 23]. This generalization, as well as the treatment of block structured matrices, will be the subject of further work.

2. Generating functions of structured matrices

Let $f(x) \in L^\infty[-\pi,\pi]$ be a real function and let

$$f(x) = \sum_{k=-\infty}^{\infty} f_k e^{\mathbf{i}kx}$$

be its Fourier series, where

$$f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-\mathbf{i}kx}\,dx$$

and $\mathbf{i}$ is the imaginary unit. For any fixed integer n, we associate the following matrices to the sequence f_k of Fourier coefficients of $f(x)$:

$$T_f \equiv (f_{i-j}) \quad H_f \equiv (f_{i+j}) \quad X_f \equiv (f_{i-j} + f_{i+j}), \qquad i,j = 0\ldots n.$$

In general, we omit an explicit mentioning of the order of the matrices, that should be made clear from the context. We say that the function $f(x)$ is the *generating function* of the matrices T_f, H_f and X_f. In what follows, we suppose $f(x) = f(-x)$, i.e., $f(x)$ be even; then it holds $f_k = f_{-k} \in \mathbf{R}$ and

$$\begin{aligned} f(x) &= f_0 + 2\sum_{k=1}^{\infty} f_k \cos(kx) \\ f_k &= \frac{1}{\pi}\int_0^{\pi} f(x)\cos(kx)\,dx. \end{aligned}$$

Hence, all matrices considered in this paper are real and symmetric. Many spectral properties of T_f are understood by considering its generating function f, by virtue of well established results due to Grenander and Szegö [6]:

THEOREM 2.1. If $\lambda_0 \le \lambda_1 \le \ldots \le \lambda_n$ are the eigenvalues of T_f, and

$$m = \inf_{-\pi\le x\le\pi} f(x) \qquad M = \sup_{-\pi\le x\le\pi} f(x),$$

then

$$m \le \lambda_k \le M \qquad k = 0\ldots n,$$

and, for any fixed integer k,

$$\lambda_k \to m \qquad \lambda_{n-k} \to M \qquad \text{as } n \to \infty.$$

Moreover, for every function $g \in C[m,M]$ it holds:

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} g(\lambda_k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(f(x))\,dx.$$

In the subsequent sections, we investigate suitable extensions of the preceding theorem to some T+H matrices. Our basic tool is the *asymptotic distribution* *function* of a triangular sequence of numbers. The following definition is adapted from [6, 14], with minor changes to fit our notations: the sequence of vectors $v^{(n)} = (v_0^{(n)}, v_1^{(n)} \dots v_n^{(n)})^t$, $n = 1, 2 \dots$ where $v_k^{(n)} \in \mathcal{I}$ and $\mathcal{I} \subset \mathbf{R}$ is bounded, is said to have the *asymptotic distribution function* $\phi(x)$, where $\phi : \mathcal{I} \to [0, 1]$ is nondecreasing, if for every function $g \in C(\mathcal{I})$ it holds

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} g(v_k^{(n)}) = \int_{\mathcal{I}} g(x)\, d\phi(x).$$

REMARK 2.1. The asymptotic distribution function of a sequence $v^{(n)}$ is strictly increasing in any interval in which the points $v_k^{(n)}$ are dense; the converse statement is also true.

Now, let $m \le f(x) \le M$; the identity

$$\int_{-\pi}^{\pi} g(f(x))\, dx = \int_{m}^{M} g(x)\, d\phi(x)$$

holds for any continuous function $g \in C[m, M]$ if $\phi(x)$ is defined as follows:

$$\phi(x) = \frac{\mu(\{t \in [-\pi, \pi] : f(t) \le x\})}{2\pi}. \tag{2.1}$$

Here, we denote $\mu(\mathcal{I})$ the Lebesgue measure of any set $\mathcal{I} \subset \mathbf{R}$. Observe that $\phi(x) = 0$ for $x < m$ and $\phi(x) = 1$ for $x \ge M$, i.e., the support of $d\phi$ is contained in $[m, M]$. Due to Theorem 2.1, the eigenvalues of the sequence of Toeplitz matrices T_f have the asymptotic distribution function $\phi(x)$ defined above. In what follows, we denote $\#\mathcal{S}$ the cardinality of the finite set $\mathcal{S}$.

REMARK 2.2. The sequence of vectors $v^{(n)}$, $n = 1, 2 \dots$ has the *Heavyside unit step function* with jump in $\bar{x}$,

$$\chi_{\bar{x}}(x) = \begin{cases} 0 & \text{when } x < \bar{x} \\ 1 & \text{when } x \ge \bar{x}, \end{cases}$$

as asymptotic distribution function if and only if for all $\epsilon > 0$ it holds:

$$\lim_{n\to\infty} \frac{\#\{k : |v_k^{(n)} - \bar{x}| > \epsilon\}}{n+1} = 0.$$

3. Spectral properties of $T_f + H_g$

We adopt the following notation: if A is a symmetric matrix of order $n+1$, its eigenvalues sorted in ascending order are denoted by $\lambda_k^{(n)}(A)$, for $k = 0 \dots n$. Moreover, its extreme eigenvalues are denoted by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$. Our main result in this section will be derived as a consequence of the following general theorem, that is a weaker version of some results in Section 2 of [23] (see also [22]):

THEOREM 3.1. Let A_n, B_n, $n = 1, 2 \ldots$ be sequences of matrices whose order is $n+1$, with

$$\|A_n\|_2 \leq c \qquad \|B_n\|_2 \leq c,$$

for some constant $c > 0$, and

$$\lim_{n\to\infty} \frac{\|A_n - B_n\|_F^2}{n+1} = 0$$

where $\|\cdot\|_F$ denotes the Frobenius matrix norm. Then, the eigenvalues of A_n have the asymptotic distribution function $\phi(x)$ if and only if the same property holds for the eigenvalues of B_n.

It is worth noting that the last part of Theorem 2.1 can be derived as a corollary of the preceding theorem (when $f \in L^\infty$), see e.g. [23], by simply letting A_n be the sequence of Toeplitz matrices T_f, and B_n be their circulant approximations built up by wrapping around or averaging diagonals [1, 9].

The following theorem is an easy consequence of Theorem 3.1:

THEOREM 3.2. The eigenvalues of the sequences of matrices $T_f + H_g$ and T_f, where f, g are bounded, have the same asymptotic distribution function.

PROOF. Theorem 2.1 implies $\|T_f\|_2 \leq \|f\|_\infty$, and the inequality $\|H_g\|_2 \leq \|g\|_\infty$ can be found in [16]. Furthermore, let g_k be the kth Fourier coefficient of $g(x)$. We have:

$$\begin{aligned}
\frac{1}{n+1}\|H_g\|_F^2 &= \sum_{k=0}^{n} \frac{k+1}{n+1} g_k^2 + \sum_{k=n+1}^{2n} \frac{2n-k+1}{n+1} g_k^2 \\
&\leq \sum_{k=0}^{n} \frac{k+1}{n+1} g_k^2 + \sum_{k=n+1}^{\infty} g_k^2 \\
&\leq \max_{i=1,2\ldots} g_i^2 \cdot \sum_{k=0}^{\lfloor \log n \rfloor} \frac{k+1}{n+1} + \sum_{k=\lceil \log n \rceil}^{\infty} g_k^2 \\
&\leq \max_{i=1,2\ldots} g_i^2 \cdot \frac{\log^2 n}{n+1} + \sum_{k=\lceil \log n \rceil}^{\infty} g_k^2.
\end{aligned}$$

By hypothesis $\sum_{k=0}^{\infty} g_k^2$ is finite, hence

$$\lim_{n\to\infty} \frac{1}{n+1}\|H_g\|_F^2 \to 0.$$

Thesis follows from a straightforward application of Theorem 3.1, with $A_n = T_f + H_g$ and $B_n = T_f$. ∎

Letting $T_f = O$ in Theorem 3.2, one sees that the asymptotic distribution function of the eigenvalues of H_g is the step function $\chi_0(x)$, regardless of $g(x)$. In view of Remark 2.2, this fact generalizes some known results on the distribution of eigenvalues of Hankel matrices that represent bounded operators in l^2, see [25]. Indeed, the infinite matrix whose finite sections are H_g is not necessarily a compact operator, under our hypotheses.

Remark 3.1. From Theorem 3.2 and Remark 2.1 it follows that for any $\epsilon > 0$:

$$\lim_{n\to\infty} \frac{\#\{k : \lambda_k^{(n)}(T_f + H_g) \in [m, m+\epsilon]\}}{n+1} = c_1$$

$$\lim_{n\to\infty} \frac{\#\{k : \lambda_k^{(n)}(T_f + H_g) \in [M-\epsilon, M]\}}{n+1} = c_2,$$

for some constants $c_1, c_2 > 0$. Hence, m and M are accumulation points for the eigenvalues of $T_f + H_g$. Moreover,

$$\lim_{n\to\infty} \frac{\#\{k : \lambda_k^{(n)}(T_f + H_g) \in [m-\epsilon, M+\epsilon]\}}{n+1} = 1.$$

Finally, observe that, whenever $\inf f > 0$, any preconditioning technique that is "good" for Toeplitz matrices is also "good" for T+H matrices: indeed, suppose preconditioned Toeplitz matrices PT_f have the Heavyside function $\chi_1(x)$ as asymptotic spectral distribution; this is the case where their eigenvalues are clustered at 1, see [23]. Now, the inequality

$$\|PH_g\|_F \le \|H_g\|_F \|P\|_2$$

allows us to deduce that $\|PH_g\|_F^2$ is sublinear in n: indeed, the norm of the preconditioner P is uniformly bounded because of the hypotesis on f, and $\|H_g\|_F^2$ is sublinear, according to the argument used in Theorem 3.2. Therefore, we can apply Theorem 3.1, with $A_n = PT_f$ and $B_n = P(T_f + H_g)$; we conclude that preconditioned T+H matrices $P(T_f + H_g)$ have the same spectral asymptotic distribution as PT_f, hence their eigenvalues are clustered at 1, regardless of $g \in L^\infty$. This result cannot be extended to the T+H matrices considered in [13], because of the particular structure of their Hankel part, whose norm is not sublinear in n; indeed, in [13] preconditioners for T+H systems are introduced as the sum of two matrices, approximating separately the Toeplitz part and the Hankel part.

4. Spectral properties of X_f

In this section we focus on matrices X_f, and we suppose $m = \inf f(x)$ and $M = \sup f(x)$, where (essential) infimum and supremum are taken with respect to $-\pi \le x \le \pi$.

We associate to any real vector $v = (v_0 \dots v_n)^t$ the trigonometric polynomial:

$$c_v(t) = \sum_{k=0}^{n} v_k \cos(kt).$$

With the above definition we have:

$$\begin{aligned} v^t X_f v &= \sum_{i=0}^{n}\sum_{j=0}^{n} v_i v_j (f_{i+j} + f_{i-j}) \\ &= \frac{1}{2\pi}\sum_{i=0}^{n}\sum_{j=0}^{n} v_i v_j \int_{-\pi}^{\pi} f(x)[\cos((i+j)x) + \cos((i-j)x)]\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{i=0}^{n}\sum_{j=0}^{n} v_i v_j f(x)\cos(ix)\cos(jx)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) c_v(x)^2\,dx. \qquad (4.1) \end{aligned}$$

Hence:

$$\frac{m}{\pi}\int_{-\pi}^{\pi} c_v(x)^2\,dt \le v^t X_f v \le \frac{M}{\pi}\int_{-\pi}^{\pi} c_v(x)^2\,dx$$

with strict inequalities whenever $m < M$. Furthermore,

$$\|v\|_2^2 \le \frac{1}{\pi}\int_{-\pi}^{\pi} c_v(x)^2\,dx = 2v_0^2 + \sum_{k=1}^{n} v_k^2 \le 2\|v\|_2^2. \qquad (4.2)$$

These relations are at the base of the following theorem:

Theorem 4.1. Let $\lambda_0 \le \lambda_1 \le \dots \le \lambda_n$ be the eigenvalues of X_f. If $m \ge 0$ then:

$$m \le \lambda_k \le M$$

for $k = 0 \dots n-1$,

$$\lambda_n \le 2M$$

and, for any fixed integer $k \ge 0$,

$$\lambda_k \to m, \quad \lambda_{n-k-1} \to M$$

as $n \to \infty$. If $m < 0$ then

$$2m \le \lambda_0, \qquad m \le \lambda_k \le M$$

for $k = 1 \dots n-1$,

$$\lambda_n \le 2M,$$

and, for any fixed integer $k \ge 0$,

$$\lambda_{k+1} \to m, \quad \lambda_{n-k-1} \to M$$

as $n \to \infty$. In both cases, for every function $g \in C[m, M]$ it holds:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\lambda_k \in [m,M]} g(\lambda_k) = \int_m^M g(x)\, d\phi(x),$$

where $\phi(x)$ is given by (2.1).

PROOF. Consider first the case $m \geq 0$. The inequalities $m \leq \lambda_k \leq 2M$ follow from relations (4.1) and (4.2), recalling the minimax characterization of eigenvalues of a symmetric matrix, see [18]. Now, let Y be the matrix obtained by deleting the first row and column from X_f:

$$Y = (f_{|i-j|} + f_{i+j}), \quad i, j = 1 \dots n.$$

For any n-vector $v = (v_1 \dots v_n)^t$ let $\tilde{c}_v(t)$ be the trigonometric cosine polynomial

$$\tilde{c}_v(t) = \sum_{k=1}^{n} v_k \cos(kt).$$

Analogously to (4.1) and (4.2), simple computations lead to the following relations:

$$\begin{aligned} v^t Y v &= \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{c}_v(t)^2 f(t)\, dt \\ v^t v &= \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{c}_v(t)^2\, dt. \end{aligned}$$

We have

$$m \leq \frac{v^t Y v}{v^t v} \leq M,$$

hence

(4.3) $m \leq \lambda_{\min}(Y) \leq \lambda_{\max}(Y) \leq M,$

and the latter inequalities are strict if $m < M$. By the Cauchy interlace theorem [18], the eigenvalues of Y separe those of X_f:

(4.4) $\lambda_k^{(n)}(X_f) \leq \lambda_k^{(n-1)}(Y) \leq \lambda_{k+1}^{(n)}(X_f) \qquad k = 0 \dots n-1,$

therefore

$$m \leq \lambda_k \leq M \qquad k = 0 \dots n-1,$$

and there exists exactly one eigenvalue of X_f in the interval $[\lambda_{\max}(Y), 2M]$. Furthermore, the limits of extreme eigenvalues follow from Remark 3.1. This proves the first part of the theorem.

If $m < 0$ then relations (4.1) and (4.2) leads to

$$2m \leq \lambda_k \leq 2M \qquad k = 0 \dots n,$$

while (4.3) and (4.4) imply

$$m \leq \lambda_k \leq M \qquad k = 1 \dots n-1.$$

By Theorem 3.2, the matrix Y have the asymptotic spectral distribution function $f(x)$, hence the last part of this theorem is a consequence of the interlacing property (4.4) for the eigenvalues of X_f and Y. ■

In particular, observe that X_f is positive definite for all n if and only if $f(x) \geq 0$ and $f(x) \not\equiv 0$, since trigonometric polynomials are dense in $C[-\pi, \pi]$.

A sufficient condition for $\lambda_n^{(n)}(X_f)$ to be greater than M is given in the next proposition:

THEOREM 4.2. If the nonlinear equation

$$\int_0^\pi \frac{1}{\mu - f(t)}\, dt = \pi$$

has a solution $\mu > M$, then

$$\lambda_n^{(n)}(X_f) \to \mu$$

as $n \to \infty$.

PROOF. Consider the selfadjoint operator in $L^2[0, \pi]$ defined as:

$$\mathcal{L}_f(v)(x) = f(x)v(x) + \frac{1}{\pi} \int_0^\pi v(t)\, dt,$$

let

$$\mathcal{C}_n = \{v_0 + 2\sum_{k=1}^{n} v_k \cos(kx),\ \ v_i \in \mathbf{R}\}$$

and let P_n be the orthogonal projection of $L^2[0, \pi]$ onto $\mathcal{C}_n$. Observe that X_f is exactly the matrix representation of the operator

$$\mathcal{X}_f : \mathcal{C}_n \mapsto \mathcal{C}_n \qquad \mathcal{X}_f(v) = P_n \mathcal{L}_f(v),$$

under the natural isomorphism between $\mathcal{C}_n$ and $\mathbf{R}^{n+1}$. Moreover, with elementary analysis one proves that μ is the largest eigenvalue of $\mathcal{L}_f$: Indeed, it holds $\mathcal{L}_f g(x) = \mu g(x)$, where

$$g(x) = \frac{1}{\mu - f(x)}.$$

The limit $\mu_n \to \mu$ follows from [24, p. 68], since spaces $\mathcal{C}_n$ are dense in $L^2[0, \pi]$. ■

Finally, we only mention here that Theorem 4.1 and other arguments found in [19, 20] can be used to prove the following proposition:

THEOREM 4.3. If f and g are nonnegative, and

$$r < \frac{f(x)}{g(x)} < R$$

then the eigenvalues of $X_g^{-1} X_f$ lie in the open interval (r, R); moreover,

$$\lambda_0^{(n)}(X_f) - m = O(n^{-k})$$

where k is the maximum order of the global minima of f.

5. An application

The problem that motivated originally the present work is the following: let $a > 0$, $\Omega = (0,1) \times (0,a)$, and $u(x,y)$ be the solution of the following PDE:

$$\begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega \\ u_x(0,y) = u_x(1,y) & = & 0 & y \in [0,a] \\ u_y(x,0) & = & -\Phi(x) & x \in [0,1] \\ u_y(x,a) + \gamma(x)u(x,a) & = & 0 & x \in [0,1] \end{array}$$

under suitable regularity conditions on $\gamma(x) \geq 0$ and $\Phi(x)$. Recovering $\gamma(x)$ from the knowledge of $\Phi(x)$ and of $u(x,0)$ is a nonlinear inverse problem of interest in the field of nondestructive corrosion detection. Indeed, this problem has been introduced in [15] as a simplified model of the electrostatics of a conductor Ω when its boundary $y = a$ is affected by corrosion.

A convergent Galerkin method for approximating $\gamma(x)$ from data $\Phi(x)$ and $u(x,0)$ is proposed and tested in [3]. The method reduces to the solution of a linear system whose matrix has T+H structure: If we let u_k, v_k, ϕ_k and γ_k be the cosine-Fourier coefficients of $u(\pi x, a), u_y(\pi x, a), \Phi(\pi x)$ and $\gamma(\pi x)$ respectively, then the numbers $\gamma_0 \ldots \gamma_n$ can be approximated by the solution of the following linear system:

$$C_n \begin{pmatrix} \tilde{\gamma}_0/2 \\ \tilde{\gamma}_1 \\ \vdots \\ \tilde{\gamma}_n \end{pmatrix} = - \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$$

where $C_n \equiv (u_{i+j} + u_{|i-j|}) = X_f$, with $f(x) = u(x,a)$. Hence, the analysis carried out in the preceding section can be used to deduce numerical stability of that method and to predict its behavior when data are noisy. In particular, one observe that the matrix C_n is positive definite, its spectral condition number does not depend on the order n,

$$\|C_n\|_2 \|C_n^{-1}\|_2 \leq 2 \frac{\max u(x,a)}{\min u(x,a)}$$

and can be estimated from the problem data, by means of suitable apriori estimates on $u(x,a)$, see [3].

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