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Divisibility by 2 of Stirling numbers of the second kind and their differences

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ABSTRACT

Let n, k, a and c be positive integers and b be a nonnegative integer. Let $\nu_2(k)$ and $s_2(k)$ be the 2-adic valuation of k and the sum of binary digits of k, respectively. Let $S(n,k)$ be the Stirling number of the second kind. It is shown that $\nu_2(S(c2^n, b2^{n+1}+a)) \geqslant s_2(a)-1$, where $0<a<2^{n+1}$ and $2 \nmid c$. Furthermore, one gets that $\nu_2(S(c2^n,(c-1)2^n+a)) = s_2(a)-1$, where $n \geqslant 2$, $1 \leqslant a \leqslant 2^n$ and $2 \nmid c$. Finally, it is proved that if $3 \leqslant k \leqslant 2^n$ and k is not a power of 2 minus 1, then $\nu_2(S(a2^n,k)-S(b2^n,k)) = n+\nu_2(a-b)-\lceil \log_2 k \rceil + s_2(k)+\delta(k)$, where $\delta(4)=2$, $\delta(k)=1$ if $k>4$ is a power of 2, and $\delta(k)=0$ otherwise. This confirms a conjecture of Lengyel raised in 2009 except when k is a power of 2 minus 1.

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1. Introduction and the statements of main results

The Stirling number of the second kind $S(n,k)$ is defined for $n \in \mathbb{N}$ and positive integer $k \leqslant n$ as the number of ways to partition a set of n elements into exactly k non-empty subsets. It satisfies the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k),$$

with initial condition $S(0,0) = 1$ and $S(n,0) = 0$ for $n > 0$. There is also an explicit formula in terms of binomial coefficients given by

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n. \tag{1}$$

Divisibility properties of Stirling numbers have been studied from a number of different perspectives. It is known that for each fixed k, the sequence $\{S(n,k),\ n \geqslant k\}$ is periodic modulo prime powers. The length of this period has been studied by Carlitz [5] and Kwong [16]. Chan and Manna [6] characterized $S(n,k)$ modulo prime powers in terms of binomial coefficients. In fact, they gave explicit formulas for $S(n,k)$ modulo 4, then for $S(n,a2^m)$ modulo 2^m, where $m \geqslant 3$, $a > 0$ and $n \geqslant a2^m + 1$, and finally for $S(n,ap^m)$ modulo p^m with p being an odd prime.

Divisibility properties of integer sequences are often expressed in terms of p-adic valuations. Given a prime p and a positive integer m, there exist unique integers a and n, with $p \nmid a$ and $n \geqslant 0$, such that $m = ap^n$. The number n is called the *p-adic valuation* of m, denoted by $n = \nu_p(m)$. The numbers $\min\{\nu_p(k!S(n,k))\colon\ m \leqslant k \leqslant n\}$ are important in algebraic topology, see, for example, [3,8,10–12,20,21]. Some work on evaluating $\nu_p(k!S(n,k))$ has appeared in above papers as well as in [7,9,24]. Amdeberhan, Manna and Moll [2] investigated the 2-adic valuations of Stirling numbers of the second kind and computed $\nu_2(S(n,k))$ for $k \leqslant 4$. They also raised an interesting conjecture on the congruence classes of $S(n,k)$, modulo powers of 2. Recently, Bennett and Mosteig [4] used computational methods to justify this conjecture if $k \leqslant 20$. But this conjecture is still kept open if $k \geqslant 21$.

This paper deals with the 2-adic valuations of the Stirling numbers of the second kind. Lengyel [17] studied the 2-adic valuations of $S(n,k)$ and conjectured, proved by Wannemacker [23], $\nu_2(S(2^n,k)) = s_2(k) - 1$, where $s_2(k)$ means the base 2 digital sum of k. Using Wannemacker's result, Hong, Zhao and Zhao [13] proved that $\nu_2(S(2^n+1, k+1)) = s_2(k) - 1$, which confirmed another conjecture of Amdeberhan, Manna and Moll [2]. Lengyel [18,19] showed that if $1 \leqslant k \leqslant 2^n$, then $\nu_2(S(c2^n,k)) = s_2(k) - 1$ for any positive integer c. Meanwhile, Lengyel [18] proved that $\nu_2(S(c2^n,k)) \geqslant s_2(k) - 1$ if $c \geqslant 1$ is an odd integer and $1 \leqslant k \leqslant 2^{n+1}$. Actually, a more general result is true. That is, one has

Theorem 1.1. *Let $n, a, b, c \in \mathbb{N}$ with $0 < a < 2^{n+1}$, $b2^{n+1} + a \leqslant c2^n$ and $c \geqslant 1$ being odd. Then*

$$\nu_2\big(S\big(c2^n, b2^{n+1} + a\big)\big) \geqslant s_2(a) - 1.$$

If one picks $b = \frac{c-1}{2}$ and $1 \leqslant a \leqslant 2^n$, then the lower bound in Theorem 1.1 is arrived as the following result shows.

Theorem 1.2. *Let $a, c, n \in \mathbb{N}$ with $c \geqslant 1$ being odd, $n \geqslant 2$ and $1 \leqslant a \leqslant 2^n$. Then*

$$\nu_2\big(S\big(c2^n, (c-1)2^n + a\big)\big) = s_2(a) - 1.$$

Another interesting property is related to the difference of Stirling numbers of the second kind. Lengyel [18] studied the 2-adic valuations of the difference $S(c2^{n+1}, k) - S(c2^n, k)$ with $1 \leqslant k \leqslant 2^n$ and $c \geqslant 1$ odd. In the meantime, Lengyel posed the following conjecture.

Conjecture 1.1. *(See [18].) Let $n, k, a, b \in \mathbb{N}$, $c \geqslant 1$ being odd and $3 \leqslant k \leqslant 2^n$. Then*

$$\nu_2\big(S\big(c2^{n+1}, k\big) - S\big(c2^n, k\big)\big) = n + 1 - f(k) \tag{2}$$

and

$$\nu_2\big(S\big(a2^n, k\big) - S\big(b2^n, k\big)\big) = n + 1 + \nu_2(a - b) - f(k)$$

for some function $f(k)$ which is independent of n.

As usual, for any real number x, let $\lceil x\rceil$ and $\lfloor x\rfloor$ denote the smallest integer no less than x and the biggest integer no more than x, respectively. Note that Lengyel [18] proved that (2) is true for any integer k with $s_2(k) \leqslant 2$. Lengyel [18] also noticed that for small values of k, numerical experimentation suggests that $f(k) = 1 + \lceil\log_2 k\rceil - s_2(k) - \delta(k)$, where $\delta(4) = 2$ and otherwise it is zero except if k is a power of two or one less, in which cases $\delta(k) = 1$. The present paper focuses on investigating Conjecture 1.1. One has the following result.

Theorem 1.3. *Let $n, k, a, b \in \mathbb{N}$, $c \geqslant 1$ being odd, $3 \leqslant k \leqslant 2^n$, and $a > b$. If k is not a power of 2 minus 1, then*

$$\nu_2\big(S\big(a2^n, k\big) - S\big(b2^n, k\big)\big) = n + \nu_2(a - b) - \lceil\log_2 k\rceil + s_2(k) + \delta(k), \tag{3}$$

where $\delta(4) = 2$, $\delta(k) = 1$ if $k > 4$ is a power of 2, and $\delta(k) = 0$ otherwise. In particular,

$$\nu_2\big(S\big(c2^{n+1}, k\big) - S\big(c2^n, k\big)\big) = n - \lceil\log_2 k\rceil + s_2(k) + \delta(k). \tag{4}$$

By Theorem 1.3, one knows that Conjecture 1.1 holds except when k is a power of 2 minus 1.

In order to prove Theorem 1.3, one needs a special case of the 2-adic valuation of $S(n,k)$, which can be stated as follows.

Theorem 1.4. *Let $a,b,c,m,n \in \mathbb{Z}^+$, $1 \leqslant a < 2^{n+1}$, $m \geqslant n+2+\lfloor \log_2 b \rfloor$ and $c \geqslant 1$ being odd. Then*

$$\nu_2\big(S\big(c2^m + b2^{n+1} + 2^n, b2^{n+2} + a\big)\big) \begin{cases} = n, & \text{if } a = 2^{n+1}-1, \\ \geqslant s_2(a), & \text{if } a < 2^{n+1}-1. \end{cases}$$

This paper is organized as follows. Some preliminary results are presented in Section 2. Then the proofs of Theorems 1.1 and 1.2 are given in Section 3. Consequently, Section 4 is devoted to the proof of Theorem 1.4. Finally, in Section 5, one uses Theorems 1.1 and 1.4 to show Theorem 1.3.

2. Lemmas

Several well-known results, which are needed for the proofs of the main results, are given in this section.

Lemma 2.1 *(Legendre). (See [22].) Let $n \in \mathbb{N}$. Then $\nu_2(n!) = n - s_2(k)$.*

Lemma 2.2 *(Kummer). (See [15].) Let k and $n \in \mathbb{N}$ be such that $k \leqslant n$. Then $\nu_2(\binom{n}{k}) = s_2(k) + s_2(n-k) - s_2(n)$. Moreover, $s_2(k) + s_2(n-k) \geqslant s_2(n)$.*

Lemma 2.3. *(See [18].) Let $k,n,c \in \mathbb{N}$ and $1 \leqslant k \leqslant 2^n$. Then $\nu_2(S(c2^n,k)) = s_2(k) - 1$.*

Lemma 2.4. *(See [18].) Let $k,n,c \in \mathbb{N}$, $2^n < k < 2^{n+1}-1$ and $c \geqslant 3$ be an odd integer. Then $\nu_2(S(c2^n,k)) \geqslant s_2(k)$ and $\nu_2(S(c2^n, 2^{n+1}-1)) = n$.*

Lemma 2.5. *(See [18].) Let $m,n,c \in \mathbb{N}$ and $0 \leqslant m < n$. Then $\nu_2(S(c2^n + 2^m, 2^n)) = n-1-m$.*

Lemma 2.6. *(See [23].) Let $k,n,m \in \mathbb{N}$ and $0 \leqslant k \leqslant n+m$. Then*

$$S(n+m,k) = \sum_{j=1}^{k} \sum_{i=0}^{j} \binom{j}{i} \frac{(k-i)!}{(k-j)!} S(n,k-i) S(m,j).$$

Lemma 2.7. *(See [1].) For $r \geqslant \max(k_1,k_2)+2$, one has*

$$\frac{k_1!k_2!(r-1)!}{(k_1+k_2+1)!} S(k_1+k_2+2,r) = \sum_{i=1}^{r-1} (i-1)!(r-i-1)! S(k_1+1,i) S(k_2+1,r-i).$$

Lemma 2.8. *(See [14].) Let $m, n, v \in \mathbb{N}$, $v \geqslant 1$ and p be a prime number. Then*

$$B_{m+np^v}(x) \equiv \sum_{j=0}^{n} \binom{n}{j} \left(x^p + x^{p^2} + \cdots + x^{p^v}\right)^{n-j} B_{m+j}(x) \mod \frac{np}{2}\mathbb{Z}_p[x], \tag{5}$$

where the Bell polynomials are defined by

$$B_n(x) = \sum_{k=0}^{n} S(n,k)x^k, \quad n \geqslant 0. \tag{6}$$

Let $n = \sum_{\lambda=0}^{\infty} \varepsilon_\lambda(n)2^\lambda$ with $\varepsilon_\lambda(n) \in \{0,1\}$. Then $s_2(n) = \sum_{\lambda=0}^{\infty} \varepsilon_\lambda(n)$. Further, one has the following result.

Lemma 2.9. *Let m and $n \in \mathbb{N}$. Then $s_2(m+n) = s_2(m) + s_2(n)$ if and only if $\varepsilon_\lambda(m) + \varepsilon_\lambda(n) = \varepsilon_\lambda(m+n)$ for all $\lambda \in \mathbb{N}$.*

Proof. This lemma follows immediately from the proof of Lemma 1 in [23]. □

Lemma 2.10. *Let $n, a \in \mathbb{N}$ and $1 \leqslant a < 2^{n+1}$. Define the set J of positive integers by $J := \{1 \leqslant j \leqslant 2^n \mid s_2(2^{n+1} + a - j) + s_2(j) = s_2(2^{n+1} + a)\}$. Then $|J| = 2^{s_2(a)} - 1$ if $1 \leqslant a \leqslant 2^n$, and $|J| = 2^{s_2(a)-1}$ if $2^n < a < 2^{n+1}$.*

Proof. For any positive integer d, define $M_d := \{\lambda \in \mathbb{N} \mid \varepsilon_\lambda(d) = 1\}$. Then $d = \sum_{\lambda \in M_d} 2^\lambda$ and $s_2(d) = |M_d|$. By Lemma 2.9 one knows that $s_2(2^{n+1} + a - j) + s_2(j) = s_2(2^{n+1} + a)$ if and only if

$$\varepsilon_\lambda(j) + \varepsilon_\lambda\left(2^{n+1} + a - j\right) = \varepsilon_\lambda\left(2^{n+1} + a\right) \tag{7}$$

for all $\lambda \in \mathbb{N}$. Therefore by (7), one has that for any given $\lambda \in \mathbb{N}$, $\varepsilon_\lambda(j) = 0$ or 1 if $\varepsilon_\lambda(2^{n+1} + a) = 1$, and $\varepsilon_\lambda(j) = 0$ if $\varepsilon_\lambda(2^{n+1} + a) = 0$. It then follows that for any given integer $1 \leqslant a \leqslant 2^n$, $j \in J$ if and only if $M_j \subseteq M_a$ and $M_j \neq \varnothing$. So $|J| = 2^{|M_a|} - 1 = 2^{s_2(a)} - 1$ if $1 \leqslant a \leqslant 2^n$.

Now let $2^n < a < 2^{n+1}$. So if $j = 2^n$, then one can check that $s_2(2^{n+1} + a - 2^n) + s_2(2^n) = s_2(2^{n+1} + a)$. This implies that $2^n \in J$. On the other hand, since $1 < a - 2^n < 2^n$, one gets that $j \in J \backslash \{2^n\}$ if and only if $M_j \subseteq M_{a-2^n}$ and $M_j \neq \varnothing$. Hence $|J| = 2^{|M_{a-2^n}|} = 2^{s_2(a)-1}$ if $2^n < a < 2^{n+1}$. The proof of Lemma 2.10 is complete. □

Lemma 2.11. *Let $n, a, c \in \mathbb{N}$ with $c \geqslant 1$ being odd and $1 \leqslant a \leqslant 2^n$. Then*

$$s_2\left(c2^n - a\right) = s_2(c) + n - \nu_2(a) - s_2(a). \tag{8}$$

Proof. If $a = 2^n$, then it is easy to check that (8) is true. Now let $1 \leqslant a < 2^n$. One can write $a = \sum_{\lambda=\nu_2(a)}^{n-1} \varepsilon_\lambda(a)2^\lambda$. Clearly $s_2(a) = \sum_{\lambda=\nu_2(a)}^{n-1} \varepsilon_\lambda(a)$ and $\varepsilon_{\nu_2(a)}(a) = 1$. Then

$$\begin{aligned} c2^n - a &= (c-1)2^n + 2^n - a \\ &= (c-1)2^n + \left(2^{\nu_2(a)} + \sum_{\lambda=\nu_2(a)}^{n-1} 2^\lambda\right) - \sum_{\lambda=\nu_2(a)}^{n-1} \varepsilon_\lambda(a)2^\lambda \\ &= (c-1)2^n + \sum_{\lambda=\nu_2(a)}^{n-1} \bigl(1-\varepsilon_\lambda(a)\bigr)2^\lambda + 2^{\nu_2(a)}. \end{aligned} \tag{9}$$

Since $s_2(c-1) = s_2(c) - 1$, by (9) one has

$$\begin{aligned} s_2\bigl(c2^n - a\bigr) &= s_2(c-1) + \sum_{\lambda=\nu_2(a)}^{n-1} \bigl(1-\varepsilon_\lambda(a)\bigr) + 1 \\ &= s_2(c) + \sum_{\lambda=\nu_2(a)}^{n-1} \bigl(1-\varepsilon_\lambda(a)\bigr) \\ &= s_2(c) + n - \nu_2(a) - s_2(a) \end{aligned}$$

as required. This completes the proof of Lemma 2.11. □

Lemma 2.12. *(See [13].) Let $N \geqslant 2$ be an integer and r, t be odd numbers. For any $m \in \mathbb{Z}^+$, one has $\nu_2((r2^N - 1)^{t2^m} - 1) = m + N$.*

3. Proofs of Theorems 1.1 and 1.2

In this section, one uses induction and Lemmas 2.1 to 2.4 and 2.6 as well as 2.7 to show Theorems 1.1 and 1.2. One begins with the proof of Theorem 1.1.

Proof of Theorem 1.1. If $b = 0$, then Theorem 1.1 is true by Lemmas 2.3 and 2.4. In what follows one lets $b \geqslant 1$. There exists a unique integer $e \geqslant 0$ such that $2^e \leqslant b < 2^{e+1}$. One shows Theorem 1.1 using induction on e. First one treats the case $e = 0$, i.e., $b = 1$. Using Lemma 2.6 with n, m and k replaced by $(c-1)2^n$, 2^n and $2^{n+1} + a$, respectively, one has

$$S\bigl(c2^n, 2^{n+1} + a\bigr) = \sum_{j=1}^{2^{n+1}+a} \sum_{i=0}^{j} f(i,j) = \sum_{j=1}^{2^n} \sum_{i=0}^{2^n} f(i,j), \tag{10}$$

where

$$f(i,j) := \binom{j}{i} \frac{(2^{n+1}+a-i)!}{(2^{n+1}+a-j)!} S\bigl((c-1)2^n, 2^{n+1}+a-i\bigr) S\bigl(2^n, j\bigr).$$

Since c is an odd integer, $\nu_2((c-1)2^n) \geqslant n+1$. It then follows from Lemmas 2.1, 2.3 and 2.4 that

$$\begin{aligned}\nu_2\big(f(i,j)\big) &\geqslant \nu_2\left(\frac{(2^{n+1}+a-i)!}{(2^{n+1}+a-j)!}\right)+\nu_2\big(S\big((c-1)2^n,2^{n+1}+a-i\big)\big)+\nu_2\big(S\big(2^n,j\big)\big)\\ &\geqslant \nu_2\big(\big(2^{n+1}+a-i\big)!\big)-\nu_2\big(\big(2^{n+1}+a-j\big)!\big)\\ &\quad+s_2\big(2^{n+1}+a-i\big)-1+s_2(j)-1\\ &=(j-i)+s_2\big(2^{n+1}+a-j\big)-s_2\big(2^{n+1}+a-i\big)\\ &\quad+s_2\big(2^{n+1}+a-i\big)+s_2(j)-2\\ &\geqslant s_2\big(2^{n+1}+a-j\big)+s_2(j)-2 \qquad (11)\end{aligned}$$

since $j\geqslant i$. By Lemma 2.2 one knows that

$$s_2(j)+s_2\big(2^{n+1}+a-j\big)\geqslant s_2\big(2^{n+1}+a\big).$$

So by (11) and noting that $0<a<2^{n+1}$, one obtains

$$\nu_2\big(f(i,j)\big)\geqslant s_2\big(2^{n+1}+a\big)-2=s_2(a)-1. \qquad (12)$$

It then follows from (10) and (12) that

$$\nu_2\big(S\big(c2^n,2^{n+1}+a\big)\big)\geqslant \min_{0\leqslant i\leqslant j\leqslant 2^n}\big\{\nu_2\big(f(i,j)\big)\big\}\geqslant s_2(a)-1.$$

Hence Theorem 1.1 is true if $e=0$. In what follows, one lets $e\geqslant 1$.

Assume that Theorem 1.1 is true for the case t with $t\leqslant e-1$. Then $\nu_2(S(c2^n, b2^{n+1}+a))\geqslant s_2(a)-1$ for any integers b with $0\leqslant b<2^e$. In the following one proves that Theorem 1.1 is true for the case e. This is equivalent to showing that Theorem 1.1 is true for all integers $b\in[2^e,2^{e+1})$, which will be done in what follows.

Let $b\in[2^e,2^{e+1})$ be any given integer. Since $c2^n\geqslant b2^{n+1}+a$, there exist two positive integers c_1 and c_2 such that $c=c_1+c_2 2^{\nu_2(b)+1}$ and $c_1<2^{\nu_2(b)+1}$. So by Lemma 2.6

$$S\big(c2^n,b2^{n+1}+a\big)=\sum_{j=1}^{c_1 2^n}\sum_{i=0}^{j}g(i,j), \qquad (13)$$

where

$$g(i,j):=\binom{j}{i}\frac{(b2^{n+1}+a-i)!}{(b2^{n+1}+a-j)!}S\big(c_2 2^{n+\nu_2(b)+1},b2^{n+1}+a-i\big)S\big(c_1 2^n,j\big).$$

Claim 1. *One has*

$$\nu_2\big(S\big(c_2 2^{n+\nu_2(b)+1},b2^{n+1}+a-i\big)\big)\geqslant s_2\big(b2^{n+1}+a-i\big)-s_2(b). \qquad (14)$$

Let's now prove Claim 1. If $\nu_2(c_2)+\nu_2(b)\geqslant e$, then $b2^{n+1}+a-i<2^{e+n+2}\leqslant 2^{\nu_2(b)+\nu_2(c_2)+n+2}$ since $a<2^{n+1}$ and $2^e\leqslant b<2^{e+1}$. So by Lemmas 2.3 and 2.4, one obtains that

$$\begin{aligned}\nu_2\big(S\big(c_2 2^{n+\nu_2(b)+1}, b2^{n+1}+a-i\big)\big) &= \nu_2\left(S\left(\frac{c_2}{2^{\nu_2(c_2)}}2^{n+\nu_2(b)+\nu_2(c_2)+1}, b2^{n+1}+a-i\right)\right)\\ &\geqslant s_2\big(b2^{n+1}+a-i\big)-1\\ &\geqslant s_2\big(b2^{n+1}+a-i\big)-s_2(b)\end{aligned}$$

as desired. So Claim 1 is proved in this case.

If $\nu_2(c_2)+\nu_2(b)\leqslant e-1$, then one can write $b=b_1 2^{\nu_2(c_2)+\nu_2(b)+1}+b_2$ for some integers $0<b_1<2^{e-\nu_2(c_2)-\nu_2(b)}$ and $2^{\nu_2(b)}\leqslant b_2<2^{\nu_2(c_2)+\nu_2(b)+1}$ since $2^e\leqslant b<2^{e+1}$. One can deduce that $s_2(b2^{n+1}+a-i)=s_2(b_2 2^{n+1}+a-i)+s_2(b_1)$. It then follows from the inductive hypothesis that

$$\begin{aligned}&\nu_2\big(S\big(c_2 2^{n+\nu_2(b)+1}, b2^{n+1}+a-i\big)\big)\\ &\quad=\nu_2\left(S\left(\frac{c_2}{2^{\nu_2(c_2)}}2^{n+\nu_2(b)+\nu_2(c_2)+1}, b_1 2^{n+\nu_2(b)+\nu_2(c_2)+2}+b_2 2^{n+1}+a-i\right)\right)\\ &\quad\geqslant s_2\big(b_2 2^{n+1}+a-i\big)-1\\ &\quad= s_2\big(b2^{n+1}+a-i\big)-s_2(b_1)-1\\ &\quad\geqslant s_2\big(b2^{n+1}+a-i\big)-s_2(b)\end{aligned}$$

as required. So Claim 1 is true for this case. This concludes the proof of Claim 1.

Claim 2. *For all the integers i and j such that $0\leqslant i\leqslant j\leqslant c_1 2^n$ with $c_1<2^{\nu_2(b)+1}$, one has*

$$\nu_2\big(g(i,j)\big)\geqslant s_2(a)-1. \tag{15}$$

Suppose that Claim 2 is true. Then from (13) and Claim 2, one deduces that

$$\nu_2\big(S\big(c2^n, b2^{n+1}+a\big)\big)\geqslant \min_{0\leqslant i\leqslant j\leqslant c_1 2^n}\big\{\nu_2\big(g(i,j)\big)\big\}\geqslant s_2(a)-1.$$

In other words, Theorem 1.1 holds if $b\in[2^e,2^{e+1})$. To finish the proof of Theorem 1.1, it remains to show that Claim 2 is true which will be done in the following.

If $1\leqslant j<2^{n+1}$, then by Lemmas 2.3 and 2.4 one has $\nu_2(S(c_1 2^n,j))\geqslant s_2(j)-1$. Thus using Lemmas 2.1–2.2 and the Claim 1, one derives from $a<2^{n+1}$ that

$$\begin{aligned}\nu_2\big(g(i,j)\big)&\geqslant \nu_2\left(\frac{(b2^{n+1}+a-i)!}{(b2^{n+1}+a-j)!}\right)+s_2\big(b2^{n+1}+a-i\big)-s_2(b)+s_2(j)-1\\ &\geqslant s_2\big(b2^{n+1}+a-j\big)-s_2\big(b2^{n+1}+a-i\big)\\ &\qquad+s_2\big(b2^{n+1}+a-i\big)-s_2(b)+s_2(j)-1\end{aligned}$$

$$\begin{aligned}
&\geqslant s_2\big(b2^{n+1}+a-j\big)+s_2(j)-s_2(b)-1\\
&\geqslant s_2\big(b2^{n+1}+a\big)-s_2(b)-1\\
&= s_2(a)-1
\end{aligned}$$

as required. Hence Claim 2 is true in this case.

If $2^{n+1}\leqslant j\leqslant c_1 2^n$, then one may let $j=j_1 2^{n+1}+j_2$ for some integers $0\leqslant j_2<2^{n+1}$ and $j_1<2^{\nu_2(b)}$ since $c_1<2^{\nu_2(b)+1}$. If $j_2=0$, i.e., $j=j_1 2^{n+1}$, then by (14) and Lemmas 2.1–2.2, noting that $a<2^{n+1}$, one yields

$$\begin{aligned}
\nu_2\big(g(i,j)\big) &\geqslant \nu_2\left(\frac{(b2^{n+1}+a-i)!}{(b2^{n+1}+a-j)!}\right)+\nu_2\big(S\big(c_2 2^{n+\nu_2(b)+1}, b2^{n+1}+a-i\big)\big)\\
&\geqslant s_2\big(b2^{n+1}+a-j\big)-s_2\big(b2^{n+1}+a-i\big)+s_2\big(b2^{n+1}+a-i\big)-s_2(b)\\
&= s_2\big((b-j_1)2^{n+1}+a\big)-s_2(b)\\
&= s_2(b-j_1)+s_2(a)-s_2(b)\\
&\geqslant s_2(a)
\end{aligned}$$

since $j_1<2^{\nu_2(b)}$ implying that $s_2(b-j_1)\geqslant s_2(b)$. Hence (15) is true if $j_2=0$. Now let $j_2\geqslant 1$. Since $j_1<2^{\nu_2(b)}\leqslant 2^e$, by the inductive hypothesis one has

$$\nu_2\big(S\big(c_1 2^n, j\big)\big)=\nu_2\big(S\big(c_1 2^n, j_1 2^{n+1}+j_2\big)\big)\geqslant s_2(j_2)-1. \tag{16}$$

Thus by Lemmas 2.1–2.2, (14) and (16) one obtains

$$\begin{aligned}
\nu_2\big(g(i,j)\big) &\geqslant \nu_2\left(\frac{(b2^{n+1}+a-i)!}{(b2^{n+1}+a-j)!}\right)+\nu_2\big(S\big(c_2 2^{n+\nu_2(b)+1}, b2^{n+1}+a-i\big)\big)+S\big(c_1 2^n, j\big)\\
&\geqslant s_2\big(b2^{n+1}+a-j\big)-s_2\big(b2^{n+1}+a-i\big)\\
&\quad+s_2\big(b2^{n+1}+a-i\big)-s_2(b)+s_2(j_2)-1\\
&= s_2\big(b2^{n+1}+a-j\big)+s_2(j_2)-s_2(b)-1\\
&= s_2\big((b-j_1)2^{n+1}+a-j_2\big)+s_2(j_2)-s_2(b)-1\\
&\geqslant s_2\big((b-j_1)2^{n+1}+a\big)-s_2(b)-1\\
&= s_2(b-j_1)+s_2(a)-s_2(b)-1\\
&\geqslant s_2(a)-1
\end{aligned}$$

since $s_2(b-j_1)\geqslant s_2(b)$. Hence Claim 2 holds if $j_2\geqslant 1$. So Claim 2 is proved.

This completes the proof of Theorem 1.1. □

Consequently, one turns attention to the proof of Theorem 1.2.

Proof of Theorem 1.2. If $a = 2^n$, then by definition of Stirling numbers of the second kind, one has

$$S\big(c2^n, (c-1)2^n + a\big) = S\big(c2^n, c2^n\big) = 1.$$

This implies that $\nu_2(S(c2^n, c2^n)) = s_2(2^n) - 1$. So Theorem 1.2 is true in this case.

Now let $1 \leqslant a < 2^n$ and $b = \frac{c-1}{2}$. Then

$$S\big(c2^n, (c-1)2^n + a\big) = S\big(b2^{n+1} + 2^n, b2^{n+1} + a\big).$$

To prove Theorem 1.2, it is sufficient to show that

$$\nu_2\big(S\big(b2^{n+1} + 2^n, b2^{n+1} + a\big)\big) = s_2(a) - 1. \tag{17}$$

For $t \in \mathbb{N}$, define

$$A_t := \big\{b \in \mathbb{N} \bigm| s_2(b) = t\big\}. \tag{18}$$

Then $\mathbb{N} = \bigcup_{t=0}^{\infty} A_t$. The proof is proceeded with induction on t. First one considers the case $t = 0$. If $b \in A_0$, then $b = 0$. By Lemma 2.3 one has

$$\nu_2\big(S\big(b2^{n+1} + 2^n, b2^{n+1} + a\big)\big) = \nu_2\big(S\big(2^n, a\big)\big) = s_2(a) - 1.$$

So Theorem 1.2 holds if $t = 0$.

In the following let $t \geqslant 1$. Assume that Theorem 1.2 is true for the case r with $r \leqslant t-1$. Then (17) holds for any positive integers $b \in A_0 \cup A_1 \cup \cdots \cup A_{t-1}$. One will prove that Theorem 1.2 is true for the case t, which is equivalent to showing (17) for all the integers $b \in A_t$.

Let $b \in A_t$ be a given integer. One first notices that

$$b2^{n+1} + a \geqslant \max\big(b2^{n+1} - 1, 2^n - 1\big) + 2.$$

Letting $k_1 = b2^{n+1} - 1$, $k_2 = 2^n - 1$ and $r = b2^{n+1} + a$ in Lemma 2.7 gives us that

$$\begin{aligned}
&\frac{(b2^{n+1} - 1)!(2^n - 1)!}{(b2^{n+1} + 2^n - 1)!}\big(b2^{n+1} + a - 1\big)!S\big(b2^{n+1} + 2^n, b2^{n+1} + a\big) \\
&\quad = \sum_{i=1}^{b2^{n+1}+a-1} (i-1)!\big(b2^{n+1} + a - i - 1\big)!S\big(2^n, i\big)S\big(b2^{n+1}, b2^{n+1} + a - i\big) \\
&\quad = \sum_{i=a}^{2^n} \frac{1}{i(b2^{n+1} + a - i)} i!S\big(2^n, i\big)\big(b2^{n+1} + a - i\big)!S\big(b2^{n+1}, b2^{n+1} + a - i\big).
\end{aligned}$$

It follows that

$$\big(b2^{n+1}+a\big)!S\big(b2^{n+1}+2^n, b2^{n+1}+a\big) = \frac{(b2^{n+1}+2^n-1)!}{(b2^{n+1}-1)!(2^n-1)!}\sum_{i=a}^{2^n} l(i), \tag{19}$$

where

$$l(i) := \frac{b2^{n+1}+a}{i(b2^{n+1}+a-i)} i!S\big(2^n, i\big)\big(b2^{n+1}+a-i\big)!S\big(b2^{n+1}, b2^{n+1}+a-i\big).$$

Write $b=(2b_0+1)2^{\nu_2(b)}$ for some $b_0\in\mathbb{N}$. Clearly $s_2(b_0)=s_2(b)-1=t-1$ since $b\in A_t$. Then $b_0\in A_{t-1}$. It then follows from Lemma 2.1 that

$$\begin{aligned}\nu_2\left(\frac{(b2^{n+1}+2^n-1)!}{(b2^{n+1}-1)!(2^n-1)!}\right) &= \nu_2\big(\big(b2^{n+1}+2^n-1\big)!\big)-\nu_2\big(\big(b2^{n+1}-1\big)!\big)-\nu_2\big(\big(2^n-1\big)!\big)\\ &= 1-s_2\big(b2^{n+1}+2^n-1\big)+s_2\big(b2^{n+1}-1\big)+s_2\big(2^n-1\big)\\ &= 1-s_2\big(b2^{n+1}\big)+s_2\big(b_02^{n+\nu_2(b)+2}+2^{n+\nu_2(b)+1}-1\big)\\ &= 1-s_2(b)+s_2(b_0)+n+\nu_2(b)+1\\ &= n+\nu_2(b)+1.\end{aligned} \tag{20}$$

On the other hand, one has

$$\begin{aligned}\nu_2\big(\big(b2^{n+1}+a\big)!\big) &= \big(b2^{n+1}+a\big)-s_2\big(\big(b2^{n+1}+a\big)\big)\\ &= b2^{n+1}+a-s_2(b)-s_2(a).\end{aligned} \tag{21}$$

So in order to show that (17) is true, by (19)–(21) one only needs to show that

$$\nu_2\left(\sum_{i=a}^{2^n} l(i)\right) = \big(b2^{n+1}+a\big)-\big(s_2(b)+\nu_2(b)+n+2\big). \tag{22}$$

To do so, one discusses the 2-adic valuation of $l(i)$ with $a\leqslant i\leqslant 2^n$ in what follows.

Since $b_0\in A_{t-1}$ and $0<2^{n+\nu_2(b)+1}+a-i\leqslant 2^{n+\nu_2(b)+1}$, by the inductive hypothesis and Lemma 2.3, one can derive that

$$\begin{aligned}&\nu_2\big(S\big(b2^{n+1}, b2^{n+1}+a-i\big)\big)\\ &\quad= \nu_2\big(S\big(b_02^{n+\nu_2(b)+2}+2^{n+\nu_2(b)+1}, b_02^{n+\nu_2(b)+2}+2^{n+\nu_2(b)+1}+a-i\big)\big)\\ &\quad= s_2\big(2^{n+\nu_2(b)+1}+a-i\big)-1\\ &\quad= s_2\big((2b_0+1)2^{n+\nu_2(b)+1}+a-i\big)-s_2(b_0)-1\\ &\quad= s_2\big(b2^{n+1}+a-i\big)-s_2(b)\end{aligned} \tag{23}$$

since $b = (2b_0+1)2^{\nu_2(b)}$ and $s_2(b) = s_2(b_0)+1$. Furthermore, by Lemmas 2.1, 2.3 and (23) one can compute that

$$\begin{aligned}
&\nu_2\bigl(i!S\bigl(2^n, i\bigr)\bigl(b2^{n+1} + a - i\bigr)!S\bigl(b2^{n+1}, b2^{n+1} + a - i\bigr)\bigr) \\
&\quad = i - s_2(i) + s_2(i) - 1 + \bigl(b2^{n+1} + a - i\bigr) \\
&\qquad - s_2\bigl(b2^{n+1} + a - i\bigr) + s_2\bigl(b2^{n+1} + a - i\bigr) - s_2(b) \\
&\quad = \bigl(b2^{n+1} + a\bigr) - s_2(b) - 1.
\end{aligned} \tag{24}$$

Then by (24) one has

$$\nu_2\bigl(l(i)\bigr) = \bigl(b2^{n+1} + a\bigr) - s_2(b) - 1 + \nu_2\bigl(b2^{n+1} + a\bigr) - \nu_2(i) - \nu_2\bigl(b2^{n+1} + a - i\bigr). \tag{25}$$

If $i = a$, then by (25) and noticing that $a \leqslant 2^n$, one gets that

$$\begin{aligned}
\nu_2\bigl(l(a)\bigr) &= \bigl(b2^{n+1} + a\bigr) - s_2(b) - 1 + \nu_2\bigl(b2^{n+1} + a\bigr) - \nu_2(a) - \nu_2\bigl(b2^{n+1}\bigr) \\
&= \bigl(b2^{n+1} + a\bigr) - \bigl(s_2(b) + \nu_2(b) + n + 2\bigr).
\end{aligned} \tag{26}$$

If $a < i \leqslant 2^n$ and $\nu_2(i) \leqslant \nu_2(b2^{n+1} + a)$, then

$$\nu_2(i) - \nu_2\bigl(b2^{n+1} + a\bigr) + \nu_2\bigl(b2^{n+1} + a - i\bigr) \leqslant \nu_2\bigl(b2^{n+1} + a - i\bigr) < n. \tag{27}$$

It then follows from (25) and (27) that

$$\nu_2\bigl(l(i)\bigr) > \bigl(b2^{n+1} + a\bigr) - s_2(b) - 1 - n > \bigl(b2^{n+1} + a\bigr) - \bigl(s_2(b) + \nu_2(b) + n + 2\bigr). \tag{28}$$

If $a < i \leqslant 2^n$ and $\nu_2(i) > \nu_2(b2^{n+1} + a)$, then one has

$$\nu_2(i) - \nu_2\bigl(b2^{n+1} + a\bigr) + \nu_2\bigl(b2^{n+1} + a - i\bigr) = \nu_2(i) \leqslant n. \tag{29}$$

So by (25) and (29) one has

$$\nu_2\bigl(l(i)\bigr) \geqslant \bigl(b2^{n+1} + a\bigr) - s_2(b) - 1 - n > \bigl(b2^{n+1} + a\bigr) - \bigl(s_2(b) + \nu_2(b) + n + 2\bigr). \tag{30}$$

Thus the desired result (22) follows immediately from (26), (28) and (30). So (17) holds if $b \in A_t$, which implies that Theorem 1.2 is true if $b \in A_t$.

The proof of Theorem 1.2 is complete. $\square$

4. Proof of Theorem 1.4

The purpose of this section is to prove Theorem 1.4. Note that its proof is different from the proofs of Theorems 1.1 and 1.2. So one provides the details of the proof of Theorem 1.4. Throughout this section, one always lets $a, b, c, m, n \in \mathbb{Z}^+$, $1 \leqslant a < 2^{n+1}$, $m \geqslant n + 2 + \lfloor \log_2 b \rfloor$ and $c \geqslant 1$ being odd. For any integers i and j with $0 \leqslant i \leqslant j \leqslant b2^{n+1} + 2^n$, one defines

$$h(i,j) := \binom{j}{i} \frac{(b2^{n+2} + a - i)!}{(b2^{n+2} + a - j)!} S\big(c2^m, b2^{n+2} + a - i\big) S\big(b2^{n+1} + 2^n, j\big). \tag{31}$$

Let

$$\Delta_1 := \sum_{j=1}^{2^n} \sum_{i=0}^{j} h(i,j), \qquad \Delta_2 := \sum_{j=2^n+1}^{2^{n+1}-2} \sum_{i=0}^{j} h(i,j),$$
$$\Delta_3 := \sum_{i=0}^{2^{n+1}-1} h\big(i, 2^{n+1} - 1\big), \qquad \Delta_4 := \sum_{j=b2^{n+1}+1}^{b2^{n+1}+2^n} \sum_{i=0}^{j} h(i,j). \tag{32}$$

First one uses the lemmas in Section 2 and Theorem 1.2 to prove the following result.

Lemma 4.1. *Each of the following is true:*

(i) *For $l = 1$ and 4, one has* $\nu_2(\Delta_l) \begin{cases} = s_2(a) - 1, & \text{if } 1 \leqslant a \leqslant 2^n \text{ and } s_2(b) = 1, \\ \geqslant s_2(a), & \text{otherwise;} \end{cases}$

(ii) $\nu_2(\Delta_2) \geqslant s_2(a)$;

(iii) $\nu_2(\Delta_3) \begin{cases} = n, & \text{if } a = 2^{n+1} - 1 \text{ and } s_2(b) = 1, \\ \geqslant s_2(a), & \text{otherwise;} \end{cases}$

(iv) $\nu_2(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) \begin{cases} = n, & \text{if } a = 2^{n+1} - 1 \text{ and } s_2(b) = 1, \\ \geqslant s_2(a), & \text{otherwise.} \end{cases}$

Proof. Evidently, part (iv) follows immediately from parts (i)–(iii). So one needs only to show parts (i)–(iii) which will be done in what follows. By Lemmas 2.1 and 2.2, one has

$$\nu_2\left(\binom{j}{i} \frac{(b2^{n+2} + a - i)!}{(b2^{n+2} + a - j)!}\right) = s_2(i) + s_2(j - i) - s_2(j) + j - i$$
$$+ s_2\big(b2^{n+2} + a - j\big) - s_2\big(b2^{n+2} + a - i\big). \tag{33}$$

(i) First one treats with Δ_1. Let $1 \leqslant j \leqslant 2^n$ and $0 \leqslant i \leqslant j$. By Lemma 2.3

$$\nu_2\big(S\big(b2^{n+1} + 2^n, j\big)\big) = s_2(j) - 1. \tag{34}$$

Let $m > n+2+\lfloor\log_2 b\rfloor$. Since $a < 2^{n+1}$ and $1 \leqslant i \leqslant 2^n$, one has $b2^{n+2}+a-i < 2^m$. By Lemma 2.3 one obtains $\nu_2(S(c2^m, b2^{n+2}+a-i)) = s_2(b2^{n+2}+a-i)-1$. Then from (31), (33), (34) and Lemma 2.2, one obtains that

$$\begin{aligned}\nu_2\big(h(i,j)\big) &= s_2(i)+s_2(j-i)+j-i+s_2\big(b2^{n+2}+a-j\big)-2\\ &\geqslant s_2(j)+s_2\big(b2^{n+2}+a-j\big)+j-i-2\\ &\geqslant s_2\big(b2^{n+2}+a\big)-2\\ &\geqslant s_2(a)-1,\end{aligned}\tag{35}$$

where equality holds if and only if $j=i$, $s_2(b)=1$ and $s_2(b2^{n+2}+a-j)+s_2(j) = s_2(b2^{n+2}+a)$. So by (32) and (35) one gets that

$$\Delta_1 = 2^{s_2(a)}\widetilde{\Delta}_1 + 2^{s_2(a)-1}\sum_{(i,j)\in\widetilde{J}}\widetilde{h}(i,j),\tag{36}$$

where $\widetilde{\Delta}_1 \in \mathbb{Z}^+$ and $\widetilde{J} := \{(i,j) \mid \widetilde{h}(i,j) \text{ is odd},\ 1\leqslant i\leqslant j\leqslant 2^n\}$. Then

$$\begin{aligned}\widetilde{J} &= \big\{(i,j)\ \big|\ j=i,\ s_2(b)=1 \text{ and } s_2\big(b2^{n+2}+a-j\big)+s_2(j)=s_2\big(b2^{n+2}+a\big)\big\}\\ &= \big\{(j,j)\ \big|\ s_2(b)=1 \text{ and } s_2\big(b2^{n+2}+a-j\big)+s_2(j)=s_2\big(b2^{n+2}+a\big)\big\}\\ &= \big\{1\leqslant j\leqslant 2^n\ \big|\ s_2(b)=1 \text{ and } s_2\big(2^{n+2}+a-j\big)+s_2(j)=s_2\big(2^{n+2}+a\big)\big\}.\end{aligned}$$

Thus by Lemma 2.10 one knows that $|\widetilde{J}| = 2^{s_2(a)}-1$ if $1\leqslant a\leqslant 2^n$ and $2^{s_2(a)-1}$ else.

Furthermore, by (36), one derives that $\nu_2(\Delta_1)$ equals $s_2(a)-1$ if $s_2(b)=1$ and $1\leqslant a\leqslant 2^n$, and is greater than $s_2(a)$ otherwise. So Lemma 4.1 (i) is true if $l=1$ and $m>n+2+\lfloor\log_2 b\rfloor$.

Now let $m = n+2+\lfloor\log_2 b\rfloor$. If either $2^n < a < 2^{n+1}$, or $1\leqslant a\leqslant 2^n$ and $1\leqslant i<a$, then one can check that the following is true:

$$2^m \leqslant b2^{n+2} < b2^{n+2}+a-i < b2^{n+2}+a \leqslant 2^{m+1}-1.$$

So Lemma 2.4 implies that

$$\nu_2\big(S\big(c2^m, b2^{n+2}+a-i\big)\big) \geqslant s_2\big(b2^{n+2}+a-i\big).\tag{37}$$

Thus by Lemma 2.2, (31), (33), (34) and (37) one deduces that

$$\begin{aligned}\nu_2\big(h(i,j)\big) &\geqslant s_2(i)+s_2(j-i)+j-i+s_2\big(b2^{n+2}+a-j\big)-1\\ &\geqslant s_2(j)+s_2\big(b2^{n+2}+a-j\big)+j-i-1\\ &\geqslant s_2\big(b2^{n+2}+a\big)-1\\ &\geqslant s_2(a).\end{aligned}\tag{38}$$

If $1 \leqslant a \leqslant 2^n$ and $a \leqslant i \leqslant j$, then $b2^{n+2}+a-i \leqslant b2^{n+2} \leqslant 2^m$. Then by Lemma 2.3 one gets $\nu_2(S(c2^m, b2^{n+2}+a-i)) = s_2(b2^{n+2}+a-i)-1$. Hence by (33), (31) and Lemma 2.2, one has

$$\begin{aligned}\nu_2\bigl(h(i,j)\bigr) &= s_2(i)+s_2(j-i)+j-i+s_2\bigl(b2^{n+2}+a-j\bigr)-2\\ &\geqslant s_2(j)+s_2\bigl(b2^{n+2}+a-j\bigr)+j-i-2\\ &\geqslant s_2\bigl(b2^{n+2}+a\bigr)-2\\ &\geqslant s_2(a)-1, \end{aligned} \tag{39}$$

with equality holding if and only if

$$j=i, \quad s_2(b)=1 \quad \text{and} \quad s_2\bigl(b2^{n+2}+a-j\bigr)+s_2(j)=s_2\bigl(b2^{n+2}+a\bigr). \tag{40}$$

Since $1 \leqslant j \leqslant 2^n$ and $a \leqslant i \leqslant j$, by Lemma 2.9 one knows that (40) holds only when $i=j=a$ and $s_2(b)=1$. It follows from (38) and (39) that $\nu_2(h(i,j)) \geqslant s_2(a)$ except for $i=j=a \in [1,2^n]$ and $s_2(b)=1$, in which case one has $\nu_2(h(a,a)) = s_2(a)-1$. Then by (32), one has $\nu_2(\Delta_1) = s_2(a)-1$ if $a \in [1,2^n]$ and $s_2(b)=1$, and $\nu_2(\Delta_1) \geqslant s_2(a)$ otherwise. Thus Lemma 4.1 (i) is true if $l=1$ and $m=n+2+\lfloor \log_2 b \rfloor$. So the statement for Δ_1 is proved.

Now one handles Δ_4. Note that $b2^{n+1}+1 \leqslant j \leqslant b2^{n+1}+2^n$ and $0 \leqslant i \leqslant j$. Let $j = b2^{n+1}+j_0$ for some integer $1 \leqslant j_0 \leqslant 2^n$. By Theorem 1.2 one has

$$\nu_2\bigl(S\bigl(b2^{n+1}+2^n, j\bigr)\bigr) = \nu_2\bigl(S\bigl(b2^{n+1}+2^n, b2^{n+1}+j_0\bigr)\bigr) = s_2(j_0)-1. \tag{41}$$

Since $m \geqslant n+2+\lfloor \log_2 b \rfloor$, one has $b2^{n+2}+a-j < b^{n+1}+a < 2^m$. So by Lemmas 2.3 and 2.4 one gets

$$\nu_2\bigl(S\bigl(c2^m, b2^{n+2}+a-i\bigr)\bigr) \geqslant s_2\bigl(b2^{n+2}+a-i\bigr)-1 \tag{42}$$

and

$$\nu_2\bigl(S\bigl(c2^m, b2^{n+2}+a-j\bigr)\bigr) = s_2\bigl(b2^{n+2}+a-j\bigr)-1. \tag{43}$$

So by (31), (33), (41)–(43) and Lemma 2.2 one obtains that

$$\begin{aligned}\nu_2\bigl(h(i,j)\bigr) &\geqslant s_2(i)+s_2(j-i)-s_2(j)+j-i+s_2\bigl(b2^{n+2}+a-j\bigr)+s_2(j_0)-2\\ &\geqslant s_2\bigl(b2^{n+1}+a-j_0\bigr)+s_2(j_0)-2\\ &\geqslant s_2(a)-1, \end{aligned} \tag{44}$$

where equality holds if and only if $j=i$, $s_2(b)=1$ and $s_2(b2^{n+1}+a-j_0)+s_2(j_0) = s_2(b2^{n+1}+a)$. It is similar to Δ_1 with $m \geqslant n+2+\lfloor \log_2 b \rfloor$, by Lemma 2.10 and (44) one has that $\nu_2(\Delta_4) = s_2(a) - 1$ if $a \in [1, 2^n]$ and $s_2(b) = 1$, and $\nu_2(\Delta_4) \geqslant s_2(a)$ otherwise. So Lemma 4.1 (i) is true if $l = 4$.

(ii) For Δ_2, noticing that $2^n < j < 2^{n+1} - 1$, $0 \leqslant i \leqslant j$ and $m \geqslant n + 2 + \lfloor \log_2 b \rfloor$, then by Lemmas 2.2–2.4, one gets

$$\nu_2\big(S\big(c2^m, b2^{n+2} + a - i\big)S\big(b2^{n+1} + 2^n, j\big)\big) \geqslant s_2\big(b2^{n+2} + a - i\big) - 1 + s_2(j).$$

So by (31) and (33), one has

$$\begin{aligned}\nu_2\big(h(i,j)\big) &\geqslant s_2(i) + s_2(j-i) + j - i + s_2\big(b2^{n+2} + a - j\big) - 1\\ &\geqslant s_2\big(b2^{n+2} + a\big) - 1\\ &\geqslant s_2(a).\end{aligned} \tag{45}$$

Hence by (32) and (45), one has $\nu_2(\Delta_2) \geqslant s_2(a)$ as desired.

(iii) For Δ_3, noting that $j = 2^{n+1} - 1$ and $0 \leqslant i \leqslant 2^{n+1} - 1$, it follows from Lemmas 2.2–2.4, (32) and (33) that

$$\begin{aligned}\nu_2\big(h\big(i, 2^{n+1} - 1\big)\big) &\geqslant s_2(i) + s_2(j-i) + j - i + s_2\big(b2^{n+2} + a - j\big) - 1\\ &\geqslant s_2\big(b2^{n+2} + a - 2^{n+1} + 1\big) + s_2\big(2^{n+1} - 1\big) - 2\\ &= s_2\big(b2^{n+2} + a - 2^{n+1} + 1\big) + n - 1\\ &\geqslant n,\end{aligned} \tag{46}$$

with equality holding if and only if $j = i = a = 2^{n+1} - 1$ and $s_2(b) = 1$. Since $1 \leqslant a < 2^{n+1}$, one has $n + 1 \geqslant s_2(a)$. So by (32) and (46), Lemma 4.1 (iii) follows immediately.

This completes the proof of Lemma 4.1. □

One can now use the lemmas presented in Section 2, Theorem 1.1 and Lemma 4.1 to show Theorem 1.3. The proof is of induction.

Proof of Theorem 1.4. By Lemma 2.6, one gets that

$$\begin{aligned}S\big(c2^m + b2^{n+1} + 2^n, b2^{n+2} + a\big) &= \sum_{j=1}^{b2^{n+1}+2^n} \sum_{i=0}^{j} h(i,j)\\ &= \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta,\end{aligned} \tag{47}$$

where $h(i,j)$ and Δ_l $(l = 1, 2, 3, 4)$ are defined in (31) and (32), respectively, and

$$\Delta := \sum_{j=2^{n+1}}^{b2^{n+1}} \sum_{i=0}^{j} h(i,j). \tag{48}$$

First one deals with the 2-adic valuation of $h(i,j)$ with $2^{n+1} \leqslant j \leqslant b2^{n+1}$ and $0 \leqslant i \leqslant j$. Let $j = j_1 2^{n+1} + j_2$ for some integers $1 \leqslant j_1 \leqslant b$ and $0 \leqslant j_2 < 2^{n+1}$.

If $j_2 = 0$, then $j = j_1 2^{n+1}$. So by Lemmas 2.2–2.4 and (31), one has

$$\begin{aligned}\nu_2\bigl(h(i,j)\bigr) &\geqslant \nu_2\left(\frac{(b2^{n+2}+a-i)!}{(b2^{n+2}+a-j)!}\right) + \nu_2\bigl(S\bigl(c2^m, b2^{n+2}+a-i\bigr)\bigr) \\ &\geqslant j-i+s_2\bigl(b2^{n+2}+a-j\bigr) - s_2\bigl(b2^{n+2}+a-i\bigr) + s_2\bigl(b2^{n+2}+a-i\bigr) - 1 \\ &\geqslant s_2\bigl(b2^{n+2}+a-j\bigr) - 1 \\ &= s_2\bigl((2b-j_1)2^{n+1}+a\bigr) - 1 \\ &\geqslant s_2(a) \end{aligned} \tag{49}$$

since $s_2(2b-j_1) \geqslant 1$ and $a < 2^{n+1}$.

If $0 < j_2 < 2^{n+1}$, by Theorem 1.1 one has

$$\nu_2\bigl(S\bigl(b2^{n+1}+2^n, j\bigr)\bigr) = \nu_2\bigl(S\bigl(b2^{n+1}+2^n, j_1 2^{n+1}+j_2\bigr)\bigr) \geqslant s_2(j_2) - 1. \tag{50}$$

Thus by Lemmas 2.2–2.3, (31), (33) and (50) one deduces

$$\begin{aligned}\nu_2\bigl(h(i,j)\bigr) &\geqslant \nu_2\left(\frac{(b2^{n+2}+a-i)!}{(b2^{n+2}+a-j)!}\right) \\ &\quad + \nu_2\bigl(S\bigl(c2^m, b2^{n+2}+a-i\bigr)\bigr) + \nu_2\bigl(S\bigl(b2^{n+1}+2^n, j\bigr)\bigr) \\ &\geqslant j-i+s_2\bigl(b2^{n+2}+a-j\bigr) + s_2(j_2) - 2 \\ &\geqslant s_2(j_2) + s_2\bigl((2b-j_1)2^{n+1}+a-j_2\bigr) - 2 \\ &\geqslant s_2\bigl((2b-j_1)2^{n+1}+a\bigr) - 2 \\ &= s_2(2b-j_1) + s_2(a) - 2. \end{aligned} \tag{51}$$

Let A_t be defined as in (18). Then $\mathbb{Z}^+ = \bigcup_{t=1}^{\infty} A_t$. One proves Theorem 1.4 by induction on t. First one considers that the case $t = 1$. Let $b \in A_1$. Then $s_2(b) = 1$. If $0 < j_2 < 2^{n+1}$, then $1 \leqslant j_1 < b$. So $s_2(2b-j_1) \geqslant 2$. Thus by (51) one has that $\nu_2(h(i,j)) \geqslant s_2(a)$ if $0 < j_2 < 2^{n+1}$. Furthermore, by (48) and (49) one gets

$$\nu_2(\Delta) \geqslant s_2(a). \tag{52}$$

By Lemma 4.1 (iv), (47) and (52), Theorem 1.4 for the case $s_2(b) = 1$ follows immediately. That is, Theorem 1.4 is proved if $t = 1$.

Now let $t \geqslant 2$. Assume that Theorem 1.4 is true for any integers $b \in A_1 \cup \cdots \cup A_{t-1}$. In what follows one proves that Theorem 1.4 is true for the case t, namely, for the case that $b \in A_t$.

For $b \in A_t$, let $b = 2^{r_1} + 2^{r_2} + \cdots + 2^{r_t}$ be the 2-adic expansion of b, where $r_1 > r_2 > \cdots > r_t$. Claim that if $1 \leqslant j_1 < b$, then $s_2(2b-j_1) = 1$ if and only if $b = 2^{r_1} + \frac{j_1}{2}$. One first notices that if $1 \leqslant j_1 < b$, then

$$2^{r_1+2} > 2b > 2b - j_1 > 2^{r_1}.$$

So $s_2(2b-j_1)=1$ if and only if $2b-j_1=2^{r_1+1}$, i.e., $b=2^{r_1}+\frac{j_1}{2}$. The claim is proved. In the following one handles Δ. For this purpose, one needs to treat with $h(i,j)$. Consider the following cases.

If $0<j_2<2^{n+1}$ and $s_2(2b-j_1)\geqslant 2$, then by (51) one derives that

$$\nu_2\bigl(h(i,j)\bigr)\geqslant s_2(a). \tag{53}$$

If $0<j_2<2^{n+1}$ and $s_2(2b-j_1)=1$, then by the claim one has $b=2^{r_1}+\frac{j_1}{2}$. It then follows that

$$S\bigl(b2^{n+1}+2^n, j_1 2^{n+1}+j_2\bigr)=S\left(2^{r_1+n+1}+\frac{j_1}{2}2^{n+1}+2^n, \frac{j_1}{2}2^{n+2}+j_2\right). \tag{54}$$

Since $2b-j_1=2^{r_1+1}$, one has $j_1=2^{r_2+1}+\cdots+2^{r_t+1}$, which implies that $s_2(\frac{j_1}{2})=t-1$ and so $\frac{j_1}{2}\in A_{t-1}$. Hence the inductive hypothesis applied to (54) gives us that

$$\nu_2\bigl(S\bigl(b2^{n+1}+2^n, j_1 2^{n+1}+j_2\bigr)\bigr)\begin{cases} =s_2(j_2)-1=n, & \text{if } j_2=2^{n+1}-1,\\ \geqslant s_2(j_2), & \text{if } 0<j_2<2^{n+1}-1.\end{cases} \tag{55}$$

For $0<j_2<2^{n+1}-1$, it follows from Lemmas 2.2–2.4, (31), (33) and (55) that

$$\begin{aligned}\nu_2\bigl(h(i,j)\bigr)&\geqslant j-i+s_2\bigl(b2^{n+2}+a-j\bigr)+s_2(j_2)-1\\ &\geqslant s_2(j_2)+s_2\bigl(b2^{n+2}-j_1 2^{n+1}+a-j_2\bigr)-1\\ &\geqslant s_2\bigl((2b-j_1)2^{n+1}+a\bigr)-1\\ &=s_2(a).\end{aligned} \tag{56}$$

For $j_2=2^{n+1}-1$, since $m\geqslant n+2+\lfloor\log_2 b\rfloor=n+2+r_1$, one has

$$b2^{n+2}+a-j=(2b-j_1)2^{n+1}+a-j_2=2^{n+2+r_1}+a-j_2\leqslant 2^{n+r_1+2}\leqslant 2^m. \tag{57}$$

Then by Lemma 2.3 and (57) one deduces that

$$S\bigl(c2^m, b2^{n+2}+a-j\bigr)=s_2\bigl(b2^{n+2}+a-j\bigr)-1. \tag{58}$$

It then follows from $1\leqslant a<2^{n+1}$, Lemmas 2.2–2.4, (31), (33), (55) and (58) that

$$\begin{aligned}\nu_2\bigl(h(i,j)\bigr)&\geqslant s_2\bigl(b2^{n+2}+a-j\bigr)-1+s_2(j_2)-1+j-i\\ &\geqslant s_2\bigl((2b-j_1)2^{n+1}+a-j_2\bigr)+s_2(j_2)-2\\ &=s_2\bigl((2b-j_1)2^{n+1}+a-2^{n+1}+1\bigr)+n-1\\ &\geqslant n,\end{aligned} \tag{59}$$

with equality holding if and only if $j=i$, $s_2(2b-j_1)=1$ and $a=2^{n+1}-1$.

Finally, by (49), (53), (56) and (59) one obtains that if $b \in A_t$, then

$$\nu_2(\Delta) \begin{cases} = n & \text{if } a = 2^{n+1} - 1, \\ \geqslant s_2(a) & \text{if } a < 2^{n+1} - 1. \end{cases} \tag{60}$$

Hence Lemma 4.1 (iv) together with (47) and (60) concludes that Theorem 1.4 is true if $b \in A_t$.

The proof of Theorem 1.4 is complete. □

5. Proof of Theorem 1.3

For any positive integer k, one defines $\theta(k)$ to be the largest integer l with $1 \leqslant l \leqslant s_2(k)$ such that $\{m_l, m_{l-1}, \ldots, m_1\}$ is a set of consecutive integers, where $k = 2^{m_1} + 2^{m_2} + \cdots + 2^{m_{s_2(k)}}$ is the 2-adic expansion of k and $m_1 > m_2 > \cdots > m_{s_2(k)}$. Then $\lceil \log_2 k \rceil = m_1 + 1$. First Theorems 1.1 and 1.4 are used to show the following lemma.

Lemma 5.1. *Let $n, k, a, c \in \mathbb{Z}^+$ be such that $3 \leqslant k \leqslant 2^n$, $s_2(k) \geqslant 2$ and $1 \leqslant a \leqslant \lceil \frac{k}{2} \rceil - 1$. Suppose that k is neither a power of 2 nor a power of 2 minus 1. Then one has*

$$\nu_2\big(S\big(c2^n - a, k - 2a\big)\big) = s_2(k) - \lceil \log_2 k \rceil + \nu_2(a)$$

if either $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$ or $a = \sum_{i=m_{\theta(k)}+1}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$, and

$$\nu_2\big(S\big(c2^n - a, k - 2a\big)\big) > s_2(k) - \lceil \log_2 k \rceil + \nu_2(a)$$

otherwise.

Proof. First, one writes

$$k = \sum_{i=m_{\theta(k)}}^{m_1} 2^i + \sum_{j=\theta(k)+1}^{s_2(k)} 2^{m_j}. \tag{61}$$

Note that the second sum in (61) vanishes if $\theta(k) = s_2(k)$. Obviously, $m_1 = m_l + l - 1$ if $1 \leqslant l \leqslant \theta(k)$ and $m_{\theta(k)} \geqslant m_{\theta(k)+1} + 2$ if $\theta(k) < s_2(k)$.

If $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$, then by (61) one infers that $k - 2a = \sum_{j=\theta(k)+1}^{s_2(k)} 2^{m_j}$ and $\nu_2(c2^n - a) = \nu_2(a) = m_{\theta(k)} - 1 = m_1 - \theta(k)$. It then follows from $\lceil \log_2 k \rceil = m_1 + 1$ that

$$s_2(k - 2a) = s_2(k) - \theta(k) \tag{62}$$

and

$$\theta(k) = m_1 - \nu_2(a) = \lceil \log_2 k \rceil - 1 - \nu_2(a). \tag{63}$$

Since $m_{\theta(k)} \geqslant m_{\theta(k)+1} + 2$, one has $k - 2a < 2^{m_{\theta(k)}-1} = 2^{\nu_2(c2^n-a)}$. It follows from Lemma 2.3, (62) and (63) that

$$\nu_2\big(S\big(c2^n - a, k - 2a\big)\big) = s_2(k - 2a) - 1 = s_2(k) - \lceil \log_2 k \rceil + \nu_2(a)$$

as required. Hence Lemma 5.1 is proved if $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$.

If $a = \sum_{i=m_{\theta(k)}+1}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$, then by (61) one deduces that $k - 2a = 2^{m_{\theta(k)}}$ and $\nu_2(c2^n - a) = \nu_2(a) = m_{\theta(k)} = m_1 + 1 - \theta(k) = \lceil \log_2 k \rceil - s_2(k)$ since $\lceil \log_2 k \rceil = m_1 + 1$. Hence $s_2(k) - \lceil \log_2 k \rceil + \nu_2(a) = 0$. It then follows from Lemma 2.3 that

$$\nu_2\big(S\big(c2^n - a, k - 2a\big)\big) = s_2\big(2^{m_{\theta(k)}}\big) - 1 = 0 = s_2(k) - \lceil \log_2 k \rceil + \nu_2(a).$$

Thus Lemma 5.1 is proved if $a = \sum_{i=m_{\theta(k)}+1}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$.

Now one treats the remaining case that neither $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$ nor $a = \sum_{i=m_{\theta(k)}+1}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$. For this remaining case, one claims that

$$\nu_2\big(S\big(c2^n - a, k - 2a\big)\big) \geqslant s_2(k) - m_1 + \nu_2(a). \tag{64}$$

From the claim (64) and noting that $\lceil \log_2 k \rceil = m_1 + 1$, one derives that

$$\nu_2\big(S\big(c2^n - a, k - 2a\big)\big) > s_2(k) - \lceil \log_2 k \rceil + \nu_2(a).$$

So Lemma 5.1 holds for the remaining case that neither $a = \sum_{i=m_{\theta(k)}}^{m_1} 2^{i-1}$ with $\theta(k) < s_2(k)$ nor $a = \sum_{i=m_{\theta(k)}+1}^{m_1} 2^{i-1}$ with $\theta(k) = s_2(k)$. Thus one needs only to prove that the claim (64) is true, which will be done in what follows.

If $\nu_2(a) < m_{s_2(k)}$, then $s_2(k) - (m_1 - \nu_2(a)) \leqslant s_2(k) - (m_1 - m_{s_2(k)} + 1) \leqslant 0$ since $s_2(k) \leqslant m_1 - m_{s_2(k)} + 1$. This concludes that the claim (64) is true if $\nu_2(a) < m_{s_2(k)}$.

If $m_{s_2(k)} \leqslant \nu_2(a) < m_{\theta(k)} - 1$, then $\theta(k) < s_2(k)$ and there is exactly one integer t with $\theta(k) < t \leqslant s_2(k)$ such that $m_t \leqslant \nu_2(a) < m_{t-1}$. Then by the definition of $\theta(k)$ one knows that $\{\nu_2(a), m_{t-1}, \ldots, m_{\theta(k)}, \ldots, m_1\}$ is not consisting of consecutive integers. This implies that $s_2(2^{m_1} + \cdots + 2^{m_{t-1}} + 2^{m_t}) = s_2(2^{m_1} + \cdots + 2^{m_{t-1}} + 2^{\nu_2(a)}) \leqslant m_1 - \nu_2(a)$. Therefore

$$\begin{aligned} s_2\big(2^{m_t} + \cdots + 2^{m_{s_2(k)}}\big) &= s_2(k) - s_2\big(2^{m_1} + \cdots + 2^{m_{t-1}} + 2^{m_t}\big) + 1 \\ &\geqslant s_2(k) - \big(m_1 - \nu_2(a)\big) + 1. \end{aligned} \tag{65}$$

Since $\nu_2(c2^n - a) = \nu_2(a)$ and $m_t \leqslant \nu_2(a) < m_{t-1}$, one may write $c2^n - a = c_1 2^{\nu_2(a)}$ and $k - 2a = c_2 2^{\nu_2(a)+1} + 2^{m_t} + \cdots + 2^{m_{s_2(k)}}$ with c_1 and c_2 being integers. Then by Theorem 1.1 and (65) one deduces that

$$\begin{aligned}\nu_2\big(S\big(c2^n-a,k-2a\big)\big)&=\nu_2\big(S\big(c_1 2^{\nu_2(a)},c_2 2^{\nu_2(a)+1}+2^{m_t}+\cdots+2^{m_{s_2(k)}}\big)\big)\\&\geqslant s_2\big(2^{m_t}+\cdots+2^{m_{s_2(k)}}\big)-1\\&\geqslant s_2(k)-m_1+\nu_2(a)\end{aligned}$$

as desired. Hence the claim (64) is proved if $m_{s_2(k)}\leqslant\nu_2(a)<m_{\theta(k)}-1$.

If $m_{\theta(k)}-1\leqslant\nu_2(a)\leqslant m_1-1$, then by (61) one can write

$$k-2a=\sum_{i=\nu_2(a)+1}^{m_1}2^i-2a+u=b2^{\nu_2(a)+2}+u \tag{66}$$

and

$$c2^n-a=c_3 2^{m_1}+\sum_{i=\nu_2(a)}^{m_1-1}2^i+2^{\nu_2(a)}-a=c_3 2^{m_1}+b2^{\nu_2(a)+1}+2^{\nu_2(a)}, \tag{67}$$

where $c_3\in\mathbb{Z}^+$ and u and b are defined as follows:

$$u:=\sum_{i=m_{\theta(k)}}^{\nu_2(a)}2^i+\sum_{j=\theta(k)+1}^{s_2(k)}2^{m_j},\qquad b:=\left(\sum_{i=\nu_2(a)}^{m_1-1}2^i-a\right)\Big/2^{\nu_2(a)+1}. \tag{68}$$

Note that the first sum of u vanishes if $\nu_2(a)=m_{\theta(k)}-1$ and the second sum of u vanishes if $\theta(k)=s_2(k)$. By (61), one has

$$s_2(u)=s_2(k)-s_2\left(\sum_{i=\nu_2(a)+1}^{m_1}2^i\right)=s_2(k)-m_1+\nu_2(a). \tag{69}$$

In the following one shows that $u<2^{\nu_2(a)+1}-1$. If $\theta(k)=s_2(k)$, then by (61) one has $k=\sum_{i=m_{\theta(k)}}^{m_1}2^i$. But k is not a power of 2 minus 1. So $m_{\theta(k)}\geqslant 1$. Thus by (68) one knows that $u=\sum_{i=m_{\theta(k)}}^{\nu_2(a)}2^i<2^{\nu_2(a)+1}-1$. If $\theta(k)<s_2(k)$, then $m_{\theta(k)}\geqslant m_{\theta(k)+1}+2$. Hence by (68) one yields that $u<2^{\nu_2(a)+1}-1$. Suppose that $b<0$. Then from (66) one deduces that

$$k-2a\leqslant -2^{\nu_2(a)+2}+u<-2^{\nu_2(a)+2}+2^{\nu_2(a)+1}-1<0,$$

which is impossible. So $b\geqslant 0$.

If $b>0$, then by (68) one has $m_1\geqslant\nu_2(a)+2+\lfloor\log_2 b\rfloor$. Since $u<2^{\nu_2(a)+1}-1$, it then follows from (66)–(69) and Theorem 1.4 that

$$\begin{aligned}\nu_2\big(S\big(c2^n-a,k-2a\big)\big)&=\nu_2\big(S\big(c_3 2^{m_1}+b2^{\nu_2(a)+1}+2^{\nu_2(a)},b2^{\nu_2(a)+2}+u\big)\big)\\&\geqslant s_2(u)=s_2(k)-m_1+\nu_2(a).\end{aligned}$$

The claim (64) is proved if $m_{\theta(k)}-1\leqslant\nu_2(a)\leqslant m_1-1$ with $b>0$.

If $b=0$, then by (66) one has $u>0$ since $k-2a>0$. In what follows one shows that $u>2^{\nu_2(a)}$. Suppose that $0<u\leqslant 2^{\nu_2(a)}$. From (68) one infers that either $\theta(k)<s_2(k)$ with $\nu_2(a)=m_{\theta(k)}-1$, or $\theta(k)=s_2(k)$ with $\nu_2(a)=m_{\theta(k)}$. If $\theta(k)<s_2(k)$ with $\nu_2(a)=m_{\theta(k)}-1$, then by (68) one gets $a=\sum_{i=m_{\theta(k)}-1}^{m_1-1}2^i$ since $b=0$. It contradicts with the assumption that $a\neq\sum_{i=m_{\theta(k)}}^{m_1}2^{i-1}$ if $\theta(k)<s_2(k)$. If $\theta(k)=s_2(k)$ with $\nu_2(a)=m_{\theta(k)}$, it then follows from (68) and $b=0$ that $a=\sum_{i=m_{\theta(k)}}^{m_1-1}2^i$, which contradicts with the assumption that $a\neq\sum_{i=m_{\theta(k)}+1}^{m_1}2^{i-1}$ if $\theta(k)<s_2(k)$. Hence $u>2^{\nu_2(a)}$. Note that $u<2^{\nu_2(a)+1}-1$. Now by (66)–(69) and Lemma 2.4 one deduces that

$$\nu_2\big(S\big(c2^n-a,k-2a\big)\big)=\nu_2\big(S\big(c_32^{m_1}+2^{\nu_2(a)},u\big)\big)\geqslant s_2(u)=s_2(k)-m_1+\nu_2(a)$$

as desired. The claim (64) is proved if $m_{\theta(k)}-1\leqslant\nu_2(a)\leqslant m_1-1$ with $b=0$.

This concludes the proof of Lemma 5.1. □

One is now in a position to show Theorem 1.3.

Proof of Theorem 1.3. Suppose that (3) is true. Then using (3) with $a=2c$ and $b=c$, one can easily derive that (4) holds. So one only needs to show that (3) is true, which will be done in the following.

To prove (3), one uses (5) and (6) with $p=2$, $m=(2b-a)2^n$, $v=1$ and n replaced by $(a-b)2^n$, and considers the coefficients of x^k:

$$\begin{aligned}
&S\big(a2^n,k\big)\\
&\quad\equiv\sum_{j=0}^{(a-b)2^n}\binom{(a-b)2^n}{j}S\big(j+(2b-a)2^n,k-2\big((a-b)2^n-j\big)\big)\\
&\quad=S\big(b2^n,k\big)+\sum_{j=(a-b)2^n-\lceil\frac{k}{2}\rceil+1}^{(a-b)2^n-1}\binom{(a-b)2^n}{j}S\big(j+(2b-a)2^n,k-2\big((a-b)2^n-j\big)\big)\\
&\quad=S\big(b2^n,k\big)+\sum_{i=1}^{\lceil\frac{k}{2}\rceil-1}\binom{(a-b)2^n}{i}S\big(b2^n-i,k-2i\big)\mod 2^{n+\nu_2(a-b)}.
\end{aligned}\tag{70}$$

It then follows from (70) that

$$S\big(a2^n,k\big)-S\big(b2^n,k\big)\equiv\sum_{i=1}^{\lceil\frac{k}{2}\rceil-1}\binom{(a-b)2^n}{i}S\big(b2^n-i,k-2i\big)\mod 2^{n+\nu_2(a-b)}.\tag{71}$$

In what follows one discusses the 2-adic valuation of a general term of (71) with $1\leqslant i\leqslant\lceil\frac{k}{2}\rceil-1$. Let $a-b=c_02^{\nu_2(a-b)}$ with $c_0\geqslant 1$ being odd. One first notices that $i\leqslant\lceil\frac{k}{2}\rceil-1<2^n$. So by Lemma 2.11 one infers that

$$s_2\big(c_02^{n+\nu_2(a-b)}-i\big)=s_2(c_0)+n+\nu_2(a-b)-\nu_2(i)-s_2(i).\tag{72}$$

It then follows from Lemma 2.2 and (72) that

$$\begin{aligned}
&\nu_2\left(\binom{(a-b)2^n}{i} S\left(b2^n-i,k-2i\right)\right) \\
&\quad = s_2(i)+s_2\left(c_0 2^{n+\nu_2(a-b)}-i\right)-s_2\left(c_0 2^{n+\nu_2(a-b)}\right)+\nu_2\left(S\left(b2^n-i,k-2i\right)\right) \\
&\quad = n+\nu_2(a-b)-\nu_2(i)+\nu_2\left(S\left(b2^n-i,k-2i\right)\right). \qquad (73)
\end{aligned}$$

One considers the following two cases.

Case 1. $s_2(k)=1$. Then one may write $k=2^m$. If $m=2$, then by (1) one has

$$\begin{aligned}
&\nu_2\left(S\left(a2^n,4\right)-S\left(b2^n,4\right)\right) \\
&\quad = \nu_2\left(\frac{1}{6}\left(4^{a2^n-1}-3^{a2^n}+3\cdot 2^{a2^n-1}-1\right)-\frac{1}{6}\left(4^{b2^n-1}-3^{b2^n}+3\cdot 2^{b2^n-1}-1\right)\right) \\
&\quad = \nu_2\left(3^{b2^n}\left(3^{(a-b)2^n}-1\right)\right)-1.
\end{aligned}$$

By Lemma 2.12, one has $\nu_2(3^{(a-b)2^n}-1)=n+\nu_2(a-b)+2$. It follows that

$$\nu_2\left(S\left(a2^n,4\right)-S\left(b2^n,4\right)\right)=n+\nu_2(a-b)-\lceil\log_2 4\rceil+s_2(4)+\delta(4)$$

since $\delta(4)=2$. Namely, Theorem 1.3 holds if $m=2$.

Now let $m\geqslant 3$. So $1\leqslant i\leqslant 2^{m-1}-1$. If $i=2^{m-2}$, then by Lemma 2.5

$$\nu_2\left(S\left(b2^n-i,2^m-2i\right)\right)=\nu_2\left(S\left(b2^n-2^{m-2},2^{m-1}\right)\right)=0. \qquad (74)$$

Thus by (73) and (74) one obtains that

$$\nu_2\left(\binom{(a-b)2^n}{i} S\left(b2^n-i,2^m-2i\right)\right)=n+\nu_2(a-b)-(m-2). \qquad (75)$$

If $i\neq 2^{m-2}$, then $\nu_2(i)<2^{m-2}$ since $i\leqslant 2^{m-1}-1$. It then follows from (73) that

$$\begin{aligned}
&\nu_2\left(\binom{(a-b)2^n}{i} S\left(b2^n-i,2^m-2i\right)\right) \\
&\quad > n+\nu_2(a-b)-(m-2)+\nu_2\left(S\left(b2^n-i,2^m-2i\right)\right) \\
&\quad \geqslant n+\nu_2(a-b)-(m-2). \qquad (76)
\end{aligned}$$

Hence by (71), (75) and (76) one derives that

$$\begin{aligned}
\nu_2\left(S\left(a2^n,2^m\right)-S\left(b2^n,2^m\right)\right) &= n+\nu_2(a-b)-m+2 \\
&= n+\nu_2(a-b)-\left\lceil\log_2 2^m\right\rceil+s_2\left(2^m\right)+\delta\left(2^m\right)
\end{aligned}$$

since $\delta(2^m)=1$. So (3) is true if $s_2(k)=1$.

Case 2. $s_2(k) \geqslant 2$. Since k is neither a power of 2 nor a power of 2 minus 1 and $1 \leqslant i \leqslant \lceil \frac{k}{2} \rceil - 1$, by Lemma 5.1, (71) and (73) one obtains that

$$\begin{aligned}\nu_2\big(S\big(a2^n, k\big) - S\big(b2^n, k\big)\big) &= n + \nu_2(a-b) - \lceil \log_2 k \rceil + s_2(k) \\ &= n + \nu_2(a-b) - \lceil \log_2 k \rceil + s_2(k) + \delta(k),\end{aligned}$$

since $\delta(k) = 0$. Hence (3) holds in this case.

The proof of Theorem 1.3 is complete. $\square$

Remark 5.1. By Theorem 1.3, one knows that Conjecture 1.1 is true if k is not a power of 2 minus 1. Theorem 1.3 tells us that $\nu_2(S(a2^{n+1}, k) - S(b2^{n+1}, k)) < n + \nu_2(a-b)$ if $k \neq 2^m - 1$ and $k \neq 4$. In fact, in the proof of Theorem 1.3, to handle the case that $k \neq 2^m - 1$ and $k \neq 4$, one makes use of the Junod congruence (5). However, for the remaining case $k = 2^m - 1$, numerical experimentation (see [18]) suggests that

$$\nu_2\big(S\big(a2^{n+1}, 2^m - 1\big) - S\big(b2^{n+1}, 2^m - 1\big)\big) = n + 1 + \nu_2(a-b) > n + \nu_2(a-b).$$

Thus, to get such result, the modulus in Lemma 2.8 (and so (71) above) is not enough. Hence one has to find a congruence stronger than (5). Unfortunately, one encounters difficulties in strengthening congruence (5). Maybe one needs some new approaches to attack Conjecture 1.1 for the remaining case $k = 2^m - 1$.

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