Results in Mathematics ~~1993~~

A q-ANALOG OF NEWTON'S SERIES, STIRLING FUNCTIONS AND EULERIAN FUNCTIONS

Jiang ZENG and Changgui ZHANG

Abstract. — Recently, Butzer et al. [BH1, BH2, BHS] have studied some classical combinatorial functions such as factorial functions, Stirling numbers and Eulerian numbers of fractional orders. In the present paper we show that much the same is true in the case of the q-analogs. Meanwhile we give some results for the convergence of a q-Newton interpolation series.

Math. Subject Classification (1991) : 05A30, 33D05.

0. Introduction

Throughout this paper q is a real number such that $0 < q < 1$. For $x \in$ define

$$[x] = \frac{q^x - 1}{q - 1},$$

where and in the sequal, we take the *principal value* of the function $x \mapsto q^x$. For $n \in$, let

$$[x]_0 = 1, \quad [x]_n = [x][x-1]\cdots[x-n+1] \qquad (n \geq 1),$$
$$[0]! = 1, \quad [n]! = [n]_n = [n][n-1]\cdots[1] \qquad (n \geq 1),$$

and define the *q-binomial coefficients* by

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = 1, \qquad \begin{bmatrix} x \\ n \end{bmatrix} = \frac{[x]_n}{[n]!} \qquad (n \geq 1).$$

Set also

$$(x;q)_\infty = \prod_{k=0}^{+\infty} (1 - xq^k), \tag{0.1}$$

and for $\alpha \in$

$$(x;q)_\alpha = \frac{(x;q)_\infty}{(xq^\alpha;q)_\infty}, \tag{0.2}$$

where $xq^\alpha \neq q^{-n}$ for any $n \in$. In particular, we have the q-*shifted factorials* :

$$(x;q)_0 = 1, \qquad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}),$$

and then the following expression [GR, p. 20] for q-binomial coefficients :

$$\begin{bmatrix} x \\ n \end{bmatrix} = \frac{(q^{-x};q)_n}{(q;q)_n}(-q^x)^n q^{-\binom{n}{2}}.$$

Recall that the q-*Gamma function* (*cf.* [GR, p. 16]) is defined for $\Re x > 0$ by

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}. \tag{0.3}$$

Note that $\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x)$, the reader is refered to Koorwinder [Ko] for a rigorous proof of this formula.

We shall require the q-*binomial formula* (*cf.* [GR, p. 7]) :

$$\sum_{n=0}^{+\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty} \qquad (|x|<1). \tag{0.4}$$

There are two classical q-analogs of the exponential function defined as follows :

$$e_q(x) = \sum_{n=0}^{+\infty} \frac{x^n}{[n]!} \qquad for \qquad |x| < \frac{1}{1-q},$$

and

$$E_q(x) = \sum_{n=0}^{+\infty} \frac{q^{\binom{n}{2}} x^n}{[n]!} \qquad for \qquad x \in .$$

By the q-binomial formula (0.4) we may write

$$e_q(x) = \frac{1}{((1-q)x;q)_\infty}, \qquad E_q(x) = ((q-1)x;q)_\infty. \tag{0.5}$$

Hence $e_q(x)\cdot E_q(-x) = 1$.

As usual, for any function $f(x)$ of x, we define the *shift operator* E : $Ef(x) = f(x+1)$, the *identity operator* I : $If(x) = f(x)$, and the q-*difference operator* by means of

$$\Delta_q^0 = I, \quad \Delta_q^n = (E - q^{n-1}I)(E - q^{n-2}I)\cdots(E - I).$$

It is readily seen that

$$\Delta_q^n f(x) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} f(x+n-k). \tag{0.6}$$

We call *q-Newton's series* a series of the form

$$\sum_{n=0}^{+\infty} a_n \begin{bmatrix} x-a \\ n \end{bmatrix}, \tag{0.7}$$

where $a \in$. We remark that if series (0.7) converges to a function $f(x)$ in an open region containing , then the coefficients a_n are given by (0.6) with $x = a$, *i.e.*, $a_n = \Delta_q^n f(a)$. For this reason, we say also that series (0.7) is the *q-Newton's interpolation series* associated to the function $f(x)$.

Definition 1. *For $\alpha \in$ and $n \in$, the q-Stirling function of second kind is defined by $S_q(\alpha, 0) = \delta_{\alpha,0}$ and for $n \geq 1$*

$$\begin{aligned} S_q(\alpha, n) &= \frac{1}{[n]!} \, \Delta_q^n [x]^\alpha \Big|_{x=0} \\ &= \frac{1}{[n]!} \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} [n-k]^\alpha. \end{aligned} \tag{0.8}$$

Remark : The q-Stirling function $S_q(\alpha, n)$ was first introduced by Carlitz [Ca1] in the case $\alpha \in$.

Definition 2. *For $\alpha \in$ and $n \in$, the $q-$Eulerian function is defined by*

$$A_q(\alpha, n) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} \alpha+1 \\ k \end{bmatrix} q^{\binom{k}{2}} [n+1-k]^\alpha. \tag{0.9}$$

Remark : The q-Eulerian function $A_q(\alpha, s)$ was also first introduced by Carlitz [Ca2] in the case $\alpha \in$. It may be interesting to recall here the combinatorial motivation of this definition. Let $\mathfrak{S}_n$ be the set of permutations of $\{1, 2, \cdots, n\}$. For $\sigma \in \mathfrak{S}_n$ we say that σ has a *descent* at i, $1 \leq i \leq n-1$, if $\sigma(i) > \sigma(i+1)$ and define

$$\operatorname{des} \sigma = \{i \,|\, \sigma(i) > \sigma(i+1)\} \quad \textit{and} \quad \operatorname{maj} \sigma = \sum_{i \in \operatorname{des} \sigma} i.$$

For exemple, if $\sigma = 4\,1\,3\,6\,5\,2 \in \mathfrak{S}_6$, then $\operatorname{des} \sigma = \{1, 4, 5\}$ and so $\operatorname{maj} \sigma = 1+4+5 = 10$. Let $\mathfrak{S}_{n,k}$ be the set of permutations with k descents. The q-Eulerian numbers

$A_q(n,s)$ $(0 \le s \le n)$ [Ca1, Ca2, Raw] is then the generating function of the maj on $\mathfrak{S}_{n,k}$:

$$A_q(n,s) = \sum_{\sigma \in \mathfrak{S}_{n,k}} q^{\mathrm{maj}\,\sigma}.$$

This paper was motivated by the recent works of BUTZER et al. [BH1, BH2, BHS], who investigated various properties of some well-known combinatorial numbers and functions such as *factorial functions*, *binomial coefficients*, *Stirling numbers* and *Eulerian numbers* in the case that the *integer* n is replaced by a *real* α. As is well-known (probably only for combinatorialists), most combinatorial functions have natural "q-analogues", so it is natural to ask whether there are some "q-analogues" of the works of BUTZER et al. [*loc. cit.*].

The purpose of this paper is to study the q-Stirling numbers $S_q(n,k)$ of second kind and the q-Eulerian numbers $A_q(n,k)$ when the integer $n \in$ is replaced by a complex $\alpha \in$. As we shall see, most properties of these numbers or functions remain true and can be proved almost *mutadis mutandis* in the same way as the restrictive case $n \in$, except the two following formulae :

$$\sum_{k=0}^{+\infty} S_q(\alpha,k)[x]_k = [x]^\alpha; \tag{0.10}$$

$$\sum_{k=0}^{+\infty} A_q(\alpha,k) = \Gamma_q(\alpha+1). \tag{0.11}$$

Note that the proofs of the above two formulae are *not easy* in the case $q = 1$. Indeed, for $q = 1$, formula (0.10) for $\Re x > -1$ was proved in [BHS] by a deep result on Newton's series, while formula (0.11) was proved in [BH2] for $x \in (-1, +\infty)$ to be equavelent to a result of fractional calculus and also proved in [Zh] in the half plane $\Re x > -1$ by combining a Tauberian theorem and some fine analytic manipulations. In this paper, we shall investigate the validity of the two formulae (0.10) and (0.11).

This paper is organized as follows. We shall first introduce a *$q-$analog of the Newton interpolation series* in the first section, where we obtain a *q-analog of the formulae of* CAHEN *and* PINCHERLE *for the abscissa of convergence* (see Theorem 1.3). These results may have some interests on themselves. In the second section, we review some results about the growth of entire functions and then apply them to the study of the convergence of q-Newton's series. The third section will be devoted to the $q-$Stirling functions, where we show especially that their generating function has *q-exponential growth of order one and of finite type* and give the *convergence region* of (0.10) (see Theorem 3.4 and Theorem 3.7). Finally, the $q-$Eulerian functions will be studied in the last section, where we shall determine the *convergence region* of (0.11) in particular (see Theorem 4.4).

1. A q-Newton interpolation series

Consider the following q-Newton series :

$$a_0 + a_1 \begin{bmatrix} x-1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} x-1 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} x-1 \\ 3 \end{bmatrix} + \cdots. \tag{1.1}$$

Note that, if $q \to 1^-$, series (1.1) reduces *formally* to

$$a_0 + a_1 \binom{x-1}{1} + a_2 \binom{x-1}{2} + a_3 \binom{x-1}{3} + \cdots,$$

whose abscissa of convergence, say σ, is well-known (*cf.* [Ge, p. 133]) :

$$\text{(Cahen)} \qquad \sigma = \limsup_{n\to+\infty} \frac{\log \left|\sum_{k=0}^{n} (-1)^k a_k\right|}{\log n} \qquad \text{if} \quad \sigma \geq 0;$$

$$\text{(Pincherle)} \qquad \sigma = \limsup_{n\to+\infty} \frac{\log \left|\sum_{k=n}^{+\infty} (-1)^k a_k\right|}{\log n} \qquad \text{if} \quad \sigma < 0.$$

For the latter use, set

$$b_n(x) = a_n \begin{bmatrix} x-1 \\ n \end{bmatrix}, \tag{1.2}$$

$$c_n(x,y) = \frac{b_n(x)}{b_n(y)} = \frac{[x-1]_n}{[y-1]_n}, \tag{1.2a}$$

where $x, y \in$ and y is a *non positive integer.*

Lemma 1.1. *For given $x, y \in$ and $y \neq 1, 2, 3, \ldots$, there exist $A_1 := A_1(x,y)$, $A_2 := A_2(x,y) > 0$ such that for all $n \in$,*

$$|c_n(x,y)| \leq A_1 q^{n\Re(x-y)}, \tag{1.3}$$

$$|c_{n+1}(x,y) - c_n(x,y)| \leq A_2 q^{n\Re(x-y)}. \tag{1.3a}$$

Proof. — By definition (1.2a) we have

$$c_n(x,y) q^{-n(x-y)} = \prod_{j=1}^{n} \frac{1-q^{j-x}}{1-q^{j-y}},$$

which is then *bounded* by a positive function $A_1 := A_1(x,y)$ since it converges to $(q^{1-x};q)_\infty/(q^{1-y};q)_\infty$ when $n \to +\infty$. Thus we have proved (1.3). As to (1.3a), it suffices to take $A_2 := 2A_1(x,y)$. ∎

Proposition 1.2. *If q-Newton's series (1.1) converges at $x_0 \neq 1, 2, \cdots$, then it converges for every x such that $\Re x > \Re x_0$.*

PROOF. — By (1.2) and (1.2a), we may rewrite series (1.1) as $\sum_{n=0}^{+\infty} b_n(x_0) c_n(x, x_0)$. Since the series $\sum_{n\geq 0} b_n(x_0)$ is convergent, so, by du BOIS-REYMOND's test (*cf.* [Ge, p. 126] or [Kn, p. 315]), it suffices to prove the convergence of the series

$$\sum_{n=0}^{+\infty} |c_{n+1}(x, x_0) - c_n(x, x_0)|$$

in the half plane $\Re x > \Re x_0$. Now, the latter is dominated by the convergent geometric series $\sum_{n=0}^{+\infty} A q^{n(\Re x - \Re x_0)}$ in view of (1.3a). □

Therefore, we may define the *convergence abscissa* of q-Newton's series (1.1) by

$$\sigma = \inf\{\sigma_0 \mid (1.1) \text{ converges for every } x \text{ such that } \Re x > \sigma_0\}.$$

The following theorem gives a q-analogue of the formulae of COHEN and PINCHERLE.

Theorem 1.3. *Let σ be the convergence abscissa of (1.1) and set*

$$\alpha = \limsup_{n\to+\infty} \frac{\log\left|\sum_{k=0}^{n} (-1)^k a_k q^{-\binom{k+1}{2}}\right|}{n \log q^{-1}}; \tag{1.4}$$

$$\beta = \limsup_{n\to+\infty} \frac{\log\left|\sum_{k=n}^{+\infty} (-1)^k a_k q^{-\binom{k+1}{2}}\right|}{n \log q^{-1}}. \tag{1.5}$$

Then $\sigma = \alpha$ if $\sigma \geq 0$, and $\sigma = \beta$ if $\sigma < 0$.

PROOF. — We shall only consider the $\sigma \geq 0$ case and leave the $\sigma < 0$ case to the interested reader, who would complete the proof without difficulty on refering to the counterpart of ordinary Newton's interpolation series in [Mi, p. 281-283].

We divide the *proof in two parts*. Suppose that series (1.1) converges at a point x_0, where x_0 is a *non positive integer*. Let $\Re x_0 = \sigma_0$.

For each $x \in$ and $n \geq 0$, set

$$\lambda_n(x) = \sum_{k=0}^{n} a_k \begin{bmatrix} x-1 \\ k \end{bmatrix}, \qquad \mu_n(x) = \sum_{k=n}^{+\infty} a_k \begin{bmatrix} x-1 \\ k \end{bmatrix}.$$

Note that

$$\lambda_n(0) = \sum_{k=0}^{n} (-1)^k a_k q^{-\binom{k+1}{2}}, \qquad \mu_n(0) = \sum_{k=n}^{+\infty} (-1)^k a_k q^{-\binom{k+1}{2}}.$$

Using the notations in (1.2), (1.2a) and by Abel's Identity [Mi, p. 276-277], we may write $\lambda_n(x)-\lambda_l(x)$, $n>l\ge 0$, as follows :

$$\text{(1.6)}\quad \sum_{k=l}^{n} b_k(x_0)c_k(x,x_0)=\sum_{k=l}^{n}(c_{k-1}(x,x_0)-c_k(x,x_0))\mu_k(x_0) - c_{l-1}(x,x_0)\mu_l(x_0)+c_n(x,x_0)\mu_{n+1}(x_0),$$

$$\text{(1.7)}\quad \sum_{k=l}^{n} b_k(x_0)c_k(x,x_0)=\sum_{k=l}^{n}(c_k(x,x_0)-c_{k+1}(x,x_0))\lambda_k(x_0) - c_l(x,x_0)\lambda_{l-1}(x_0)+c_{n+1}(x,x_0)\lambda_n(x_0),$$

where $x_0\in$ and $x_0\neq 1,2,3,\cdots$.

(i) We prove that if $\sigma_0>0$, then $\alpha\le\sigma_0$; and consequently that $\alpha\le\sigma$, if $\sigma\ge 0$.

Since (1.4) can be written as

$$\alpha=\limsup_{n\to+\infty}\frac{\log|\lambda_n(0)|}{n\log q^{-1}},$$

it is then sufficent (*cf.* [Mi, p. 277-278]) to prove

$$\text{(1.8)}\qquad \lim_{n\to+\infty} q^{n\sigma_0}\lambda_n(0)=0.$$

By hypothesis, the series $\sum b_n(x_0)$ converges, and hence given $\varepsilon>0$, we have $l\in$ such that $|\mu_n(x_0)|<\varepsilon$ if $n\ge l$. Thus we deduce from (1.6) and (1.3a) that for $\Re x<\sigma_0$,

$$\text{(1.9)}\quad \begin{aligned} |\lambda_n(x)| &\le |\lambda_l(x)|+A_2\varepsilon\left(\sum_{k=l+1}^{n} q^{k(\Re x-\sigma_0)}+q^{l(\Re x-\sigma_0)}+q^{n(\Re x-\sigma_0)}\right)\\ &\le |\lambda_l(x)|+2A_2\varepsilon\frac{q^{(n+1)(\Re x-\sigma_0)}-q^{l(\Re x-\sigma_0)}}{q^{\Re x-\sigma_0}-1}. \end{aligned}$$

Noting that $\lim_{n\to+\infty} q^{n(\sigma_0-\Re x)}=0$, therefore, if we multiply (1.9) by $q^{n(\sigma_0-\Re x)}$ and let first $n\to+\infty$, and then $\varepsilon\to 0$, we get then

$$\limsup_{n\to+\infty} q^{n(\sigma_0-\Re x)}\lambda_n(x)=0.$$

which yields clearly (1.8) by setting $x=0$.

(ii) Without the loss of generality, we suppose that α is finite. We prove that series (1.1) converges at $\Re x=\varepsilon+\alpha$ for any $\varepsilon>0$. This implies that $\alpha\ge\sigma$.

Let $\Re x = \alpha + \varepsilon$ with $\varepsilon > 0$. It suffices to prove that the partial sum $\lambda_n(x)$ of (1.1) converges when $n \to +\infty$.

To prove this, setting $x_0 = 0$ and $l = 1$ in (1.7) we have

$$\lambda_n(x) = \sum_{k=1}^{n} (c_k(x,0) - c_{k+1}(x,0))\lambda_k(0) + (1 - c_1(x,0))\lambda_0(0) - c_{n+1}(x,0)\lambda_n(0).$$

Since α is finite, it follows from (1.4) that for sufficiently large n, we have

$$|\lambda_n(0)| \leq q^{-n(\alpha+\varepsilon/2)}. \tag{1.10}$$

Combining (1.3) and (1.10), we have, for sufficiently large n,

$$|c_{n+1}(x,0)\lambda_n(0)| \leq A_1 q^{\alpha+(1+n/2)\epsilon}.$$

Since the second member of the above inequality tends to 0 when $n \to +\infty$, it remains to prove the convergence of the series $\sum_{k=0}^{+\infty}(c_k(x,0) - c_{k+1}(x,0))\lambda_k(0)$, but this is obvious in view of (1.3a) and (1.10). □

Proposition 1.4. *Let σ be the convergence abscissa of (1.1). If σ is finite, then series (1.1) defines a bounded regular function in every δ-half plane $\Re x \geq \sigma + \delta$, where $\delta > 0$.*

Proof. — Let x_0 be a non integral point and $\Re x_0 = \sigma_0 = \sigma + \delta/2$ with $\delta > 0$. Set $M = \sup_{n\geq 0}\{|\mu_n(x_0)|\}$, then M is finite, since series (1.1) with $x = x_0$ is convergent. As in (1.9), it follows from (1.6) with $l = 0$ and (1.3a) that

$$\begin{aligned} |\lambda_n(x)| &< |a_0| + 2A_2 M \frac{q^{(n+1)(\Re x - \sigma_0)} - 1}{q^{\Re x - \sigma_0} - 1} \\ &< |a_0| + \frac{2A_2 M}{1 - q^{\Re x - \sigma_0}}. \end{aligned}$$

where $\Re x > \sigma_0$ and $n \geq 0$. The sum of series (1.1) is then bounded by $|a_0| + 2A_2 M/(1 - q^{\delta/2})$ in the half plane $\Re x \geq \sigma + \delta$. □

In the same manner, we can define the *absolute convergence abscissa* of a q-Newton's series (1.1) by

$$\bar{\sigma} = \inf\{\sigma_0 \mid (1.1) \text{ converges absolutely for every } x \text{ such that } \Re x > \sigma_0\}.$$

It is obvious that $\bar{\sigma} \geq \sigma$. Similarly, we can prove the following

Theorem 1.5. *The absolute convergence abscissa $\bar\sigma$ of (1.1) is given by*

$$\bar\sigma = \limsup_{n\to+\infty} \frac{\log \sum_{k=0}^{n} \left| a_k q^{-\binom{k+1}{2}} \right|}{n \log q^{-1}}, \qquad \text{if } \bar\sigma \geq 0; \tag{1.11}$$

$$\bar\sigma = \limsup_{n\to+\infty} \frac{\log \sum_{k=n}^{+\infty} \left| a_k q^{-\binom{k+1}{2}} \right|}{n \log q^{-1}}, \qquad \text{if } \bar\sigma < 0. \tag{1.12}$$

It should mention that there remains a lot to do with q-Newton's series. Some natural questions are : What condition should a function satisfy in order to be represented by a q-Newton's series ? What is the relation between the convergence abscissa and the absolute convergence abscissa of a q-Newton's series ? Actually, there exist several q-analogues of Newton's series (see, for exemple, [Wa]). It should be interesting to compare these different q-Newton series and study their relations.

2. Some lemmas about the growth of entire functions

We shall first give some definitions about the *q-exponential growth* of entire functions following Ramis [Ram].

Definition 3. *Let be given a non zero real number k. A power series*

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \tag{2.1}$$

is said to be q-Gevrey of order $s = 1/k$, if there exist real numbers $A, K > 0$ such that

$$|a_n| < K A^n q^{\frac{n(n+1)}{2k}}. \tag{2.2}$$

Thus formulae (1.11) and (1.12) imply immediately that if the power series (2.1) is q-Gevrey of *order one*, then the absolute convergence abscissa of the series (1.1) is finite. The following lemma has been given in [Ram, Lemma 2.2, ii)].

Lemma 2.1. *Let $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ be an entire function. If there exist real numbers $K, k > 0$, μ such that*

$$|f(x)| \leq K e^{\frac{k}{2}(\log|x|)^2 + \mu \log|x|} \qquad \text{when} \quad |x| \to +\infty,$$

then

$$|a_n| \leq K e^{-\frac{(n-\mu)^2}{2k}} \qquad \text{for} \quad n \geq 0.$$

Definition 4. *Let f be an entire function. If there exist real numbers $k \neq 0$ and μ such that, for a suitably chosen $K > 0$,*

$$|f(x)| < K e^{-\frac{1}{2k}\frac{\log^2|x|}{\log q} + \mu \log|x|} \qquad \text{when} \quad |x| \to +\infty, \tag{2.3}$$

we say that f has q-exponential growth of order k and of finite type μ.

Remark : Comparing the above definition with that in Ramis [Ram, p. 71], we have changed his q^{-1} by q. Note also that for the same entire function $f(x)$, there may exist several pairs of numbers (k, μ) satisfying (2.3). The largest k is called the *precise order of q-exponential growth.*

The following result has been given in [Ram, Prop. 5.5] and [Wa], but the proof given below seems *new*.

Proposition 2.2. *The q-exponential function, $E_q(x)$, has q-exponetial growth of order 1 and of finite type, more precisely*

$$|E_q(x/(1-q))| = O(e^{-\frac{\log^2|x|}{2\log q}+\frac{\log|x|}{2}}) \qquad \textit{when} \quad |x| \to +\infty. \tag{2.4}$$

Proof. — Since $|E_q(x)| \leq E_q(|x|)$ in view of (0.5), and in particular $|E_q(x)| = E_q(x)$ for real numbers $x > 0$, we then need only to consider the growth of the function $E_q(x)$ along the real positive axe. For $x \in (0, +\infty)$, define the function

$$f(x) = E_q(x/(1-q))e^{\frac{\log^2 x}{2\log q}-\frac{\log x}{2}}\prod_{n=1}^{+\infty}(1+q^n/x). \tag{2.5}$$

It is easy to check that

$$f(qx) = f(x) = (-x;q)_\infty(-q/x;q)_\infty \frac{1}{\sqrt{x}}e^{\frac{\log^2 x}{2\log q}},$$

so the function $f(x)$ is somewhat "q-periodic". Therefore, if we define

$$H = \min_{x\in[q,1]}\{f(x)\} \qquad \textit{and} \qquad K = \max_{x\in[q,1]}\{f(x)\},$$

we should have

$$0 < H \leq f(x) \leq K \qquad \text{for all} \quad x > 0.$$

On the other hand, noticing that

$$\lim_{x\to+\infty}\prod_{n=1}^{+\infty}(1+q^n/x) = 1,$$

we derive from (2.5) that for sufficiently large x,

$$\frac{1}{2}He^{-\frac{\log^2 x}{2\log q}+\frac{\log x}{2}} < E_q(x/(1-q)) < 2Ke^{-\frac{\log^2 x}{2\log q}+\frac{\log x}{2}}, \tag{2.6}$$

which is clearly equivalent to (2.4) for $x \in (0, +\infty)$ by definition. □

Proposition 2.3. *Let $f(x)$ be an entire function. If there exist real positive numbers μ, k, K, $d > 0$ and $p > 1$ such that for $|x| \in \{d, dp, dp^2, \cdots\}$*

$$|f(x)| < K e^{-\frac{\log^2 |x|}{2k \log q} + \mu \log |x|} \qquad when \quad |x| \to +\infty,$$

then $f(x)$ has q-exponential growth of order k and of finite type

$$\mu' = \mu - \frac{\log p}{k \log q}. \tag{2.7}$$

Proof. — Let $r_n = dp^n$ for $n \in$. By hypothesis, for any $x \in$, there exists an integer $n \geq 1$ such that

$$r_n \leq |x| < r_{n+1}.$$

Therefore, if $|x|$ is sufficiently large, by the maximum principle of analytic functions (*cf.*, for exemple, [Ti, p. 165]), we have

$$\begin{aligned} |f(x)| &< K e^{-\frac{\log^2 |r_{n+1}|}{2k \log q} + \mu \log |r_{n+1}|} \\ &< K e^{-\frac{\log^2 |xp|}{2k \log q} + \mu \log |xp|} \\ &\leq K' e^{-\frac{\log^2 |x|}{2k \log q} + (\mu - \frac{\log p}{k \log q}) \log |x|}, \end{aligned}$$

where $K' = K e^{-\frac{\log^2 p}{2k \log q}}$. □

Now we formulate a sufficient condition for the absolute convergence abscissa of (1.1) to be *finite* in terms of the *q-exponential growth* of functions.

Proposition 2.4. *Let $\bar{\sigma}$ be the absolute convergence abscissa of series (1.1). If the entire funcion $f(x) = \sum_{n=0}^{+\infty} a_n x^n$ has q-exponential growth of order 1 and of finite type μ, then*

$$\bar{\sigma} \leq \mu + \frac{1}{2}. \tag{2.8}$$

Proof. — By hypothesis and (2.3) we have

$$|f(x)| < K e^{-\frac{1}{2} \frac{\log^2 |x|}{\log q} + \mu \log |x|} \qquad when \quad |x| \to +\infty.$$

Applying Lemma 2.1 with $k = -1/\log q$ yields that there exists $K > 0$ such that

$$|a_n| \leq K q^{\frac{(n-\mu)^2}{2}}.$$

We then get (2.8) from Theorem 1.5. □

In order to illustrate the above proposition, we end this section with the following example. Consider the q-Newton's interpolation series associated to the q-exponential function $E_q(x)$:

$$\sum_{n=0}^{+\infty} \frac{q^{\binom{n}{2}}}{[n]!} \begin{bmatrix} x-1 \\ n \end{bmatrix} \tag{2.9}$$

By Proposition 2.2, we have

$$|E_q(x)| = O(e^{-\frac{\log^2|x|}{2\log q}+\mu\log|x|}) \qquad (|x| \to +\infty),$$

where $\mu = \frac{1}{2} - \log_q(1-q)$. Therefore, in view of (2.8), the absolute convergence abscissa of (2.9) is less than or equal to $1 - \log_q(1-q)$.

3. q-Stirling function of second kind

By definition 1 and (0.8), it is readily seen that the $q-$Stirling functions satisfy the following recurrence :

$$S_q(\alpha+1, k+1) = q^k S_q(\alpha, k) + [k+1] S_q(\alpha, k+1),$$

where $\alpha \in$ and $k \geq 1$.

The first four values of $q-$Stirling functions are as follows :

$$\begin{aligned} S_q(\alpha,0) &= \delta_{\alpha,0}, \\ S_q(\alpha,1) &= 1, \\ S_q(\alpha,2) &= (1+q)^{\alpha-1} - 1, \\ S_q(\alpha,3) &= \frac{1}{1+q}((1+q+q^2)^{\alpha-1} - q^3(1+q)^{\alpha} + q). \end{aligned}$$

For convenience, we set $[0]^\alpha = \delta_{\alpha,0}$ in what follows.

Let us introduce for $\alpha \in$, $|x| < 1/(1-q)$ the generalized q-Stirling "polynomials"

$$f(\alpha, x) = \frac{1}{e_q(x)} \sum_{k=0}^{+\infty} \frac{[k]^\alpha}{[k]!} x^k, \tag{3.1}$$

the radius of convergence of the series being $1/(1-q)$.

Proposition 3.1. *For $|x| < 1/(1-q)$ and $\alpha \in$ we have*

$$f(\alpha, x) = \sum_{n=0}^{+\infty} S_q(\alpha, n) x^n. \tag{3.2}$$

PROOF. — This follows by comparing the coefficients of x^k of both sides in (3.2) and then the definition of q-Stirling functions (0.8). □

Corollary 3.2. *For $\alpha \in$, $|x| < 1$ we have*

$$\sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix} [l]! S_q(\alpha, l) = [k]^\alpha \qquad (k \geq 0), \tag{3.3}$$

$$\sum_{k=0}^{+\infty} \frac{S_q(\alpha, k)[k]! x^k}{(x;q)_{k+1}} = \sum_{k=0}^{+\infty} x^k [k]^\alpha. \tag{3.4}$$

PROOF. — Multiplying both sides of (3.2) by $e_q(x)$ yields

$$e_q(x) \sum_{k=0}^{+\infty} S_q(\alpha, k) x^k = \sum_{k=0}^{+\infty} \frac{[k]^\alpha}{[k]!} x^k,$$

which is equivalent to (3.3) by identifying the coefficients of x^k. Next, note first that the series on the right side of (3.4) is convergent for $|x| < 1$. By the q-binomial formula (0.4), for $|x| < 1$ we have

$$\frac{1}{(x;q)_{k+1}} = \sum_{l=0}^{+\infty} \begin{bmatrix} k+l \\ l \end{bmatrix} x^l.$$

Substituting this into (3.4) and extracting the coefficients of x^k of both sides, we see that (3.4) is equivalent to (3.3). □

REMARK : If $x = 1$, Proposition 3.1 reduces to

$$\sum_{k=0}^{+\infty} S_q(\alpha, k) = \frac{1}{e_q(1)} \sum_{k=0}^{+\infty} \frac{[k]^\alpha}{[k]!}. \tag{3.5}$$

Note further that if $\alpha \in$ and $q = 1$, the left-hand side of (3.5) is actually the *Bell numbers* and (3.5) reduces to *Dobinski's formula* (*cf.*, [Co, p. 45]). In the case of $\alpha \in$, Proposition 3.2 has been proved in [Ze] as a q-analogue of Touchard's formula refining *Dobinski's formula*. Therefore, as suggested by the referer, (3.5) could be denoted as *q-Bell numbers*.

Corollary 3.3. *For all $\alpha \in$, the function $f(\alpha, x)$ is an entire function of x.*

Proof. — Proposition 3.1 shows that the function $f(\alpha, x)$ is analytic in the disc $|x| < 1/(1-q)$. Therefore, in order to prove the analyticity of $f(\alpha, x)$ in the whole plane, it suffices to do this for the right-hand side of (3.2). Indeed, for any $n \in$, we may write

$$[k]^\alpha = \left(\frac{1-q^k}{1-q}\right)^\alpha = (1-q)^{-\alpha} \sum_{j=0}^{n} (-1)^j \binom{\alpha}{j} q^{jk} + R(k,n) \quad (k \geq 1),$$

where, by Taylor's formula,

$$R(k,n) = (1-q)^{-\alpha} \binom{\alpha}{n+1} \theta^{n+1} (1-\theta q^k)^{\alpha-n-1} q^{kn+k}$$

for some $\theta \in (0,\, 1)$. Set

$$I_n(x) = \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} \frac{e_q(q^k x)}{e_q(x)}, \qquad J_n(x) = \frac{1}{e_q(x)} \sum_{k=0}^{+\infty} \frac{R(k,n)}{[k]!} x^k, \tag{3.6}$$

and write

$$f(\alpha, x) = \frac{1}{e_q(x)} \sum_{k=0}^{+\infty} \frac{[k]^\alpha}{[k]!} x^k = I_n(x) + J_n(x). \tag{3.7}$$

Note that for $0 < \theta < 1$ and $k \geq 1$, we have $1 > 1 - \theta q^k > 1 - q^k \geq 1 - q$. Let $T_0 = \max\left\{(1-q)^{\Re\alpha}, 1\right\}$, a constant independent of k and n, then

$$\left|(1-\theta q^k)^{\alpha-n-1}\right| < T_0 (1-q^k)^{-n-1}.$$

It follows that

$$|R(k,n)| \leq T_0 (1-q)^{-\Re\alpha} \left(\frac{q^k}{1-q^k}\right)^{n+1} \left|\binom{\alpha}{n+1}\right| \qquad (k \geq 1), \tag{3.8}$$

which implies the convergence of the series $\sum_{k\geq 0} R(k,n) x^k/[k]!$ for $|x| < 1/q^{n+1}(1-q)$. Besides, the functions $1/e_q(x)$, $e_q(q^j x)/e_q(x)$ ($j \in$) are clearly regular for all $x \in$, so $I_n(x)$ and $J_n(x)$ are both analytic in the disc $|x| < 1/q^{n+1}(1-q)$ for every $n \in$, which implies the analyticity of $f(\alpha, x)$ in the whole plane. ∎

Theorem 3.4. *If $\Re\alpha > -1$, then $f(\alpha, x)$ has q-exponential growth of order 1 and of type*

$$\mu = \frac{3}{2} + \log_q(1-q). \tag{3.9}$$

PROOF. — In view of (3.7) we have

$$|f(x,\alpha)| \le |I_n(x)| + |J_n(x)|. \tag{3.10}$$

Let us first estimate $|I_n(x)|$. By *Stirling's formula* about *Gamma function* (*cf.*, [Mi, p. 254]), we have

$$\left|\binom{\alpha}{n}\right| = O(n^{-\Re\alpha-1}) \qquad when \quad n \to +\infty.$$

Hence, for $\Re\alpha > -1$, there exists a constant $K_1 > 0$ such that

$$\left|\binom{\alpha}{n}\right| < K_1 \qquad \forall n \in . \tag{3.11}$$

On the other hand, by (0.5) we have

$$\left|\frac{e_q(q^k x)}{e_q(x)}\right| = |(1-(1-q)x)\cdots(1-(1-q)q^{k-1}x^k)| \le E_q(|x|).$$

It then follows from (3.6) and (3.11) that

$$|I_n(x)| \le \sum_{k=1}^{n} \left|\binom{\alpha}{k}\right| \left|\frac{e_q(q^k x)}{e_q(x)}\right| \le nK_1 E_q(|x|). \tag{3.12}$$

Next, we shall estimate $|J_n(x)|$ on the circles :

$$|x| = r_n = \frac{1}{2} q^{-n-1}. \tag{3.13}$$

Applying (3.11) to (3.8) implies that there exists a constant $K_2 > 0$, independent of k and n, such that

$$|R(k,n)| < K_2 \left(\frac{q^k}{1-q^k}\right)^{n+1} \qquad (k \ge 1).$$

Noting also that $1/|e_q(x)| = |E_q(-x)| \le E_q(|x|)$, therefore, it follows from (3.6) that

$$|J_n(x)| \le \frac{K_2}{|e_q(x)|} \sum_{k=1}^{+\infty} \frac{\left|q^{n+1}x\right|^k}{[k]!(1-q^k)^{n+1}} \le K_2 E_q(1/2)(1-q)^{-n-1} E_q(r_n), \tag{3.14}$$

since $|q^{n+1}x| = 1/2$ by (3.13). Combining (3.12) and (3.14) yields that for $|x| = r_n$,

$$|f(\alpha,x)| \le \left(nK_1 + K_2 E_q(1/2)(1-q)^{-n-1}\right) E_q(r_n). \tag{3.15}$$

Now, by (2.4), there exists a constant $C > 0$ such that

$$|E_q(x)| < Ce^{-\frac{\log^2|x|}{2\log q}+(\frac{1}{2}-\log_q(1-q))\log|x|} \tag{3.16}$$

Note also that for sufficiently large n,

$$n < (1-q)^{-n-1} = e^{2(\log_q(1-q))\log r_n}. \tag{3.17}$$

It then follows from (3.15), (3.16) and (3.17) that for sufficiently large n and $|x| = r_n$, we have

$$|f(\alpha, x)| < Ke^{-\frac{\log^2|x|}{2\log q}+(\frac{1}{2}+\log_q(1-q))\log|x|}.$$

The theorem follows then according to Proposition 2.3 with $p = q^{-1}$. ∎

Corollary 3.5. *If $\Re\alpha > -1$, then the series*

$$\sum_{n=0}^{+\infty} S_q(\alpha, n)[x]_n \tag{3.18}$$

converges absolutely in the half plane $\Re x > 1 + 2\log_q(1-q)$.

Proof. — It is easy to see that if $\Re\alpha > -1$, the series

$$\sum_{n=0}^{+\infty}[n]!S_q(\alpha, n)x^n$$

represents an entire function having a q-exponential growth of order 1 and of finite type $(\frac{3}{2}+2\log_q(1-q))$. Hence, by Proposition 2.4 and noting that $[x]_n = [n]!\left[{(x+1)-1 \atop n}\right]$, the proof is complete. ∎

In the rest of this section we shall study the validity of (0.10), *i.e.*, determine the sum of series (3.18). The following lemma is classical (*cf.*, [Ge, p.144]).

Lemma 3.6. *Let $\beta > \sigma$, $0 < \eta < \pi$. Let $f(x)$ be an analytic function in the half plane $\Re x > \sigma$. Assume that $f(\beta + n) = 0$ for all $n \in$ and that there exists $A > 0$ satisfying*

$$|f(\sigma + re^{i\theta})| < Ae^{\eta r} \qquad (r \to +\infty),$$

for all $\theta \in [-\pi/2, \pi/2]$. Then, the function f is identically null.

Theorem 3.7. *If $\Re\alpha > -1$, then we have for $\Re x > 1 + 2\log_q(1-q)$*

$$[x]^\alpha = \sum_{n=0}^{+\infty} S_q(\alpha, n)[x]_n. \tag{3.19}$$

PROOF. — In view of Proposition 1.4 and Corollary 3.5, we see that if $\Re\alpha > -1$, series (3.15) represents a bounded function in every δ-half plane

$$\Re x \geq 1 + 2\log_q(1-q) + \delta, \qquad \delta > 0.$$

Therefore, the difference

$$[x]^\alpha - \sum_{n=0}^{+\infty} S_q(\alpha, n)[x]_n \tag{3.20}$$

is bounded in every δ-half plane. On the other hand, by (3.3) function (3.20) takes value zero at every real integral point $x \in$. The theorem follows then by Lemma 3.6. ∎

Comparing (3.19) with its counterpart for $q = 1$ [BH1], it is reasonable to conjecture that (0.10) has convergence abscissa zero if $\Re\alpha > -1$. By (1.3) or (1.4) this would be proved if the following identity

$$\sum_{n=0}^{N} (-1)^n S_q(\alpha, n)[n]! q^{-\binom{n+1}{2}} = O(q^{-N}) \qquad (N \to +\infty)$$

is true.

4. q-Eulerian function

By (0.9) of Defintion 2, it is easy to verify that the q-Eulerian functions satisfy the following recurrence :

$$A_q(\alpha+1, s) = [s+1]A_q(\alpha, s) + q^s[\alpha+1-s]A_q(\alpha, s-1),$$

where $\alpha \in$ and $s \geq 1$.

The first four q-Eulerian functions are

$$\begin{aligned} A_q(\alpha, 0) &= \delta_{\alpha,0}, \\ A_q(\alpha, 1) &= [2]^\alpha - [\alpha+1], \\ A_q(\alpha, 2) &= [3]^\alpha - [\alpha+1][2]^\alpha + q\begin{bmatrix}\alpha+1\\2\end{bmatrix}, \\ A_q(\alpha, 3) &= [4]^\alpha - [\alpha+1][3]^\alpha + q\begin{bmatrix}\alpha+1\\2\end{bmatrix}[2]^\alpha - q^3\begin{bmatrix}\alpha+1\\3\end{bmatrix}. \end{aligned}$$

Let us introduce for $\Re\alpha > -1$, $|x| < 1$ the generalized q-Eulerian "polynomials"

$$P_\alpha(x; q) = \frac{(x; q)_\infty}{(q^{\alpha+1}x; q)_\infty} \sum_{n=0}^{+\infty} [n+1]^\alpha x^n, \tag{4.1}$$

the radius of convergence of the series being obviously 1 since $\lim_{n\to+\infty} \sqrt[n]{|[n+1]^\alpha|} = 1$.

Lemma 4.1. *For $\Re\alpha > -1$ and $|z| < 1$, the following identity holds*

$$P_\alpha(z;q) = \sum_{s=0}^{+\infty} A_q(\alpha,s)z^s. \tag{4.2}$$

Proof. — By Definition 2, we have for $|z| < 1$

$$\sum_{s=0}^{+\infty} A_q(\alpha,s)z^s = \sum_{i=0}^{+\infty}(-1)^i \begin{bmatrix}\alpha+1\\ i\end{bmatrix} q^{\binom{i}{2}} z^i \cdot \sum_{j=0}^{+\infty}[j+1]^\alpha z^j, \tag{4.3}$$

the two series of the second member being both convergent in the open disc $|z| < 1$. By the q-binomial formula (0.4), we have for $|z| < 1$

$$\sum_{i=0}^{+\infty}(-1)^i \begin{bmatrix}\alpha+1\\ i\end{bmatrix} q^{\binom{i}{2}} x^i = \sum_{i=0}^{+\infty} \frac{(q^{-\alpha-1};q)_i}{(q;q)_i}(xq^{\alpha+1})^i = \frac{(x;q)_\infty}{(xq^{\alpha+1};q)_\infty}.$$

The proof is thus complete by putting this into (4.3). □

Proposition 4.2. *We have for $\alpha \in$ and $n \geq 0$*

$$[n+1]^\alpha = \sum_{k=0}^{n} \begin{bmatrix}\alpha+k\\ k\end{bmatrix} A_q(\alpha, n-k), \tag{4.4}$$

$$\sum_{k=0}^{n} A_q(\alpha,k) = \sum_{k=0}^{n}(-1)^k \begin{bmatrix}\alpha\\ k\end{bmatrix} q^{\binom{k+1}{2}}[n-k+1]^\alpha. \tag{4.5}$$

Proof. — By Lemma 4.1, we have, for $\Re\alpha > -1$ and $|x| < 1$,

$$\frac{(q^{\alpha+1}x;q)_\infty}{(x;q)_\infty}\sum_{s=0}^{+\infty} A_q(\alpha,s)x^s = \sum_{j=0}^{+\infty}[j+1]^\alpha x^j,$$

and also

$$\frac{1}{1-x}\sum_{s=0}^{+\infty} A_q(\alpha,s)x^s = \frac{(qx;q)_\infty}{(q^{\alpha+1}x;q)_\infty}\sum_{j=0}^{+\infty}[j+1]^\alpha x^j.$$

We obtain (4.4) and (4.5) for $\Re\alpha > -1$ by identifying respectively the coefficients of x^n in the above two identities. Finally, the restriction $\Re\alpha > -1$ could be released to $\alpha \in$ by analytic continuation. □

Proposition 4.3. *For $\Re\alpha > -1$, $\Re\beta > 0$, $z \in$ and $|x| < 1$, we have*

$$\sum_{n=0}^{+\infty} \frac{P_{\alpha+n\beta}(x;q)}{(x;q)_{\alpha+n\beta+1}} \frac{z^n}{n!} = \sum_{n=0}^{+\infty} [n+1]^\alpha \exp([n+1]^\beta z) x^n. \tag{4.6}$$

Proof. — By (0.2) and Lemma 4.1, the left side of (4.6) equals, after substitution of (4.2),

$$\sum_{n=0}^{+\infty} \frac{P_{\alpha+n\beta}(x;q)}{(x;q)_{\alpha+n\beta+1}} \frac{z^n}{n!} = \sum_{n=0}^{+\infty} \frac{z^n}{n!} \sum_{j=0}^{+\infty} [j+1]^{\alpha+n\beta} x^j,$$

which proves immediately (4.6) after an exchange of the order of summation. □

We now come to the main result of this section.

Theorem 4.4. *We have*

$$\Gamma_q(x+1) = \sum_{s=0}^{+\infty} A_q(x,s) \qquad \textit{for} \qquad \Re x > -1. \tag{4.7}$$

Proof. — Observe that for $|z| < 1$

$$\frac{(z;q)_\infty}{(zq^{x+1};q)_\infty} = \frac{(zq;q)_\infty}{(zq^{x+1};q)_\infty}(1-z)\cdot$$

So, by Lemma 4.1 we have, for $|z| < 1$,

$$\sum_{s=0}^{+\infty} A_q(x,s) z^s = \frac{(zq;q)_\infty}{(zq^{x+1};q)_\infty} \Big(1 + \sum_{j=1}^{+\infty} \left([j+1]^x - [j]^x\right) z^j\Big). \tag{4.8}$$

On the other hand, since the function $(zq;q)_\infty/(zq^{x+1};q)_\infty$ is analytic for $z \in$ such that

$$|zq| < 1, \quad \left|zq^{1+x}\right| < 1,$$

it is then regular at $z = 1$ if $\Re x > -1$.

Furthermore, note that

$$\lim_{j\to+\infty} \sqrt[j]{|[j+1]^x - [j]^x|} = q \ (< 1),$$

hence the convergence radius of the power series $\sum_{j=1}^{+\infty} \left([j+1]^x - [j]^x\right) z^j$ is equal to q^{-1} (> 1). Therefore, the function defined by (4.8) is actually regular at the point $z = 1$. Finally, since

$$\sum_{j=0}^{+\infty} \left([j+1]^x - [j]^x\right) = \lim_{j\to\infty} [j+1]^x = (1-q)^{-x},$$

we obtain (4.7) by putting $z=1$ in (4.8) and referring to (0.3). ∎

REMARK : When $q\to 1^-$, formula (4.7) *formally* reduces to

$$\Gamma(x+1)=\sum_{s=0}^{+\infty}\sum_{j=0}^{s}(-1)^j(s+1-j)^x\binom{x+1}{j}\qquad(\Re x>-1),$$

which has been respectively established by BUTZER and HAUSS [Bu-Ha] and ZHANG [Zh].

The following proposition gives a relation between the $q-$Stirling and the $q-$Eulerian functions.

Proposition 4.5. *For $\alpha\in$, $|x|<1$ we have*

$$\sum_{k=0}^{+\infty}\frac{S_q(\alpha,k)[k]!x^k}{(x;q)_{k+1}}=\sum_{k=0}^{+\infty}[k]^\alpha x^k=\frac{(xq^{\alpha+1};q)_\infty}{(x;q)_\infty}\sum_{k=0}^{+\infty}A_q(\alpha,k)x^{k+1}.$$

PROOF. — It follows immediately from Lemma 4.1 and Corollary 3.2. ∎

If $\alpha=n\in$, the above formula reduces to the *q-Frobenius formula* (see [Ga]) :

$$\sum_{k=0}^{n}\frac{S_q(n,k)[k]!x^k}{(x;q)_{k+1}}=\sum_{k=0}^{+\infty}[k]^n x^k=\frac{\sum_{\sigma\in\mathfrak{S}_n}x^{\mathrm{des}\,\sigma+1}q^{\mathrm{maj}\,\sigma}}{(x;q)_{n+1}}.\tag{4.9}$$

Proposition 4.6. *For $\alpha\in$, $|x|<1$ we have*

$$A_q(\alpha,n)=(-1)^{n+1}\sum_{k=1}^{n+1}(-1)^k\begin{bmatrix}\alpha-n\\ n+1-k\end{bmatrix}q^{\binom{n-k+2}{2}}[k]!S_q(\alpha,k).\tag{4.10}$$

$$S_q(\alpha,n)=\frac{1}{[n]!}\sum_{k=0}^{n-1}q^{n(n-k-1)}\begin{bmatrix}\alpha-k-1\\ \alpha-n\end{bmatrix}A_q(\alpha,k).\tag{4.11}$$

PROOF. — By Proposition 4.5 and the q-binomial formula (0.4), we have for $|x|<1$

$$\begin{aligned}\sum_{n=0}^{+\infty}A_q(\alpha,k)x^{n+1}&=\sum_{k=1}^{+\infty}\frac{(xq^{k+1};q)_\infty}{(xq^{\alpha+1};q)_\infty}S_q(\alpha,k)[k]!x^k\\ &=\sum_{k=1}^{+\infty}S_q(\alpha,k)[k]!x^k\sum_{l=0}^{+\infty}\frac{(q^{k-\alpha};q)_l}{(q;q)_l}(xq^{\alpha+1})^l.\end{aligned}$$

Formula (4.10) follows then by equating the coefficients of x^n of the above identity.

Next, by (0.8) and (4.4), we have

$$\begin{aligned} S_q(\alpha,n) &= \frac{1}{[n]!}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}}[k]^{\alpha} \\ &= \frac{1}{[n]!}\sum_{k=0}^{n}(-1)^{n-k}\begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}}\sum_{l=0}^{k-1}\begin{bmatrix} \alpha+l\\ l\end{bmatrix} A_q(\alpha,k-l-1) \\ &= \frac{1}{[n]!}\sum_{j=1}^{n}\left\{\sum_{k=j}^{n}(-1)^{n-k}\begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}}\begin{bmatrix} \alpha+k-j\\ k-j\end{bmatrix}\right\} A_q(\alpha,j-1). \end{aligned}$$

We can evaluate the second summation by the *q-Chu-Vandermonde formula* (see [GR, p. 236]) and get

$$\sum_{k=j}^{n}(-1)^{n-k}\begin{bmatrix} n\\ k\end{bmatrix} q^{\binom{n-k}{2}}\begin{bmatrix} \alpha+k-j\\ k-j\end{bmatrix} = q^{n(n-j)}\begin{bmatrix} \alpha-j\\ n-j\end{bmatrix}.$$

Substituting this in the above identity and rescaling $j-1$ to j yield (4.11). □

We close this paper by pointing out that it is possible to generalize the formulae of this section to the multiple index's Eulerian functions as one did for the multiple index's Eulerian numbers (see [Raw] and [FZ]). For exemple, we can generalize (4.6) in the following manner.

Let $\mathbf{x}=(x_1,x_2,\ldots,x_m)$ be a sequence of complex numbers. Define the *multiple q-Eulerian functions* $A_q(\mathbf{x},s)$ by

$$A_q(\mathbf{x},s)=\sum_{l=0}^{s}(-1)^l\begin{bmatrix} x_1+\cdots+x_m+1\\ l\end{bmatrix} q^{\binom{l}{2}}\prod_{k=1}^{m}\begin{bmatrix} s+x_k-l\\ s-l\end{bmatrix}. \tag{4.12}$$

Theorem 4.7. *If* $\Re x_i>-1$ $(i=1,2,\ldots,m)$ *and* $\Re(x_1+\cdots+x_m)>-1$, *then we have*

$$\frac{\Gamma_q(x_1+x_2+\cdots+x_m+1)}{\Gamma_q(x_1+1)\Gamma_q(x_2+1)\cdots\Gamma_q(x_m+1)}=\sum_{s=0}^{+\infty}A_q(\mathbf{x},s). \tag{4.13}$$

Proof. — By definition (4.12) and the q-binomial formula, we have

$$\begin{aligned} \sum_{s=0}^{+\infty}A_q(\mathbf{x},s)z^s &= \sum_{l=0}(-1)^l\begin{bmatrix} x_1+\cdots+x_m+1\\ l\end{bmatrix} q^{\binom{l}{2}}z^l\sum_{s=0}^{+\infty}\prod_{k=1}^{m}\begin{bmatrix} s+x_k\\ s\end{bmatrix} z^s \\ &= \frac{(z;q)_\infty}{(zq^{x_1+\cdots+x_m+1};q)_\infty}\sum_{s=0}^{+\infty}\prod_{k=1}^{m}\begin{bmatrix} s+x_k\\ n_k\end{bmatrix} z^s \\ &= \frac{(zq;q)_\infty}{(zq^{x_1+\cdots+x_m+1};q)_\infty}\sum_{s=0}^{+\infty}\left(\prod_{k=1}^{m}\begin{bmatrix} s+1+x_k\\ n_k\end{bmatrix}-\prod_{k=1}^{m}\begin{bmatrix} s+x_k\\ n_k\end{bmatrix}\right)z^s. \end{aligned}$$

As in the proof of Theorem 4.4, the above series defines a function which is regular at $z = 1$. So putting $z = 1$ in the above identity yields

$$\sum_{s=0}^{+\infty} A_q(\mathbf{x}, s) = \frac{(q;q)_\infty}{(q^{x_1+\cdots+x_m+1};q)_\infty} \prod_{k=1}^{m} \frac{(q^{1+x_k};q)_\infty}{(q;q)_\infty},$$

which is clearly equivalent to (4.13). ∎

Acknowledgement : The authors are indebted to Professor Butzer for sending them the preprint [BH2] before its publication and the anonymous referer for the suggestions on the first version of this paper.

REFERENCES

[BH1] Butzer (P.) and Hauss (M.). — *On Stirling functions of the second kind*, Stud. Appl. Math., t. **84**, 1991, p. 71–91.

[BH2] Butzer (P.) and Hauss (M.). — *Eulerian numbers with fractional order parameters*, Aequationes Math., t. **1/2**, 1993, p. 119–142.

[BHS] Butzer (P.), Hauss (M.) and Schmidt (M.). — *Factorial functions and Stirling numbers of fractional orders*, Results in Math., t. **16**, 1989, p. 16–48.

[Ca1] Carlitz (L.). — *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., t. **76**, 1954, p. 332–350.

[Ca2] Carlitz (L.). — *A note on q-Eulerian numbers*, J. Comb. Th. A, t. **25**, 1978, p. 90–94.

[Co] Comtet (L.). — *Analyse Combinatoire (Vol. 2)*. — Presses Universitaires de France, Paris, 1970.

[FZ] Foata (D.) and Zeilberger (D.). — *Multibasic Eulerian polynomials*, Trans. Amer. Math. Soc., t. **328**, 1991, p. 843–862.

[Ga] Garsia (A.). — *On the "maj" and "inv" q-analogues of Eulerian numbers*, J. Lin. Multilin. Alg., t. **8**, 1979, p. 21–34.

[GR] Gasper (G.) and Rahman (M.). — *Basic Hypergeometric Series*. — Cambridge Univ. Press, 1990.

[Ge] Gelfond (A.O.). — *Calcul des différences finies*. — Dunod, Paris, 1963.

[Kn] Knopp (K.). — *Theory and Application of Infinite Series*. — Dover Publications, INC., New York, 1990.

[Ko] Koornwinder (T. H.). — *Jacobi functions as limit cases of q-Ultraspherical polynomials*, J. Math. Ana. Appl., t. **148**, 1990, p. 44–54.

[Mi] Milne–thomson (L. M.). — *The calculus of finite differences.* — Macmillan and Co Ltd., New York, 1960.

[Ram] Ramis (J. P.). — *About the growth of entire functions solutions of linear algebraic $q-$difference equations*, Annales de la Faculté des Sciences de Toulouse, Série 6, t. **1**, 1992, p. 53–94.

[Raw] Rawlings (D.). — Permutation and Multipermutation statistics, *Publ. de l'IRMA*, Strasbourg, 1979/P–23.

[Ti] Titchmarsh (E. C.). — *The Theory of Functions* (Second Edition). — Oxford Univ. Press, London - New York, 1948.

[Wa] Wallisser (R.). — *Über ganze Funktionen, die in einer geometrischen Folge ganze Werte annehmen*, Monatsh. für Math., t. **100**, 1985, p. 329–335.

[Ze] Zeng (J.). — *The q-Stirling numbers, continued fractions and the q-Charlier and q-Laguerre polynomials*, J. of Comp. and Applied. Math., *to appear*.

[Zh] Zhang (C.). — *La fonction Gamma considérée comme la somme de nombres eulériens généralisés*, Europ. J. Combi., *to appear*.

Eingegangen am 5. Juli 1993

Institut de Recherche Mathématique Avancée
Université Louis Pasteur et CNRS
7, rue René Descartes
67084 Strasbourg Cedex
France