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ON q-ANALOGUES OF STIRLING SERIES

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ABSTRACT. In this short note, we construct another form of Stirling's asymptotic series by new form of Carlitz's q-Bernoulli numbers.

0. Introduction

We use the notation

$$[x] = [x; q] = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \to 1}[x; q] = x$, for any $x \in \mathbb{C}$.

Carlitz's q-Bernoulli numbers $\beta_k = \beta_k(q)$ can be determined inductively by

$$\beta_0 = 1, \quad q(q\beta + 1)^k - \beta_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing β^i by β_i. Let u be a complex number with $|u| > 1$, then the Carlitz's q-Euler numbers $H_k(u; q)$ is defined inductively by

$$H_0(u; q) = 1, \quad (qH + 1)^k - uH_k(u; q) = 0 \tag{1}$$

for $k \geq 1$, with the usual convention of replacing H^k by $H_k(u, q)$. We remark that Carlitz's q-Euler numbers are reduced to the ordinary Euler numbers $H_k(u)$ when $q = 1$.

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Let $q, u \in \mathbb{C}$ with $|q| < 1$, $|u| > 1$. For $s \in \mathbb{C}$, the complex function $l_q(s, u)$ is constructed by

$$l_q(s,u) = \sum_{n=1}^{\infty} \frac{u^{-n}}{[n]^s}. \tag{2}$$

For any positive integer k, it can be found in [2] that

$$l_q(-k,u) = \begin{cases} \frac{1}{u-1}, & \text{if } k = 0, \\ \frac{u}{u-1} H_k(u,q), & \text{if } k \geq 1. \end{cases} \tag{3}$$

In [6], q-Riemann ζ-function was defined by

$$\zeta_q(s) = \frac{2-s}{s-1}(1-q)\sum_{n=1}^{\infty} \frac{q^n}{[n]^{s-1}} + \sum_{n=1}^{\infty} \frac{q^n}{[n]^s}.$$

In this paper, we will give new relations on Carlitz's q-Bernoulli numbers and new properties of the function $G_q(x)$ in [2].

1. Relations on Carlitz's q-Bernoulli numbers

The new form of Carlitz's q-Bernoulli numbers $\mathcal{B}_k = \mathcal{B}_k(u;q)$ can be determined inductively by

$$\mathcal{B}_0(u;q) = 1, \;\; u^{-1}(q\mathcal{B}+1)^k - \mathcal{B}_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing $\mathcal{B}^i$ by $\mathcal{B}_i$. The new form of Carlitz's q-Bernoulli numbers are reduced to the ordinary Carlitz's q-Bernoulli numbers $\beta_k(q)$ when $u^{-1} = q$. We shall explicitly determine the generating function $F_{u;q}(t)$ of $\mathcal{B}_k$:

$$F_{u;q}(t) = \sum_{k=0}^{\infty} \mathcal{B}_k(u;q) \frac{t^k}{k!}.$$

This is the unique solution of the following (u;q)-difference equation

$$F_{u;q}(t) = u^{-1} e^t F_{u;q}(qt) + 1 - u^{-1} - t. \tag{4}$$

Indeed, if the power series $\sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ satisfies (4), then

$$\begin{aligned}
&\sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \\
&= u^{-1} e^t \sum_{k=0}^{\infty} q^k a_k \frac{t^k}{k!} + 1 - u^{-1} - t \qquad (5)\\
&= u^{-1} a_0 + 1 - u^{-1} + \{u^{-1}(qa_1 + 1) - 1\}t + \sum_{k=2}^{\infty} u^{-1}(qa+1)^k \frac{t^k}{k!},
\end{aligned}$$

where a^i means a_i. By comparing the coefficients of both sides of (5), we see that a_k satisfies the recurrence formula of the new form of Carlitz's q-Bernoulli numbers.

Lemma 1. *For $u, q, t \in \mathbb{C}$ with $|u| > 1,\ |q| < 1$,*

$$F_{u;q}(t) = \sum_{n=0}^{\infty} u^{-n} e^{[n]t} (1 - u^{-1} - q^n t). \qquad (6)$$

Proof. It is easy to show that the right side of (6) satisfies (4). □

Note that the series on the right hand side of (6) is uniformly convergent in a wider sense. For $k \geq 0$, we have

$$\begin{aligned}
&\mathcal{B}_k(u; q) \\
&= \frac{d^k}{dt^k} F_{u;q}(t) \bigg|_{t=0} \qquad (7)\\
&= (1 - u^{-1}) \sum_{n=0}^{\infty} u^{-n} [n]^k - k \sum_{n=0}^{\infty} u^{-n} q^n [n]^{k-1} \\
&= (1 - u^{-1}) \sum_{n=0}^{\infty} u^{-n} [n]^k + (1-q)k \sum_{n=0}^{\infty} u^{-n} [n]^k - k \sum_{n=0}^{\infty} u^{-n} [n]^{k-1},
\end{aligned}$$

since we used the relation $\frac{q^n}{[n]} = \frac{1}{[n]} + q - 1$.

Hence we can define the similar form of q-analogue ζ-function as follows:

Definition 1. For $s \in \mathbb{C}$,

$$\zeta_q(u;s) = \frac{1-u^{-1}}{s-1}\sum_{n=1}^{\infty}\frac{u^{-n}}{[n]^{s-1}} + (q-1)\sum_{n=1}^{\infty}\frac{u^{-n}}{[n]^{s-1}} + \sum_{n=1}^{\infty}\frac{u^{-n}}{[n]^{s}}. \tag{8}$$

Note that $\zeta_q(u;s)$ is a meromorphic function on $\mathbb{C}$ with only one simple pole at $s=1$. The values of $\zeta_q(u;s)$ at non-positive integers are obtained by the following theorem.

Theorem 1. *For any positive integer k, we have*

$$\zeta_q(u;1-k) = \begin{cases} \frac{1-q}{u-1}\mathcal{B}_1(u;q), & \text{if } k=1, \\ -\frac{\mathcal{B}_k(u;q)}{k}, & \text{if } k>1. \end{cases}$$

Proof. It is clear by (7) and (8). □

If $u = q^{-1}$ then we obtain the following:

Corollary 1. *For any positive integer k, we have*

$$\zeta_q(1-k) = \begin{cases} q\beta_1(q), & \text{if } k=1, \\ -\frac{\beta_k(q)}{k}, & \text{if } k>1. \end{cases}$$

2. Results

The Stirling's asymptotic series

$$\log\frac{\Gamma(x)}{\sqrt{2\pi}} \approx (x-\frac{1}{2})\log x - x + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}B_{k+1}\frac{1}{x^k}$$

is well known. Here, we treat similar form of q-analogue of the above formula in [2].

We have the function from [2];

$$G_{u;q}(x) = \sum_{n=0}^{\infty} u^{-n}(x+[n])\log(x+[n]), \tag{9}$$

for $x \in \mathbb{C}$ with $|u|>1$, $|q|<1$. Then $G_{u;q}(x)$ is a locally analytic function on $\mathbb{C}$.

Lemma 2 [2].

(1) *For* $x \in \mathbb{C}$ *with* $|x| > \frac{1}{(1-|q|)^2}$, *we have*

$$G_{u,q}(x) = \frac{1}{1-u^{-1}}\left(x\log x + \frac{1}{u-q}(\log x + 1) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{x^k}\right).$$

(2) *Let* ${G'}_{u;q}(x) = \frac{d}{dx}G_{u;q}(x)$. *Then*

$${G'}_{u;q}(x) = \frac{1}{1-u^{-1}}(\log x + 1) + \frac{1}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{x^k}.$$

From the Theorem 1, (3) and (8), for $k \geq 2$ we have

$$\begin{aligned}
&-\frac{\mathcal{B}_k(u;q)}{k} \\
=&\zeta_q(u;1-k) \\
=&-\frac{1-u^{-1}}{k}\sum_{n=1}^{\infty}u^{-n}[n]^k + (q-1)\sum_{n=1}^{\infty}u^{-n}[n]^k + \sum_{n=1}^{\infty}u^{-n}[n]^{k-1} \\
=&-\left(\frac{1-u^{-1}}{k} + (1-q)\right)\frac{1}{1-u^{-1}}H_k(u;q) + \frac{1}{1-u^{-1}}H_{k-1}(u;q) \\
=&-\left(\frac{1}{k} + \frac{1-q}{1-u^{-1}}\right)H_k(u;q) + \frac{1}{1-u^{-1}}H_{k-1}(u;q).
\end{aligned}$$

Hence we obtain the following:

Lemma 3. *For integer* $k \geq 2$, *we have*

$$-\frac{\mathcal{B}_k(u;q)}{k} = -\left(\frac{1}{k} + \frac{1-q}{1-u^{-1}}\right)H_k(u;q) + \frac{1}{1-u^{-1}}H_{k-1}(u;q).$$

Now we define the function $G_q(u;x)$ by

$$\begin{aligned}
G_q(u;x) =&(qx - x - 1){G'}_{u;q}(x) + (2 - q - u^{-1})G_{u;q}(x) \\
&+ \frac{1-u^{-1}}{1-qu^{-1}} + \frac{1-q}{1-u^{-1}}H_1(u;q).
\end{aligned}$$

Then $G_q(u;x)$ is a locally analytic function on $\mathbb{C}$. If $u = q^{-1}$ then $G_q(q^{-1},x) = G_q(x)$ which is the function $G_q(x)$ in [2].

By Lemma 2, we have

$$
\begin{aligned}
&(qx-x-1){G'}_{u;q}(x) + (2-q-u^{-1})G_{u;q}(x)\\
=&-(1-q)x{G'}_{u;q}(x) - {G'}_{u;q}(x) + (1-q)G_{u;q}(x) + (1-u^{-1})G_{u;q}(x)\\
=&-(1-q)x\left\{\frac{1}{1-u^{-1}}(\log x+1) + \frac{1}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{x^k}\right\}\\
&-\left(\frac{1}{1-u^{-1}}(\log x+1) + \frac{1}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{x^k}\right)\\
&+(1-q)\left\{\frac{1}{1-u^{-1}}\left(x\log x + \frac{1}{u-q}(\log x+1)\right.\right.\\
&\qquad\qquad\qquad\left.\left.+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{x^k}\right)\right\}\\
&+(1-u^{-1})\left\{\frac{1}{1-u^{-1}}\left(x\log x + \frac{1}{u-q}(\log x+1)\right.\right.\\
&\qquad\qquad\qquad\left.\left.+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{x^k}\right)\right\}\\
=&-\frac{1-q}{1-u^{-1}}x\log x - \frac{1-q}{1-u^{-1}}x - \frac{1}{1-u^{-1}}\log x - \frac{1}{1-u^{-1}}\\
&+\frac{1-q}{1-u^{-1}}x\log x + \frac{1-q}{1-u^{-1}}\frac{1}{u-q}\log x + \frac{1-q}{1-u^{-1}}\frac{1}{u-q}\\
&+x\log x + \frac{1}{u-q}\log x + \frac{1}{u-q}\\
&-\frac{1-q}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{x^{k-1}}\\
&-\frac{1}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{x^k}\\
&+\frac{1-q}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{x^k} + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{x^k}
\end{aligned}
$$

$$
\begin{aligned}
=&x\log x-\left(\frac{1}{1-u^{-1}}-\frac{1-q}{1-u^{-1}}\frac{1}{u-q}-\frac{1}{u-q}\right)\log x-\frac{1-q}{1-u^{-1}}x\\
&-\left(\frac{1}{1-u^{-1}}-\frac{1-q}{1-u^{-1}}\frac{1}{u-q}-\frac{1}{u-q}\right)-\frac{1-q}{1-u^{-1}}H_1(u;q)\\
&+\frac{1-q}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k+1}\frac{k}{k}H_{k+1}(u;q)\frac{1}{x^k}\\
&-\frac{1}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{x^k}\\
&+\frac{1-q}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{x^k}+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{x^k}\\
=&x\log x-\frac{1-u^{-1}}{1-qu^{-1}}\log x-\frac{1-q}{1-u^{-1}}x-\frac{1-u^{-1}}{1-qu^{-1}}-\frac{1-q}{1-u^{-1}}H_1(u;q)\\
&+\frac{1-q}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_{k+1}(u;q)\frac{1}{x^k}\\
&+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{x^k}-\frac{1}{1-u^{-1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{x^k}\\
=&\left(x-\frac{1-u^{-1}}{1-qu^{-1}}\right)\log x-\frac{1-q}{1-u^{-1}}x-\frac{1-u^{-1}}{1-qu^{-1}}-\frac{1-q}{1-u^{-1}}H_1(u;q)\\
&+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left\{\frac{1-q}{1-u^{-1}}H_{k+1}(u;q)\right.\\
&\qquad\qquad\left.+\frac{1}{k+1}H_{k+1}(u;q)-\frac{1}{1-u^{-1}}H_k(u;q)\right\}\frac{1}{x^k}.
\end{aligned}
$$

By Lemma 3, we have

$$
\begin{aligned}
&G_q(u;x)\\
&=\left(x-\frac{1-u^{-1}}{1-qu^{-1}}\right)\log x-\frac{1-q}{1-u^{-1}}x+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\mathcal{B}_{k+1}(u;q)\frac{1}{x^k}.
\end{aligned}
$$

Hence we obtain the following:

Theorem 2. *For* $x \in \mathbb{C}$ *with* $|x| > \frac{1}{(1-|q|)^2}$

$$G_q(u;x) = \left(x - \frac{1-u^{-1}}{1-qu^{-1}}\right)\log x - \frac{1-q}{1-u^{-1}}x + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\mathcal{B}_{k+1}(u;q)\frac{1}{x^k}.$$

If $u = q^{-1}$, then we obtain the following:

Corollary 1 [2]. *For* $x \in \mathbb{C}$ *with* $|x| > \frac{1}{(1-|q|)^2}$

$$G_q(x) = \left(x - \frac{1}{[2]}\right)\log x - x + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\beta_{k+1}(q)\frac{1}{x^k}.$$

Now, we give another result.

We define the function $\mathcal{G}_{u;q}(x)$ which is the q-analogue of x in (9):

$$\mathcal{G}_{u;q}(x) = \sum_{n=0}^{\infty} u^{-n}([x]+[n])\log([x]+[n]),$$

for $x \in \mathbb{C}$, with $|u| > 1$, $|q| < 1$. Then $\mathcal{G}_{u;q}(x)$ is a locally analytic function on $\mathbb{C}$.

For $|[x]| > \frac{1}{(1-|q|)^2}$, we have

$$\begin{aligned}
&\mathcal{G}_{u;q}(x)\\
&= \sum_{n=1}^{\infty} u^{-n}([x]+[n])\log([x]+[n])\\
&= \sum_{n=1}^{\infty} u^{-n}([x]+[n])\log[x] + \sum_{n=1}^{\infty} u^{-n}([x]+[n])\log\left(1+\frac{[n]}{[x]}\right) + [x]\log[x]\\
&= \left(\sum_{n=1}^{\infty} u^{-n}\right)[x]\log[x] + \left(\sum_{n=1}^{\infty} u^{-n}[n]\right)\log[x]\\
&\quad + \sum_{n=1}^{\infty} u^{-n}\left([n] + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\frac{[n]^{k+1}}{[x]^k}\right) + [x]\log[x]\\
&= \frac{1}{u-1}[x]\log[x] + \frac{u}{u-1}H_1(u;q)\log[x] + \frac{u}{u-1}H_1(u;q)\\
&\quad + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\frac{u}{u-1}H_{k+1}(u;q)\frac{1}{[x]^k} + [x]\log[x].
\end{aligned}$$

Since $H_1(u;q)=\frac{1}{u-q}$, we have

$$\begin{aligned}
&\mathcal{G}_{u;q}(x)\\
&=\frac{u}{u-1}[x]\log[x]+\frac{u}{u-1}\left(\frac{1}{u-q}(1+\log[x])\right)\\
&\quad+\frac{u}{u-1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{[x]^k}\\
&=\frac{u}{u-1}\left\{[x]\log[x]+\frac{1}{u-q}(1+\log[x])+\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{[x]^k}\right\}.
\end{aligned}$$

Let $\mathcal{G}'_{u;q}(x)=\frac{d}{dx}\mathcal{G}_{u;q}(x)$, then we have

$$\begin{aligned}
&\mathcal{G}'_{u;q}(x)\\
&=\frac{u}{u-1}\Bigg\{-\frac{q^x}{1-q}\log q\log[x]+\left(-\frac{q^x}{1-q^x}\log q\right)[x]\\
&\qquad\qquad+\frac{1}{u-q}\left(-\frac{q^x}{1-q^x}\log q\right)\Bigg\}\\
&\quad+\frac{u}{u-1}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\left(\frac{q^x}{1-q}k\log q\frac{1}{[x]^{k+1}}\right)\\
&=-\frac{u}{u-1}\frac{q^x}{1-q}\log q(\log[x]+1)-\frac{u}{u-1}\frac{q^x}{1-q}\log qH_1(u;q)\frac{1}{[x]}\\
&\quad-\frac{u}{u-1}\frac{q^x}{1-q}\log q\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{[x]^k}\\
&=-\frac{u}{u-1}\frac{q^x}{1-q}\log q(\log[x]+1)\\
&\quad-\frac{u}{u-1}\frac{q^x}{1-q}\log q\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{[x]^k}.
\end{aligned}$$

Therefore we obtain the following:

Proposition 1.

(1) *For* $x \in \mathbb{C}$ *with* $|[x]| > \frac{1}{(1-|q|)^2}$, *we have*

$$\mathcal{G}_{u;q}(x) = \frac{1}{1-u^{-1}}\left\{[x]\log[x] + \frac{1}{u-q}(\log[x]+1) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}H_{k+1}(u;q)\frac{1}{[x]^k}\right\}.$$

(2)

$$\mathcal{G}'_{u;q}(x) = -\frac{1}{1-u^{-1}}\frac{q^x}{1-q}\log q\left\{(\log[x]+1) + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}H_k(u;q)\frac{1}{[x]^k}\right\}.$$

Now we rewrite the function $\mathcal{G}_q(u;x)$ by

$$\mathcal{G}_q(u;x) = (q[x]-[x]-1)\mathcal{G}'_{u;q}(x) - \frac{q^x}{1-q}\log q(2-q-u^{-1})\mathcal{G}_{u;q}(x) - \frac{q^x}{1-q}\log q\left(\frac{1-u^{-1}}{1-qu^{-1}} + \frac{1-q}{1-u^{-1}}H_1(u;q)\right).$$

Then $\mathcal{G}_q(u;x)$ is a locally analytic function on $\mathbb{C}$. By Theorem 2, we have

$$\begin{aligned}
&\mathcal{G}_q(u;x)\\
&= (q[x]-[x]-1)\mathcal{G}'_{u;q}(x) - \frac{q^x}{1-q}\log q(2-q-u^{-1})\mathcal{G}_{u;q}(x)\\
&\quad - \frac{q^x}{1-q}\log q\left(\frac{1-u^{-1}}{1-qu^{-1}} + \frac{1-q}{1-u^{-1}}H_1(u;q)\right)\\
&= -\frac{q^x}{1-q}\log q\Big\{(q[x]-[x]-1)G'_{u;q}([x]) + (2-q-u^{-1})G_{u;q}([x])\\
&\qquad + \frac{1-u^{-1}}{1-qu^{-1}} + \frac{1-q}{1-u^{-1}}H_1(u;q)\Big\}
\end{aligned}$$

$$= -\frac{q^x}{1-q}\log q\; G_q(u;[x])$$
$$= -\frac{q^x}{1-q}\log q\Bigg\{\left([x]+\frac{1-u^{-1}}{1-qu^{-1}}\right)\log[x]-\frac{1-q}{1-u^{-1}}[x] + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\mathcal{B}_{k+1}(u;q)\frac{1}{[x]^k}\Bigg\}.$$

Hence we obtain the following:

THEOREM 3. *For $x \in \mathbb{C}$ with $|[x]| > \frac{1}{(1-|q|)^2}$ we have*

$$\mathcal{G}_q(u;x) = -\frac{q^x}{1-q}\log q\Bigg\{\left([x]+\frac{1-u^{-1}}{1-qu^{-1}}\right)\log[x]-\frac{1-q}{1-u^{-1}}[x] + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\mathcal{B}_{k+1}(u;q)\frac{1}{[x]^k}\Bigg\}.$$

Note that if $u = q^{-1}$ and $q \to 1$, then we have

$$\begin{aligned}
&\lim_{q\to 1}\mathcal{G}_q(q^{-1},x)\\
&= \lim_{q\to 1}\Bigg\{-\frac{q^x}{1-q}\log q\Bigg(\left([x]+\frac{1}{[2]}\right)\log[x]-[x] + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}\beta_{k+1}(q)\frac{1}{[x]^k}\Bigg)\Bigg\}\\
&= \left(x-\frac{1}{2}\right)\log x - x + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}B_{k+1}\frac{1}{x^k},
\end{aligned}$$

which is the well known Stirling asymptotic series.

References

[1] Jack Diamond, *The p-adic log Gamma function and p-adic Euler constants*, Trans. Amer. Math. Soc. **233** (1977), 321-337.

[2] Han Soo Kim and T. Kim, *On q-log-Gamma-Function*, Bull. Korean Math. Soc. **33** (1996), no. 1, 111-118.

[3] N. Kobilitz, *q-extension of the p-adic Gamma function*, Trans. Amer. Math. Soc. **260** (1980), 449-457.

[4] ———, *p-adic analysis: a short course on recent work*, Mathematical Society Lecture Notes, Series 46, London: Cambridge Univ. Press, 1980.

[5] T. Kim, *On explicit formulas of p-adic q-L-function*, Kyushu J. Math. **48** (1994), 73-86.

[6] Junya Satoh, *q-Analogue of Riemann's zeta-Function and q-Euler Numbers*, J. of Number Theory **31** (1989), 346-362.

[7] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th Ed., Cambridge University Press, 1964.

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