On ψ-umbral extensions of Stirling numbers and Dobinski-like formulas

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February 1, 2008

Abstract

A so called ψ-umbral extensions of the Stirling numbers of the second kind are considered and the resulting Dobinski-like various formulas - including *new* ones - are presented. These extensions naturally encompass the two well known q-extensions. The further consecutive *ψ- umbral extensions* of Carlitz-Gould-Milne q-Stirling numbers are therefore realized here in a two-fold way. The fact that the umbral q-extended Dobinski formula may also be interpreted as the average of powers of random variable X_q with the q-Poisson distribution singles out the q-extensions which appear to be a kind of "singular point" in the domain of ψ-umbral extensions as expressed by Observations 2.1 and 2.2. Other relevant possibilities are tackled with the paper's closing down questions and suggestions with respect to other already existing extensions while a brief limited survey of these other type extensions is being delivered. There the Newton interpolation formula and divided differences appear helpful and inevitable along with umbra symbolic language in describing properties of general exponential polynomials of Touchard and their possible generalizations. Exponential structures or algebraically equivalent prefabs with their exponential formula appear to be also naturally relevant.

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1. In the q-extensions realm

At first let us make a remark on notation (see also Appendix). ψ is a number or functions' sequence - sequence of functions of a parameter q. ψ denotes an extension of $\langle \frac{1}{n!} \rangle_{n \geq 0}$ sequence to quite arbitrary one (the so called - "admissible" [1, 2]). The specific choices are for example : Fibonomialy-extended sequence $\langle \frac{1}{F_n!} \rangle_{n \geq 0}$ ($\langle F_n \rangle$ - Fibonacci sequence) or just "the usual" ψ-sequence $\langle \frac{1}{n!} \rangle_{n \geq 0}$ or Gauss q-extended $\langle \frac{1}{n_q!} \rangle_{n \geq 0}$ admissible sequence of extended umbral operator calculus, where $n_q = \frac{1-q^n}{1-q}$ and $n_q! = n_q(n-1)_q!, 0_q! = 1$ - see more below. With such type extension we may "ψ-mnemonic" repeat with exactly the same simplicity this what was done by Rota forty one years ago. Namely forty one years ago Gian-Carlo Rota [3] proved that the exponential generating function for Bell numbers B_n is of the form

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} B_n = \exp(e^x - 1) \tag{1}$$

using the linear functional L such that

$$L(X^{\underline{n}}) = 1, \qquad n \geq 0. \tag{2}$$

Bell numbers (see: formula (4)in [3]) are then defined by

$$L(X^n) = B_n, \qquad n \geq 0. \tag{3}$$

The above formula is exactly the Dobinski formula [4] if L is interpreted as the average value functional for the random variable X with the Poisson distribution where $L(X) = 1$. As a matter of fact it is Blissard calculus inspired umbral formula [3] (see [5] for umbral nature of Poisson random variables and the introduction in [5] for historical remarks on Blissard's calculus roots). On this occasion let us recall that the Stirling numbers of the second kind are relatives of the Poisson distribution in the known way. Namely if X is a random variable with a Poisson distribution with expected value λ, then its $n - th$ moment is the exponential polynomial φ_n value at λ i.e.

$$E(X^n) = \varphi_n(\lambda) = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} \lambda^k.$$

Hence in particular, the $n - th$ moment of the Poisson distribution with expected value 1 is precisely the number of partitions of a set of size n i.e. it is the $n - th$ Bell number (this fact is Dobinski's formula as stated by

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the formula (3)). The formula (3) is tempting to be ψ-extended somehow as the ψ-Poisson process distribution is known [2, 1]. Before doing this let us remind that recently an interest to extensions of Stirling numbers and consequently to Bell numbers was revived among "q-coherent states physicists" [6, 7, 8] with several important generalizations already at hand such as in [9, 10, 11]. The merit of such applications is in that the expectation value with respect to coherent state $|\gamma>$ with $|\gamma| = 1$ of the n-th power of the number of quanta operator [6, 7, 8] is "just" the n-th Bell number B_n and the explicit formula for this expectation number of quanta is "just" Dobinski formula [6]. The same holds for q-coherent states case [6] i.e. the expectation value with respect to q-coherent state $|\gamma>$ with $|\gamma| = 1$ of the n-th power of the number operator is the n-th q-Bell number [8, 6] defined as the sum of q-Stirling numbers $\left\{ {n \atop k} \right\}_q$ introduced by Carlitz and Gould and recently exploited among others in [12, 6, 7, 8]. Note there then that for the **two standard** [12] q**-extensions** of the Stirling numbers of the second kind we have as the **first** ones the q-Stirling numbers:

$$x_q^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q x_q^{\underline{k}},\tag{4}$$

where $x_q = \frac{1-q^x}{1-q}$ and $x_q^{\underline{k}} = x_q(x-1)_q...(x-k+1)_q$ and then the **second** ones called $q^\sim$-Stirling numbers. Both correspond to the ψ sequence choice in the q-Gauss form $\langle \frac{1}{n_q!} \rangle_{n \geq 0}$. Here the $q^\sim$-Stirling numbers of the second kind are introduced as coefficients in the famous Newton interpolation formula (Liber III, Lemma V, pp. 481-483 in [13]) now applied to the polynomial sequence $\langle e_n \rangle_{n \geq 0}, \quad e_n(x) = x^n, n \geq 0$, i.e.

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q^\sim \chi_{\underline{k}}(x), \quad i.e. \quad \left\{ {n \atop k} \right\}_q^\sim = [0, 1_q, 2_q, ..., k_q; e_n],\tag{5}$$

where $\chi_{\underline{k}}(x) = x(x - 1_q)(x - 2_q)...(x - [k - 1]_q)$, and

$$[x_0, x_1, x_2, ..., x_k; f] = \frac{[x_0, x_1, x_2, ..., x_{k-1}; f] - [x_1, x_2, ..., x_k; f]}{x_0 - x_k}$$

denotes the $k - th$ divided difference with

$$[x_0, x_1; f] = \frac{f(x_0) - f(x_1)}{x_0 - x_1}.$$

These two kinds of now classical q-extensions of Stirling numbers of the second kind as defined by (4) and (5) are related in a simple way through re-scaling [14]. They satisfy the known respective recurrences:

$$\left\{ {n+1 \atop k} \right\}_q = \sum_{l=0}^{n} \binom{n}{l}_q q^l \left\{ {l \atop k-1} \right\}_q; n \geq 0, k \geq 1,$$

$$\left\{ {n+1 \atop k} \right\}_q^{\sim} = \sum_{l=0}^{n} \binom{n}{l}_q q^{l-k+1} \left\{ {l \atop k-1} \right\}_q^{\sim}; n \geq 0, k \geq 1.$$

From the above it follows immediately that corresponding q-extensions of B_n Bell numbers satisfy respective recurrences on their own:

$$B_q(n+1) = \sum_{l=0}^{n} \binom{n}{l}_q q^l B_q(l); n \geq 0,$$

$$B_q^{\sim}(n+1) = \sum_{l=0}^{n} \binom{n}{l}_q q^{l+1} \overline{B}_q^{\sim}(l), n \geq 0$$

where

$$\overline{B}_q^{\sim}(l) = \sum_{k=0}^{l} q^k \left\{ {l \atop k} \right\}_q^{\sim}.$$

Different definitions via (4) and (5) equations correspond consequently to different q-counting [14]. With any other choice out of countless choices of the ψ sequence the equation (5) becomes just the definition of $\psi^{\sim}$-Stirling (vide "Fibonomial-Stirling") numbers of the second kind $\left\{ {n \atop k} \right\}_\psi^{\sim}$ and then $\psi^{\sim}$-Bell numbers $B_n^{\sim}(\psi)$ are defined as usual as sums of the corresponding Stirling-like numbers - where now $\chi_{\underline{k}}(x)$ in (5) is to be replaced by $\psi_{\underline{k}}(x) = x(x-1_\psi)(x-2_\psi)...(x-[k-1]_\psi)$. These $\psi^{\sim}$-Stirling numbers of the second kind for q case identified as Comtet numbers in Wagner's terminology [15, 14] satisfy familiar recursion and are given by familiar formulas to be presented soon. The extension of definition (4) of the q- Stirling numbers of the second kind beyond this q-case i.e. beyond the $\psi = \langle \frac{1}{n_q!} \rangle_{n \geq 0}$ choice is not that mnemonic at all and the problem of recursion appears. This part of alternative treatment is to be considered later on after we exploit a little bit more some consequences of (4). Namely - due to (4) one immediately notices that the expectation value with respect to q-coherent state $|\gamma >$ with $|\gamma| = 1$ of the n-th power of the number operator is exactly the popular q-Dobinski formula which can be given Blissard calculus inspired umbral form - like in (3). It is enough to apply to (4) L_q - the average value

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functional for q-Poisson distribution [1, 2]. The formula thus obtained may be also treated as a definition of q-extended Bell numbers $B_n(q)$

$$L_q(X_q^n) = B_n(q), \qquad n \geq 0 \tag{6}$$

due to the fact that this linear functional L_q interpreted as the average value functional for the random variable X_q with the q-Poisson distribution [1] ($L_q(X_q) = 1$) satisfies

$$L_q(X_q^{\frac{n}{q}}) = 1, \qquad n \geq 0. \tag{7}$$

Then with the q-exponential polynomials

$$\varphi_n(x, q) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q x^k \qquad (q - exp - pol - I)$$

one obtains for $x = 1$ in correspondence with $L_q(X_q) = 1$ the q - formula of Dobinski type (compare with [12] and see (5.28) in [15]):

$$\varphi_n(1, q) = B_n(q) = L_q(X_q^n) = e_q^{-1} \sum_{k=0}^{\infty} \frac{k_q^n}{k_q!}, \quad n \geq 0, \quad e_q^{-1} \equiv [exp_q\{1\}]^{-1}.$$

We arrive to this simple conclusion using Jackson derivative difference operator in place of $D = d/dx$ in $q = 1$ case and the power series generating function $G(t)$ for q-Poisson probability distribution:

$$p_n = [exp_q\lambda]^{-1} \frac{\lambda^n}{n_q!}, G(t) = \sum_{n \geq 0} p_n t^n, \tag{8}$$

where $exp_q\lambda \equiv exp_{\psi(q)}\lambda = \sum_{n \geq 0} \psi_n(q)t^n, \quad \psi_n(q) \equiv \frac{1}{n_q!}$. Naturally

$$p_n = [\frac{\partial_q^n G(t)}{n_q!}]_{t=0}, \quad [\partial_q G(t)]_{t=1} = 1 \quad for \quad \lambda = 1. \tag{9}$$

In order to arrive at the q-Dobinski formula apply (7) to (4) with (6) in mind. (As for ψ-Poisson probability distribution - see [1,2].) There are many ways leading to q-extended Stirling numbers according to their weighted counting interpretation i.e. various statistics are counted by q-Stirling numbers of the second kind [16]. For example $w(\pi) = q^{cross(\pi)}, \quad w(\pi) = q^{inv(\pi)}$ from [17] give the Carlitz-Gould-Milne q-Stirling numbers $\left\{ {n \atop k} \right\}_q$ - after being summed over the set of k-block partitions while $w(\pi) = q^{nin(\pi)}$ from [18] gives rise

to the Carlitz-Gould -Milne $q^\sim$-Stirling numbers $\left\{ {n \atop k} \right\}_q^\sim$ after being summed over the set of k-block partitions (see also maj and $maj^\sim$ statistics in [19] as well as other statistics [14] giving also both extensions). The choice of weight $w(\pi) = q^{i(\pi)}$ [20] gives rise to another statistics counted by new kind of q-extended Stirling numbers of the second kind. Let us consider - for the sake of illustration this example from [20] in more detail. This is the example of weighted counting $\sum_{\sigma \in \Pi_n} q^{i(\sigma)}$ of partitions of [n]. The weight w of such a partition $\pi \in \Pi_n$ is determined by inversions' i function in the form $w(\pi) = q^{i(\pi)}$. Here Π_n denotes the lattice of all partitions of the set [n] while $A_{n,k}$ denotes the family of all k -block partitions. A k-block partition $\pi \in A_{n,k} \subseteq \Pi_n$ is represented in the standard form: $\pi = B_1/B_2/.../B_k$ with the convention that $maxB_1 < maxB_2 < ..., maxB_k = n$. For $i \in [n]$ let b_i denotes a number of a block to which i pertains. Define an inversion of partition π to be a pair $< i, j >$ such that $b_i < b_j$ and $i > j$. The inversion set of π is $I(\pi) = \{ < i, j >; < i, j > is\ an\ inversion\ of\ \pi \}$. Then $i(\pi) = |I(\pi)|$ and the *inversion* q-Bell numbers are naturally defined as

$$B_n^{inv}(q) = \sum_{\sigma \in \Pi_n} q^{i(\sigma)} = \sum_{k \geq 0} \sum_{\pi \in A_{n,k}} q^{i(\pi)}$$

while *inversion* q-Stirling numbers of the second kind are identified with

$$\left\{ {n \atop k} \right\}_q^{inv} \equiv \sum_{\pi \in A_{n,k}} q^{i(\pi)}.$$

The *inversion* q-Bell number $B_n^{inv}(q)$ is the generating function for $I(s) =$ the number of all partitions from Π_n with s of the above inversions because

$$B_n^{inv}(q) = \sum_{\sigma \in \Pi_n} q^{i(\sigma)} = \sum_{s \geq 0} q^s \sum_{\pi \in \Pi_n, i(\pi)=s} 1 \equiv \sum_{s \geq 0} I(s) q^s.$$

Recursions for both *inversion* q-Bell numbers and *inversion* q-Stirling numbers of the second kind are not difficult to be derived. Also in a natural way the *inversion* q-Stirling numbers of the second kind from [20] satisfy a q-analogue of the standard recursion for Stirling numbers of the second kind to be written via mnemonic adding "q" subscript to the binomial and second kind Stirling symbols in the the standard recursion formula i.e.

$$\left\{ {n+1 \atop k} \right\}_q^{inv} = \sum_{l=0}^{n} \binom{n}{l}_q \left\{ {n-l \atop k-1} \right\}_q^{inv}; \quad n \geq 0, k \geq 1.$$

Another q-extended Stirling numbers much different from Carlitz "q-ones" were introduced in the reference [21] from where one infers [22] the *cigl*-analog of (6). Let Π denotes the lattice of all partitions of the set $\{0, 1, .., n-1\}$. Let $\pi \in \Pi$ be represented by blocks $\pi = \{B_o, B_1, ...B_i, ...\}$, where B_o is the block containing zero: $0 \in B_o$. The weight adapted by Cigler defines weighted partitions' counting according to the content of B_o. Namely

$$w(\pi) = q^{cigl(\pi)}, cigl(\pi) = \sum_{l \in B_0} l, \sum_{\pi \in A_{n,k}} q^{cigl(\pi)} \equiv \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{cigl}.$$

Therefore $\sum_{\pi \in \Pi} q^{cigl(\pi)} \equiv B_n(q)$. Here $A_{n,k}$ stays for subfamily of all k-block partitions. With the above relations one has defined the *cigl-q-Stirling* and the *cigl-q-Bell* numbers. The *cigl-q-Stirling* numbers of the second kind are expressed in terms of q-binomial coefficients and $q = 1$ Stirling numbers of the second kind [21] as follows

$$\left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}_q^{cigl} = \sum_{l=0}^{n} \binom{n}{l}_q q^{\binom{n-l+1}{2}} \left\{ \begin{matrix} n-l \\ k-1 \end{matrix} \right\}_q^{cigl}; n \geq 0, k \geq 1.$$

As seen above these are **new** q-extended Stirling numbers. The corresponding *cigl-q-Bell* numbers recently have been equivalently defined via *cigl-q-Dobinski* formula [22] - which now in more adequate notation reads:

$$L(X^{\overline{q^n}}) = \overline{B}_n(q), \qquad n \geq 0, X^{\overline{q^n}} \equiv X(X+q-1)...(X-1+q^{n-1}).$$

The above *cigl-q-Dobinski* formula is interpreted as the average of this specific $n - th$ *cigl-q-power* random variable $X^{\overline{q^n}}$ with the $q = 1$ Poisson distribution such that $L(X) = 1$. For that to see use the identity by Cigler [21]

$$x(x-1+q)...(x-1+q^{n-1}) = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{cigl} x^{\underline{k}}.$$

2. Beyond the q-extensions realm

The further consecutive ψ-umbral extension of Carlitz-Gould q-Stirling numbers $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{\sim}$ is realized two-fold way - one of which leads to a surprise in contrary to the other way.

2.1. The first way

The first "easy way" consists in almost mnemonic sometimes replacement of q subscript by ψ after having realized that in equation (5) we are dealing with the specific case of the so called Comtet numbers [14, 15] (Comtet L. in *Nombres de Stirling generaux et fonctions symtriques* C.R. Acad. Sci. Paris, Series A, 275 (1972):747-750 formula (2) refers to **Wronski**). This array of Stirling-like numbers $\left\{ {n \atop k} \right\}_\psi^\sim$ - *"alephs de Wronski"* as Comtet refers to it or these Comtet numbers in terminology of Wagner [14, 15] or as a matter of fact [13] these **Newton** interpolation coefficients for $e_n, n \geq 0$ i.e. divided differences $[0, 1_\psi, 2_\psi, ..., k_\psi; e_n]$ are defined accordingly as such coefficients - below.

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_\psi^\sim \psi_{\underline{k}}(x), \quad n \geq 0, \tag{10}$$

i.e. equivalently (recall that $e_n(x) = x^n, n \geq 0$)

$$\left\{ {n \atop k} \right\}_\psi^\sim = [0, 1_\psi, 2_\psi, ..., k_\psi; e_n] = \sum_{l=0}^{k} \frac{e_n(l_\psi)}{\psi_{\underline{k+1}}^{\backprime}(l_\psi)}, \quad n \geq 0, \qquad (Newton)$$

where

$$\psi_{\underline{k}}(x) = x(x - 1_\psi)(x - 2_\psi)...(x - [k-1]_\psi)$$

and $\psi_{\underline{s}}^{\backprime}$ denotes the first derivative. Let then $f = \langle f_n \rangle_{n \geq 0}$ be an arbitrary sequence of polynomials. In the following we shall call $S(f; n, k)$ defined below

$$[d_0, d_1, d_2, ..., d_k; f_n] \equiv S(f; \langle d_l \rangle_{l \geq 0}, n, k) \qquad (N - W - C \quad Stirling)$$

the Newton-Wronski-Comptet Stirling numbers (N-W-C for short)- compare with Appendix A.2.

The $\psi^\sim$ - Stirling numbers $\left\{ {n \atop k} \right\}_\psi^\sim$ defined by (10) are specification of $N - W - C$ Stirling array for which we naturally define $\psi^\sim$-exponential polynomials $\varphi_n(x, \psi)$ as follows

$$\varphi_n^\sim(x, \psi) = \sum_{k=0}^{n} [0, 1_\psi, 2_\psi, ..., k_\psi; e_n] x^k, \quad n \geq 0. \qquad (\psi^\sim - exp - pol)$$

Note the trivial but important fact that in the N-W-C Stirling numbers case we are dealing with not equidistant nodes' interpolation in general and

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note that (*Rescal*) from the subsection 2.2. below is no more valid beyond q-extension case - both with an impact on the way to find out the Dobinski-like formulae - see more below.

As a consequence of (10) we have "for granted" the following extensions of recurrences for Stirling numbers of the second kind:

$$\left\{ {n+1 \atop k} \right\}_\psi^\sim = \left\{ {n \atop k-1} \right\}_\psi^\sim + k_\psi \left\{ {n \atop k} \right\}_\psi^\sim; \quad n \geq 0, k \geq 1, \tag{11}$$

where $\quad \left\{ {n \atop 0} \right\}_\psi^\sim = \delta_{n,0}, \quad \left\{ {n \atop k} \right\}_\psi^\sim = 0, \quad k > n;$ and the recurrence for ordinary generating function reads

$$G_{\widetilde{k_\psi}}(x) = \frac{x}{1 - k_\psi} G_{\widetilde{k_\psi - 1}}(x), \quad k \geq 1, \tag{12}$$

where naturally

$$G_{\widetilde{k_\psi}}(x) = \sum_{n \geq 0} \left\{ {n \atop k} \right\}_\psi^\sim x^n, \quad k \geq 1$$

from where one infers that

$$G_{\widetilde{k_\psi}}(x) = \frac{x^k}{(1 - 1_\psi x)(1 - 2_\psi x)...(1 - k_\psi x)} \quad, \quad k \geq 0. \tag{13}$$

Hence we arrive in the standard extended text-book way [22] at the following explicit **new** formula (compare with (2.3) in [15])

$$[0, 1_\psi, 2_\psi, ..., k_\psi; e_n] = \left\{ {n \atop k} \right\}_\psi^\sim = \frac{1}{k_\psi!} \sum_{r=1}^{k} (-1)^{k-r} \binom{k_\psi}{r_\psi} r_\psi^n; \quad n \geq k \geq 0, \tag{14}$$

where

$$\sum_{r=1}^{k} (-1)^{k-r} \binom{k_\psi}{r_\psi} r_\psi^n; \quad n, k \geq 0$$

is readily recognized as the ψ-extension of the formula for surjections in its - after inclusion-exclusion principle had been applied - form.

Expanding the right hand side of (13) results in another explicit formula for these ψ-case Newton-Wronski-Comtet array of Stirling numbers of the second kind i.e. we have

$$\left\{ {n \atop k} \right\}_\psi^\sim = \sum_{1 \leq i_1 \leq i_2 \leq ... \leq i_{n-k} \leq k} (i_1)_\psi (i_2)_\psi ... (i_{n-k})_\psi; \quad n \geq k \geq 0 \tag{15}$$

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or equivalently (compare with [13, 14])

$$\left\{ {n \atop k} \right\}_\psi^\sim = \sum_{d_1+d_2+...+d_k=n-k, \quad d_i \geq 0} 1_\psi^{d_1} 2_\psi^{d_2} ... k_\psi^{d_k}; \quad n \geq k \geq 0. \qquad (16)$$

N-W-C case $\psi^\sim$ - **Stirling numbers** of the second kind being defined equivalently by (10), (*Newton*), (14), (15) or (16) yield N-W-C case $\psi^\sim$ - **Bell numbers**

$$B_n^\sim(\psi) = \sum_{k=0}^n \left\{ {n \atop k} \right\}_\psi^\sim = \sum_{k=0}^n [0, 1_\psi, 2_\psi, ..., k_\psi; e_n], \qquad n \geq 0 \qquad (B^\sim).$$

Naturally $\exists!$ functional $L^\sim$ such that on the basis of persistent root polynomials $\psi_{\underline{k}}(x)$ it takes the value 1:

$$L^\sim(\psi_{\underline{k}}(x)) = 1, \quad k \geq 0.$$

Then from (10) we get an analog of (3)

$$B_n^\sim(\psi) = L^\sim(x^n) \qquad\qquad (L^\sim).$$

Problem: which distribution the functional $L^\sim$ is related to is an open technical question by now. More - the recurrence for $B_n^\sim(\psi)$ is already quite involved and complicated for the q-extension case (see: the first section)- and no acceptable readable form of recurrence for the ψ-extension case is known to us by now.

Nevertheless after adapting the standard text-book method [23] we have the following formulae for two variable ordinary generating function for $\left\{ {n \atop k} \right\}_\psi^\sim$ Stirling numbers of the second kind and the ψ-exponential generating function for $B_n^\sim(\psi)$ Bell numbers

$$C_\psi^\sim(x, y) = \sum_{n \geq 0} \varphi_n^\sim(\psi, y) x^n, \qquad\qquad (17)$$

where the ψ- exponential polynomials $\varphi_n^\sim(\psi, y)$

$$\varphi_n^\sim(\psi, y) = \sum_{k=0}^n \left\{ {n \atop k} \right\}_\psi^\sim y^k$$

do satisfy the recurrence (compare with formulas (28) in Touchard's [24] from 1956)

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$$\varphi_n^\sim(\psi, y) = [y(1 + \partial_\psi]\varphi_{n-1}^\sim(\psi, y) \qquad n \geq 1,$$

hence

$$\varphi_n^\sim(\psi, y) = [y(1 + \partial_\psi)]^n 1, \qquad n \geq 0.$$

The linear operator ∂_ψ acting on the algebra of formal power series is being called (see: [1, 2] and references therein) the "ψ-derivative" as $\partial_\psi y^n = n_\psi y^{n-1}$.

The $\psi^\sim$ - exponential generating function

$$B_\psi^\sim(x) = \sum_{n \geq 0} B_n^\sim(\psi) \frac{x^n}{n_\psi!} \qquad (\psi^\sim - e.g.f.)$$

for $B_n^\sim(\psi)$ Bell numbers - after cautious adaptation of the method from the Wilf's generatingfunctionology book [23] can be seen to be given by the following **new** formula

$$B_\psi^\sim(x) = \sum_{r \geq 0} \epsilon(\psi, r) \frac{e_\psi[r_\psi x]}{r_\psi!} \qquad (18)$$

where (see: [1,2] and references therein)

$$e_\psi(x) = \sum_{n \geq 0} \frac{x^n}{n_\psi!}$$

while

$$\epsilon(\psi, r) = \sum_{k=r}^{\infty} \frac{(-1)^{k-r}}{(k_\psi - r_\psi)!} \qquad (19)$$

and the **new** Dobinski - like formula for the ψ-extensions here now reads

$$B_n^\sim(\psi) = \sum_{r \geq 0} \epsilon(\psi, r) \frac{r_\psi^n}{r_\psi!}. \qquad (20)$$

The $\psi^\sim$-exponential polynomials are therefore given correspondingly by

$$\varphi_n^\sim(\psi, x) = \sum_{r \geq 0} \epsilon(\psi, r) \frac{r_\psi^n}{r_\psi!} x^r. \qquad (\psi^\sim - exp - pol - II)$$

In the case of Gauss q-extended choice of $\langle \frac{1}{n_q!} \rangle_{n \geq 0}$ admissible sequence of extended umbral operator calculus equations (19) and (20) take the form

$$\epsilon(q, r) = \sum_{k=r}^{\infty} \frac{(-1)^{k-r}}{(k-r)_q!} q^{-\binom{r}{2}} \qquad (21)$$

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and the **new** N-W-C case $q^\sim$-Dobinski formula is given by

$$B_n^\sim(q) = \sum_{r \geq 0} \epsilon(q, r) \frac{r_q^n}{r_q!}, \tag{22}$$

which for $q = 1$ becomes the Dobinski formula from 1887 [4]. Note the appearance of re-scaling factor $q^{-\binom{r}{2}}$ in (21). In its absence we would get **not** $q^\sim$-Dobinski but q-Dobinski formula

$$B_n(q) = \frac{1}{\exp_q(1)} \sum_{0 \leq k} \frac{k_q^n}{k_q!} \qquad (q - Dobinski)$$

- see [15] and formula (5.28) there coinciding with N-W-C case of Dobinski formula after re-scaling in correspondence with (Rescal) below in subsection 2.2. Correspondingly we would the have **not** $(q^\sim - exp - pol)$ formula but $(q - exp - pol)$ formula:

$$\varphi_n(x, q) = \sum_{k=0}^{n} q^{\binom{k}{2}} [0, 1_q, 2_q, ..., k_q; e_n] x^k = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q x^k. \qquad (q - exp - pol)$$

The interpretation problem. Combinatorial interpretations of the known up to now various q-extensions of Stirling numbers of both kinds - are briefly reported on in the Appendix. The problem of how eventually one might interpret - beyond the q-extensions' realm - for example the $\psi^\sim$-Dobinski formulae (20) and (22) also in the Rota-like way represented here by equation (3) we leave **opened** - see the discussion in Appendix A.2 and A.3.II. Naturally there exist a unique linear functional $L_\psi^\sim$ such that

$$L_\psi^\sim(\psi_{\underline{k}}(x) = x(x - 1_q)(x - 2_q)...(x - [k-1]_\psi)) = 1, \quad k \geq 0.$$

It is also to be noted that in the exceptional case of q-extensions and **only for q-extension** we have equivalence of $\left\{ {r \atop l} \right\}_\psi^\sim$ and $\left\{ {r \atop l} \right\}_\psi$ by re-scaling. For the latter q-Stirling numbers we have Dobinski formula and simultaneously q-Poisson average functional interpretation as represented by the definition (6). Namely - recall the fact that the linear functional L_q there is interpreted as the average value functional for the random variable X_q with the q-Poisson distribution which is specific case of the ψ-Poisson distributions from [2, 1]. Note again that this re-scaling takes place for q-extensions and **only for q-extension**. This is so because the relation

$$n_\psi - k_\psi = f(k)(n - k)_\psi, \quad 1_\psi = 1$$

holds for and only for $f(1) \equiv q$ when it becomes the identity (2.6) from [25] i.e. $f(k) = q^k$. It is our conviction that this is the very reason that q-extensions seem to appear as a kind of "a bifurcation point" in the domain of ψ-umbral extensions. This **conviction** is supported by the corresponding considerations in [26]- section 3 - on possibility of ψ-analogue of the so called "quantum" q-plane formulation of q-umbral calculus. The parallel treatment of the Newton-Wronski-Comtet $\begin{bmatrix} n \\ k \end{bmatrix}^\sim_\psi$ Stirling numbers of the *first kind* is now not difficult (consult [25, 12, 8] for example and Wagner's recent treatment of the well established q-case in [14]).

In the inversion-dual way to our equation (10) above we define the $\psi^\sim$-Stirling numbers of the first kind as coefficients in the following expansion

$$\psi_{\underline{k}}(x) = \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}^\sim_\psi x^r \tag{23}$$

where - recall $\psi_{\underline{k}}(x) = x(x - 1_\psi)(x - 2_\psi)...(x - [k-1]_\psi)$. (Attention: see equations (10)-(16) in [8] and note the difference with the present definition). Therefore from the above we infer that

$$\sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}^\sim_\psi \left\{ \begin{matrix} r \\ l \end{matrix} \right\}^\sim_\psi = \delta_{k,l}. \tag{24}$$

Another natural counterpart to $\psi^\sim$-Stirling numbers of the second are ψ^c-Stirling numbers of the first kind defined here down as coefficients in the following expansion ("c" because of cycles in non-extended case)

$$\psi_{\overline{k}}(x) = \sum_{r=0}^{k} \begin{bmatrix} k \\ r \end{bmatrix}^c_\psi x^r \tag{25}$$

where - now $\psi_{\overline{k}}(x) = x(x + 1_\psi)(x + 2_\psi)...(x + [k-1]_\psi)$. These are to be studied elsewhere.

On interpretation. For possible **unified** combinatorial interpretations of binomial coefficients of both kinds, the Stirling numbers of both kinds and the Gaussian coefficients of the first and second kind - i.e for the specific choices of $\psi = \langle \frac{1}{n_{\psi!}} \rangle_{n \geq 0}$ - see [27, 28]. As for q-analogue of Stirling cycle numbers see [29] and Sect. 5.3. in [30]. The problem of eventual combinatorial interpretation of other ψ-extensions (vide Fibonomial - for example)

- remains opened.

2.2. The second way.

We shall come over now to inspect the outcomes of **the second way** - announced at the start of this section - after having realized that in the equation (4) we are **not** dealing straightforwardly with Newton-Wronski-Comtet array of Stirling-like numbers [13, 14, 15] - except for the $q = 1$ case "extension"- of course. Though it is to be noted that still the re-scaling takes place for q-extensions and **only for q-extension**. Not beyond. (This is so because the relation

$$n_\psi - k_\psi = f(k)(n-k)_\psi, \quad 1_\psi = 1$$

holds for and only for $f(1) \equiv q$ when it becomes the identity (2.6) from Gould's [25] i.e. $f(k) = q^k$). Thus after the above Gould re-scaling we would recover Newton-Wronski-Comtet array of Stirling-like numbers re-scaled - anyhow i.e.

$$y^n = \sum_{k=0}^{n} q^{-\binom{k}{2}} \left\{ {n \atop k} \right\}_q y^{\underline{k}}, \qquad (Rescal)$$

where $y = x_q = \frac{1-q^x}{1-q}$ and $\psi_{\underline{k}}(y) = y(y - 1_q)(y - 2_q)...(y - [k-1]_q)$. At first let us recall that the definition (4) of q-Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_q$ is equivalent to the definition by the recursion

$$\left\{ {n+1 \atop k} \right\}_q = q^{k-1} \left\{ {n \atop k-1} \right\}_q + k_q \left\{ {n \atop k} \right\}_q; \quad n \geq 0, k \geq 1, \qquad (26)$$

where $\left\{ {n \atop 0} \right\}_q = \delta_{n,0}, \quad \left\{ {n \atop k} \right\}_q = 0, \quad k > n$.

These in turn is equivalent to (just use the standard Q-Leibniz rule [1, 2, 31] for Jackson derivative ∂_q)

$$(\hat{x}\partial_q)^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_q \hat{x}^k \partial_q^k, \qquad (27)$$

where $\left\{ {n \atop 0} \right\}_q = \delta_{n,0}, \quad \left\{ {n \atop k} \right\}_q = 0, \quad k > n$. Here $\hat{x}$ denotes the multiplication by the argument of a function. The formula (27) is a special case of the typical for GHW algebra [32, 1, 2, 33] expression investigated by Carlitz in

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1932 [34] (compare with GHW formulae (1), (31), (32) in [7] and see also [35]). The idea now is to extend eventually the definition by equation (27) *via* **replacing** q-extended operators by the corresponding ψ-extended elements of the Graves-Heisenberg-Weyl (*GHW*) algebra representation [32, 1, 2, 33]. However we note at once (see: Appendix for particulars of the up-side down notation) that the two following observations hold.

Observation 2.1 The equivalent definitions (28) and (29)

$$(\hat{x}\partial_\psi)^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_\psi \hat{x}^k \partial_\psi^k \tag{28}$$

where $\left\{ {n \atop 0} \right\}_\psi = \delta_{n,0}$, $\left\{ {n \atop k} \right\}_\psi = 0$, $k > n$ and

$$x_\psi^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_\psi x_{\frac{k}{\psi}} \tag{29}$$

lead to the one first order recurrence of the type (26) **for and only for** q**-extension**.

This again is so because the relation

$$n_\psi - k_\psi = f(k)(n-k)_\psi, \quad 1_\psi = 1$$

holds for and only for $f(1)$ i.e. for q-extension, where it becomes the identity (2.6) from [25] i.e. $f(k) = q^k$. The next observation now comes as a would be surprise.

Observation 2.2 The equivalent definitions (28) and (29) have no non-trivial realizations beyond the q-extension case.

In order to arrive at this observation let us act appropriately on x^N $N \geq 0$ monomials by both sides of the *GHW* algebra representation definition (28) thus getting an *infinite* **sequence** *of recurrences*

$$\sum_{k \geq 0} \left\{ {n+1 \atop k} \right\}_\psi N_{\frac{k}{\psi}} = \sum_{k \geq 0} \left\{ {n \atop k} \right\}_\psi N_\psi N_{\frac{k}{\psi}}, \quad N \geq 0, \tag{30}$$

with no nontrivial solutions as spectacularly evident with the choice of - for example - Fibonomialy-extended sequence $\langle \frac{1}{F_n!} \rangle_{n \geq 0}$ ($\langle F_n \rangle$ - Fibonacci sequence) **unless** $\psi = \langle \frac{1}{n_q!} \rangle_{n \geq 0}$. And for this and only for this choice $\psi = \langle \frac{1}{n_q!} \rangle_{n \geq 0}$ we have

$$N_q = q^k((N-k)_q + k_q$$

which after being applied in (30) results in **one** recurrence which is exactly the recurrence (26).

As expected - the equation (29) becomes equivalent to the one first order recurrence of the type (26) **for and only for q-extension**.

Closing remark. We see that the Carlitz-Gould q-Stirling numbers $\left\{ {n \atop k} \right\}_q$ make q-umbral extension to appear as a kind of "a bifurcation point" in the domain of respective ψ-umbral extensions. This in statu nascendi conviction is also supported by the corresponding considerations in [26]- section 3 - considerations about possibility of ψ-analogue of the so called "quantum" q-plane formulation of q-umbral calculus. As for eventual second way's ψ-extensions *beyond* the q-extension case where the *rescaling* does not take place we are left with an opened problem how to eventually find the way to get round this inspiring obstacle. The selective comparison of the presented umbral extensions of Stirling numbers, Bell numbers and Dobinski-like formulas with other existing extensions (as well as relevant information in brief) serves the purpose of seeking analogies and is to be find in the Appendix that follows now.

Appendix - for remarks, discussion and brief comparative review of ideas.

A.1. Notation.

The necessary commutation relations' representation for the GHW (Graves-Heisenberg-Weyl) algebra generators is provided in [31, 1, 2, 33]. Applications of these might be worthy of the further study [25, 36, 37]. The simplicity of the first steps to be done while identifying general properties of such ψ-extensions consists in notation i.e. here - in writing objects of these extensions in mnemonic convenient **upside down notation** [1], [2]

$$\frac{\psi_{(n-1)}}{\psi_n} \equiv n_\psi, \quad n_\psi! = n_\psi(n-1)_\psi!, \quad n > 0, \quad x_\psi \equiv \frac{\psi(x-1)}{\psi(x)}, \qquad (31)$$

$$x_\psi^{\underline{k}} = x_\psi(x-1)_\psi(x-2)_\psi...(x-k+1)_\psi \qquad (32)$$

$$x_\psi(x-1)_\psi...(x-k+1)_\psi = \frac{\psi(x-1)\psi(x-2)...\psi(x-k)}{\psi(x)\psi(x-1)...\psi(x-k+1)}. \qquad (33)$$

If one writes the above in the form $x_\psi \equiv \frac{\psi(x-1)}{\psi(x)} \equiv \Phi(x) \equiv \Phi_x \equiv x_\Phi$, one sees that the name upside down notation is legitimate. You may consult [1, 2, 26,

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33, 36, 37] for further development and usefulness of this notation. In this notation the ψ-extension of binomial incidence coefficients read familiar:

$$\binom{n}{k}_\psi = \frac{n_\psi!}{k_\psi!(n-k)_\psi!} = \binom{n}{n-k}_\psi.$$

A.2. Discussion, remarks, questions.

q-umbral extensions are expected to be of distinguished character - also due to what was stated in Section 2. Being so they pay to us with simplicity of formulae and elegance of q - weighting combinatorial interpretations and thus various statistics of the combinatorial origin.

Because of that and because of the importance of q-umbral extensions in coherent states mathematics we here adjoin a remark on simplicity based on the remark of Professor Cigler (in private).

Namely Katriel indicates in the very important source paper [6] that his derivation of the Dobinski formula is the simplest. And really it is simple and wise. Possibly then this may be occasionally and profitably confronted with the also extremely simple derivation by Cigler (see p. 104 in [38]) based on GHW-algebra properties. Let then $\hat{x}$ denotes the multiplication by x operator while D denotes differentiation - both acting on the prehilbert space P of polynomials. Then due to the recursion for Stirling numbers of the second kind and the identity (operators act on P)

$$\hat{x}(D+1) \equiv \frac{1}{\exp(x)}(\hat{x}D)\exp(x)$$

one defines in GHW - algebra manner the **exponential polynomials**

$$n \geq 0, \qquad \varphi_n(x) = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} x^k \qquad\qquad (ExPol)$$

introduced by Acturialist J.F. Steffensen [39, 40] (see: Bell's "Exponential polynomials" in umbra-symbolic language [41] p. 265 and his symbolic formula (4.7) for now Bell numbers). These exponential polynomials were substantially investigated by Touchard in Blissard umbra-symbolic language [24]. Here now comes the GHW-definition [38] of these basic polynomials

$$\varphi_n(x) = \frac{1}{\exp(x)}(\hat{x}D)^n \exp(x)$$

resulting in the formula which becomes Dobinski one for $x = 1$ i.e.

$$\varphi_n(x) = \frac{1}{\exp(x)} \sum_{0 \leq k} \frac{k^n x^k}{k!}.$$

Note: The q-case as well as ψ-case **formal** mnemonic counterpart formulae are automatically arrived at with the mnemonic attaching of q or ψ indices to nonnegative numbers [1, 2] - vide:

$$\varphi_n(x, \psi) = \frac{1}{\exp_\psi(x)} \sum_{0 \leq k} \frac{k_\psi^n x^k}{k_\psi!} \tag{34}$$

which for $\psi = \langle \frac{1}{n_q!} \rangle_{n \geq 0}$ and $x = 1$ becomes the well known q-Dobinski formula as of course $\varphi_n(x = 1, q) = B_n(q)$ - see in [15] the formula (5.28) and note that this is **not** $q^\sim$-Dobinski formula (22) as noticed right after (22). As for eventual second way's ψ-extensions beyond the q-extension case where the *rescaling* does not take place - we are left with an opened problem how to eventually find the way to get round this inspiring obstacle. Perhaps instead of the second beyond the q-extension way we might follow Alexander the great in his Gordian Knot problem solution and define $S(\psi, n, k)$ as follows (whenever one may prove that the object being defined is really a polynomial):

$$\varphi_n(x, \psi) = \sum_{k=0}^{n} S(\psi, n, k) x^k = \frac{1}{\exp_\psi(x)} \sum_{0 \leq k} \frac{k_\psi^n x^k}{k_\psi!}. \qquad (S(\psi) - exp - pol)$$

An alternative good idea perhaps would be an attempt to ψ-extend the celebrated Newton interpolation formula (use ∂_ψ instead D, then exp_ψ instead of exp and then you will be faced with ψ-Leibniz rule application problem though... see [1, 2, 33] for Leibnitz rules). Let us then make - also for the sake of comparison with existing knowledge - let us then make us wonder on the intrinsic presence and assistance of Newton interpolation which corresponds to the first "easy" way as described in Subsection 2.1.

The intrinsic presence and assistance of Newton interpolation formula in derivation of Dobinski formula for exponential polynomials and their binomial analogues was underlined and used in [42] for specific presentation of the $q = 1$ case from the umbral point of view of the classical finite operator calculus. In [42] a Dobinski-like formula was derived being as a matter of fact the particular ("binomial") case of formula (30) from Touchard's 1956

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year paper [24]. In more detail. Choosing any binomial polynomial sequence $\langle b_n \rangle_{n \geq 0}$ consider its Newton interpolation formula

$$b_n(x) = \sum_{k=0}^{n} [0, 1, 2, ..., k; b_n] x^{\underline{k}}.$$

Then apply an umbral operator sending the binomial basis $\langle x^{\underline{n}} \rangle_{n \geq 0}$ of delta operator Δ to the binomial basis $\langle x^n \rangle_{n \geq 0}$ of delta operator D. Then use

$$[0, 1, 2, ..., k; b_n] = \frac{\Delta^k b_n|_{x=0}}{k!} = \sum_{l=0}^{k} \frac{(-1)^{k-l} b_n(l)}{(k-l)! l!} \quad (Newton - Stirling)$$

so as to arrive (thanks to binomial convolution) at Dobinski like formula from [42] i.e.

$$b_n(\varphi(x)) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k) x^k}{k!},$$

where φ is the umbral symbol satisfying [24]

$$\varphi_{n+1} = x(\varphi + 1)^n, \quad \varphi^{\underline{k}} = x^k. \qquad (Touchard)$$

In order to see that this is just the particular ("binomial") case of umbra-symbolic formula (30) from Touchard's 1956 year paper [24] just choose in Touchard formula (30) the *arbitrary* polynomial f to be any binomial one $b_n = f$. Then $f(\varphi) = b_n(\varphi) = b_n(\varphi(x))$ is binomial also and we have

$$f(\varphi) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k) x^k}{k!}. \qquad (Dobinski - Touchard)$$

Equidistant nodes Newton's interpolation array of coefficients $[0, 1, 2, ..., k; b_n]$ - here the connection constants of the general exponential polynomial $p_n(x) = b_n(\varphi(x))$ are to be called in the following the **Newton-Stirling** numbers of the second kind and are consequently given by

$$p_n(x) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k) x^k}{k!} = \sum_{k=0}^{n} [0, 1, 2, ..., k; b_n] x^k, \qquad (N - S - Dob)$$

where $\langle b_n \rangle_{n \geq 0}$ is any sequence of polynomials. These are - in their turn - the special case of N-W-C Stirling numbers.

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Coherent States' Example I. Take the $b_m(x) = f(x)$ in the (*Dobinski-Touchard*) formula to be of the form resulting from normal ordering problem (see A.3.II. - below) i.e. let (see: [10])

$$f(x) = b_{ns}(x; r, s) = \prod_{j=1}^{n} [x + (j-1)(r-s)]^{\underline{s}}$$

Then we get (2.8) from [10] i.e.

$$[0, 1, 2, ..., k; b_{ns}(...; r, s)] = \frac{1}{k!} \sum_{l=s}^{k} (-1)^{k-l} b_{ns}(l; r, s) \binom{k}{l} \equiv S_{r,s}(n, k)$$

becomes the definition of the generalized Stirling numbers (see A.3.II. - below), which appear to be special case of general Newton-Stirling numbers of the second kind. (Here $b_{ns}(...; r, s)(x) = b_{ns}(x; r, s)$.) Naturally the Dobinski-like formula (2.1) from [10] for exponential polynomials determined by $[0, 1, 2, ..., k; b_{ns}(.; r, s] = S_{r,s}(n, k)$ is special case of (N-S-Dob) Dobinski-like formula with counting adapted to the choice $f = b_{ns}$. Along with Bell numbers' sequence or Bessel numbers's sequence this special case of Newton-Bell numbers' sequence

$$B_{r,s}(n) = \sum_{l=s}^{ns} S_{r,s}(n, k)$$

is a moment sequence [43].

Example II The next example of Newton-Stirling numbers $d_{n,k}$ comes from the paper [44] on interpolation series related to the Abel-Goncharov problem. There the divided difference functional Δ_k is applied to e_n yielding $d_{n,k}$ accordingly:

$$\Delta_k e_n = [0, \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}, 1; e_n] = d_{n,k}.$$

The general rules for Newton-Stirling arrays allow us to notice that

$$d_{n,k} = [0, \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}, 1; e_n] = \frac{k^k}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{r^n}{k^n}; \quad n \geq k \geq 0,$$

hence for corresponding exponential polynomials we have

$$\varphi_n(x) = \sum_{k=0}^{n} \frac{k^k}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{r^n}{k^n} x^k,$$

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in accordance with the fact [44] that $k^{n-k}d_{n,k} = \left\{ {n \atop k} \right\}$. Derivation of the Dobinski-like formula we leave as an exercise.

On ψ-extension. A ψ-extension of the above Touchard's symbolic definition of exponential polynomials would start with the defining formula

$$\varphi_{n+1} = x(\varphi +_\psi 1)^n, \quad \varphi^{\underline{k}} = x^k. \qquad (\psi - Exp - Pol)$$

resulting in analogous umbra-symbolic identities and with corresponding Dobinski-like formula as (35) below, where $b_n = e_n$. Compare these with (10) from where we have for this case of $b_n(x) = e_n(x) = x^n, n \geq 0$ the Newton interpolation formula

$$x^n = \sum_{k=0}^n [0, 1_\psi, 2_\psi, ..., k_\psi; e_n]\psi_{\underline{k}}(x), \quad n \geq 0.$$

For the meaning of the ψ-shift "$+_\psi$" see [1, 2, 26, 31, 33]. This we shall develop elsewhere. Meanwhile let us continue the limited review of other extensions.

Plethystic Stirling numbers' extension The above umbral extensions as well as the other extensions to be mentioned in what follows are to be confronted with inventions of plethystic exponential polynomials, plethystic Stirling numbers of the second kind and plethystic extension of Bell numbers from [45] which constitutes an advanced and profound way to reach the merit of the finite operator calculus representations - this time realized with vector space of polynomials in the *infinite* sequence of variables. In [45] Mendez had derived profits from Nava's combinatorics of plethysm then developed by Chen to become an elegant plethystic representation of umbral calculus so as to find out also umbral inverses of plethystic exponential polynomials and related plethystic Stirling numbers of the first kind (for references see: [45]). The plethystic exponential polynomials are then there expressed via Dobinski-like (plethystic) formula (see: (35) in [45]) and the plethystic Stirling numbers of the second kind (see: (38) in [45]) are expressed via formula extending the formula for Stirling numbers of the second kind resulting from the formula for surjections in its after inclusion-exclusion principle had been applied form. Whether ψ-extension of plethystic constructs as above is interesting and possible - we leave as an inquiry for the future. Occasionally note that though Mendez's Stirling numbers of the first and second kind are not Whitney numbers of an appropriate poset they do bear a striking resemblance to the latter.

Whitney numbers, statistic, interpretation. It is well known [46] that

denoting set of n elements partition lattice by Π_n the arrays $\begin{bmatrix} k \\ r \end{bmatrix}$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ are identified (see also Theorem 1.3 in [18]) as follows

$$\begin{bmatrix} n \\ n-k \end{bmatrix} = w_k(\Pi_n) \quad and \quad \left\{ \begin{matrix} n \\ n-k \end{matrix} \right\} = W_k(\Pi_n)$$

where $w_k(\Pi_n)$ and $W_k(\Pi_n)$ denote Whitney numbers of the first and second kind correspondingly. In order to recognize the possible evolvement of state of affairs while the combinatorics is concerned let us come back for a while to q-extensions realm. There are several available ways to define combinatorially $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{\sim}$ arrays. Most of these ways are based on on static on set partitions (see for example [47], [19], [18], [14], [16], [17], [48], [49]). For example Gessel in [48] gave to $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{\sim}$ combinatorial interpretation as generating functions for an inversion statistics. In another source paper [49] Milne demonstrated that $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q$ may be viewed in terms of inversions on partitions and that they count restricted growth functions using various statistics (see also [16]). We owe to Milne also the interpretation of $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q$ as sequences of lines in a corresponding vector space over finite field. In [19] Sagan delivered the major index statistics' interpretation of $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q$ array of q-Stirling numbers of the second kind. After that the authors of [18] constructed a family $\wp_n(q)$ of posets as q-analogues of the set partition lattice (different from Dowling q-analogue) in such a manner that (Theorem 5.3 in [18])

$$\begin{bmatrix} n \\ n-k \end{bmatrix} = w_k(\wp) \quad and \quad \left\{ \begin{matrix} n \\ n-k \end{matrix} \right\}_q = W_k(\wp) \quad \forall \wp \in \wp_n(q)$$

become Whithey numbers $w_k(\wp)$ and $W_k(\wp)$ of the first and second kind respectively. Whitney numbers for any graded poset may be looked at as Stirling like numbers. We shall indicate at the end of this survey a class of substantially new examples of such Stirling like arrays - after we inform on prefabs' structures. Meanwhile let us come back to the main challenge of ψ-extensions where we are faced with an ispiring obstacle.

Surprise ? In [1], [2] a family of the so called ψ-Poisson processes was introduced i.e. the corresponding choice of the function sequence ψ leads to the Poisson-like ψ-Poisson process. Accordingly one would expect the extension of Dobinski formula to the ψ- case - to be automatic. Of course

it makes no problem to call the numbers

$$B_n(\psi) = \varphi_n(x = 1, \psi)$$

the ψ - Bell numbers - whenever it makes sense - for example either the sequence of these numbers has combinatorial interpretation and/or the defining series below are convergent:

$$B_n(\psi) \equiv \varphi_n(x = 1, \psi) = exp_\psi^{-1} \sum_{0 \leq k} \frac{k_\psi^n}{k_\psi!}. \qquad (35)$$

The above might be a far reaching generalization of the standard case [23]. For example - what about the spectacularly natural and number theoretic important choice: $\psi = \langle \frac{1}{F_n!} \rangle_{n \geq 0}$ ($\langle F_n \rangle$ - Fibonacci sequence)? In this connection (Fibonacci binomial coefficients [50] are natural numbers!) a question arises whether one can prescribe eventual arithmetic properties of some of ψ-Bell numbers beyond q-extensions to any kind of composite modules as in [24], [51] or [52] and [53, 54] - see references therein. The papers just mentioned perform their investigation mostly in umbra symbolic Blissard language (see the introduction in [5] for historical remarks on Blissard's calculus roots). Note then (see **On ψ-extension** remark above) that in the ψ-extensions realm one may formally introduce the ψ-extended umbra symbol B_ψ by analogy to the Bell's source of the idea article [41] as follows

$$B(\psi)_{n+1} = (B(\psi) +_\psi 1)^n, \quad \varphi^{\underline{k}} = x^k. \qquad (\psi - B - umbra)$$

(see symbolic formula (4.7) in [41] p. 264). The above definition is equivalent to

$$B(\psi)_{n+1} = \sum_{k=0}^{n} \binom{n}{l}_\psi B(\psi)_k, \quad n \geq 0.$$

For the meaning of the ψ-shift "$+_\psi$" operator - already implicit in Ward's paper [55] - see [1, 2, 26, 31, 33]. See occasionally substantial reference to Ward [55] in Wagner's article [15] on generalized Stirling and Lah numbers. **Question.** Summarizing the discussion above - would we then - beyond the q-umbral extensions' realm - would we have ψ-Bell numbers with Poisson - like processes background - and *not related to* a kind of Stirling numbers extension - at least in a way we are acquainted with? Or should we introduce extended Stirling numbers in another way so as to be *not related to* Poisson - like processes beyond the q-umbral extensions' realm?

On this occasion note also that all ψ extensions of umbral calculus do not exhaust all possible representations of GHW. For GHW (Graves-Heisenberg-Weyl) algebra the most general representation of its' defining commutation relation is already implicit in [56] which serves [1, 2, 33] as the algebraic operator formulation of Ward's calculus of sequences [55]. Namely from the Rodrigues formula (Theorem 4.3. in [56]) with

$$\hat{x}q_{n-1}(x) = q_n(x) \quad , \quad Qq_n = nq_{n-1}$$

it follows that

$$[Q, \hat{x}] = \mathbf{1} \ , \quad \hat{x} = xQ^{`-1}$$

where Q - a differential operator [56] is a linear operator lowering degree of any polynomial by one. Q needs not to be a delta neither ψ-delta operator [2, 1, 33]. We deal with such a case after the choice of admissible sequence [56, 1, 2, 33] different from ψ-sequence $\langle\frac{1}{n!}\rangle_{n\geq 0}$ or $\langle\frac{1}{(2n)!}\rangle_{n\geq 0}$ in $(B^{\sim})$ and $(L^{\sim})$ for $(B_n^{\sim}(\psi)$ Bell numbers. Then from [57] we know that basis consisting of the persistent root polynomials $\langle\psi_{\underline{k}}(x)\rangle_{k\geq 0}$ does not correspond to ψ-delta operator. However it determines [56] a differential operator i.e. the linear one lowering degree of any polynomial by one. Another possible "rescue" in seeking for the convenient, efficient structure with natural objects corresponding to Stirling or Bell numbers and Dobinski-like formulas in special cases are the exponential structures and prefabs. For example reading [30] one notices (section 3) that the q-analog of the Stirling numbers of the second kind description developed by Morrison (compare with Section 4 in [15] to see in which way it is complementary) constitutes **the same** example of Ward'ian - prefab'ian" extension as the Bender - Goldman [58] prefab example to be considered next right now. As noticed by Morrison the relevant prefab exponential formula may equally well be derived from the corresponding Stanley's exponential formula in [59].

Exponential structures versus prefabs. A subcase of Two General Classes. Exponential structures and exponential prefabs are - in Stanley's words - basically two ways of looking at the same phenomenon [59]. Before coming over to inspect [59] from the "Stirling point of view" let us give at first a family of decisive examples showing that prefabs are all around us in combinatorics especially when quite free extensions of Stirling numbers are concerned. The following example contains such a family.

Bender - Goldman - Wagner Ward - prefab example. If corresponding "prefabian" $\hat{q}$-Bell numbers $B_n^{pref}(\gamma)$ are defined as sums over k of $\hat{S}_q(n,k)$ Stirling numbers of the q-lattice of unordered direct sums decompositions of the n-dimensional vector space $V_{q,n}$ over $GF(q) \equiv F_q$ in sect.

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2 of [15] then the formula (2.5) in [15] shows up equivalent to the Bender-Goldman exponential formula (17) from [58] - the source paper on prefabs - and in our ψ-extensions' notation formula (17) with $D_n(q)$ from [58] now reads:

$$B_\gamma^{pref}(x) = \sum_{n \geq 0} B_n^{pref}(\gamma) \frac{x^n}{n_\gamma!} = exp\{exp_\gamma(x) - 1\}. \qquad (\gamma - e.g.f.)$$

Here
$$n_\gamma! = (q^n - 1)(q^n - q^1)...(q^n - q^{n-1}) = |GL_n(F_q)|,$$

$D_o(q) = 1$ by convention while $D_n(q) \equiv B_n^{pref}(\gamma) =$ number of unordered direct sums decompositions of the vector space $V_{q,n}$. Compare with formulae (4-6) in [60] representing the completely new class of combinatorial prefab structures with noncommutative and *nonassociative* composition (synthesis) of its objects. Note "The natural hint" on ψ extensions remark there right below these formulae. Coming back to the Bender - Goldman - Wagner Ward - prefab example it is to be noticed that this is a special case of the First Class formula according to the terminology of three paths of generalizations being developed in [15]. According to us Wagner justly refers his First class to Ward [55]. We propose to call this Wagner's First Class a "Ward'ian - prefab'ian" Class of extensions as the characterization formula (1.15) in [15] *after being summed* over k yields exactly ψ- extension [60] of prefab exponential formula (12) from [58] where $\psi = \langle \frac{1}{f_n} \rangle_{n \geq 0}$ in Wagner's notation [15]. Note that our notation [1, 2, 33] is consequently always "Ward'ian". We also advocate by means of the present paper the attitude of **Two General Classes**. The Wagner's Class I is in our terms "**Ward'ian - prefab'ian**" (see $(\gamma - e.g.f.)$ above) with $F(n, k)$ Ward-prefab Stirling numbers and with $\hat{S}_q(n, k)$ as example. The second general class in our terms is "**Newtonian**" and it incorporates Wagner's Class II and Class III with N-W-C Stirling numbers $U(n, k)$ and with Newtonian $S_q^\sim(n, k)$ and Gould-Carlitz-Milne $S_q(n, k)$ as examples mutually expressible each by the other one with help of re-scaling. One may see that really we are dealing here with the **Newtonian** way notifying that our N-W-C formula (20) extends (1.12) from the Class III of [15] and our N-W-C formula (14) extends (1.10) from the Class III of [15]. Note also that (5.28) from [15] via re-scaling coincides with q - N-W-C case of (20) i.e. with $(q - Dobinski)$ formula. Here inevitable questions arise. For the Newtonian General class we have the extension (20) of Dobinski formula. So what about the corespondent formula for the Ward'ian - prefab'ian general class? ... And what about

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The Two General Ways of this paper? The one "easy" way is Newtonian. The other way seams to contain the q-extension as a kind of "singular point" in the domain of ψ-umbral extensions. Is there at and beyond this "singular point" of the second way an another non-Newtonian second path - all-embracing what was left beyond the first way ? Before an attempt to answer some of these questions let us encourage ourselves by just recalling another distinguished example.

This another crucial "Ward'ian - prefab'ian" example we owe to Gessel [48] with his q-analog of the exponential formula as expressed by the Theorem 5.2 from [48].

We also recall that the q-analog of the Stirling numbers of the second kind investigated by Morrison in Section 3 of [30] constitute **the same** example of Ward'ian - prefab'ian extension as in the Bender - Goldman - Wagner Ward - prefab example. As noticed there by Morrison the $(\gamma - e.g.f.)$ prefab exponential formula may equally well be derived from the corresponding Stanley's exponential formula in [59]. Let us then now come over to these exponential structures of Stanley with an expected impact on the current considerations (for definitions, theorems etc. see [59]). In this connection we recall quoting (notation from [59]) an important class of Stanley's Stirling - like numbers $\frac{S_{nk}}{M(n)}$ of the second and those of the first kind Stanley's Stirling - like numbers $\frac{s_{nk}}{M(n)}$. Both kinds are characteristic immanent for counting of exponential structures (or equivalently - corresponding exponential prefabs) and inheriting from there their combinatorial meaning. This is due to the fact [59] that "with each exponential structure is associated an "exponential formula" and more generally a "convolution formula" which is an analogue of the well known exponential formula of enumerative combinatorics" [59]. Consequently with each exponential structure are associated Stirling-like , Bell-like numbers and Dobinski - like formulas are expected also.

Exponential structures. Let $\{Q_n\}_{n \geq 0}$ be any exponential structure and let $\{M(n)\}_{n \geq 0}$ be its denominator sequence i.e. $M(n) =$ number of minimal elements of Q_n. Let $|Q_n|$ be the number of elements of the poset Q_n

$$|Q_n| = \sum_{\pi \in Q_n} 1.$$

Example: For $Q = \langle \Pi_n \rangle_{n \geq 1}$ where Π_n is the partition lattice of $[n]$ we have $M(n) = 1$.

Define "Whitney-Stanley" number $S_{n,k}$ to be the number of $\pi \in Q_n$ of degree equal to $k \geq 1$ i.e.

$$S_{n,k} = \sum_{\pi \in Q_n, |\pi|=k} 1.$$

Define $S_{n,k}$ - generating characteristic polynomials (vide exponential polynomials) in standard way

$$W_n(x) = \sum_{\pi \in Q_n} x^{|\pi|} = \sum_{k=1}^{n} S_{n,k} x^k.$$

Then the exponential formula $(W_0(x) = 1 = M(0))$ becomes

$$\sum_{n=0}^{\infty} \frac{W_n(x) y^n}{M(n) n!} = exp\{x q^{-1}(y)\},$$

where

$$q^{-1}(y) = \sum_{n=1}^{\infty} \frac{y^n}{M(n) n!} \equiv exp_\psi - 1,$$

with the obvious identification of ψ-extension choice here. Hence the polynomial sequence $\langle p_n(x) = \frac{W_n(x)}{M(n)} \rangle_{n \geq 0}$ constitutes the sequence of binomial polynomials i.e. the basic sequence of the corresponding delta operator $\hat{Q} = q(D)$. We observe then that

$$p_n(x) = \sum_{k=0}^{n} \frac{S_{n,k} x^k}{M(n)} \equiv \sum_{k=0}^{n} [0, 1, 2, ..., k; b_n] x^k$$

are just exponential polynomials' sequence for the equidistant nodes case i.e. **Newton-Stirling** numbers of the second kind $S_{n,k}^\sim \equiv \frac{S_{n,k}}{M(n)}$. Both numbers and the exponential sequence are being bi-univocally determined by the exponential structure Q. This is a special case of the one already considered and we have as in this "Lupas case" the Newton-Stirling-Dobinski formula:

$$p_n(x) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k) x^k}{k!} = \sum_{k=0}^{n} [0, 1, 2, ..., k; b_n] x^k, \qquad (N-S-Dob)$$

where $\langle b_n \rangle_{n \geq 0}$ is defined by

$$b_n(x) = \sum_{k=0}^{n} S_{n,k}^\sim x^{\underline{k}}.$$

Note the **identification** $b_n(x) = \frac{w_n(x)}{M(n)}$, where

$$w_n(x) = - \sum_{\pi \in Q_n} \mu(\hat{0}, \pi) \lambda^{|\pi|}.$$

μ is Möbius function and $\hat{0}$ is unique minimal element adjoined to Q_n. Corresponding Bell-like numbers are then given by

$$p_n(1) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k)}{k!} = \sum_{k=0}^{n} [0, 1, 2, ..., k; b_n], \qquad (N - S - Bell).$$

Besides those above - in Stanley's paper [59] there are implicitly present also inverse-dual "Whitney-Stanley" numbers $s_{n,k}$ of the first kind i.e.

$$s_{n,k} = - \sum_{\pi \in Q_n, |\pi| = k} \mu(\hat{0}, \pi).$$

On this occasion and to the end of considerations on exponential structures and Stirling like numbers let us make few remarks. q-extension of exponential formula applied to enumeration of permutations by inversions is to be find in Gessel's paper [48] (see there Theorem 5.2.) where among others he naturally arrives at the q-Stirling numbers of the first kind giving to them combinatorial interpretation. Recent extensions of the exponential formula in the prefab language [58] are to be find in [60]. Then **note**: exponential structures, prefab exponential structures (extended ones - included) i.e. schemas where exponential formula holds-imply the existence of Stirling like and Bell like numbers. As for the Dobinski-like formulas one needs binomial or extended binomial coefficients' convolution as it is the case with ψ-extensions of umbral calculus in its operator form.

Information. On the basis of [60] the present author introduces new prefab posets' Whitney numbers in [61]. Two extreme in a sense constructions are proposed there. Namely the author of [61] introduced two natural partial orders: one $\leq$ in grading-natural subsets of cobweb's prefabs sets [60] and in the second proposal one endows the set sums of the so called "prefabiants" with such another partial order that one arrives at Bell-like numbers including Fibonacci triad sequences introduced by the present author in [62].

A.3

Other Generalizations in brief. We indicate here THREE kinds of extensions of Stirling and Bell numbers - including those which appear in coherent

states' applications in quantum optics on one side or in the extended rook theory on the other side. In the supplement for this brief account to follow on this topics let us note that apart from applications to extended coherent states' physics of quantum oscillators or strings [6 - 11, 63, 64] and related Feymann diagrams' description [65] where we face the spectacular and inevitable emergence of extended Stirling and Bell numbers (consult also [66]) there exists a good deal of work done on *discretization* of space - time [67] and/or Schrodinger equation using umbral methods [68] and GHW algebra representations in particular (see: [67, 68] for references).

A.3.I. An analog of logarithmic algebra. An extension of binomiality property from the algebra of formal power series to the algebra of formal Laurent series *and then beyond* leading to the Loeb's [69] iterated logarithmic algebra - was realized by Roman [70, 71] with the basic Logarithmic Binomial Formula at the start. The Logarithmic Binomial Formula and the iterated logarithmic algebra may be given their ψ-analog including q-analog of the Logarithmic Fib-binomial Formula as shown in [72].

The extension of the iterated logarithmic algebra from [69] is the logarithmic algebra of Loeb-Rota [73]. This generalization of the formal Laurent series algebra retains main features and structure of an umbral calculus. Among others it allows for logarithmic analog of Appell or Sheffer polynomials. These give rise to Stirling- type formulas already in [69]. In [74] Kholodov has invented an analog of the logarithmic algebra from [69] in the shape of an umbral calculus on logarithmic algebras. Specifically (Example 3.1 in [74]) the basic logarithmic algebra constructed via Jackson derivative ∂_q gives rise to the analog of q-Stirling formula. The *mnemonic natural question* arises: are similar constructs performable for ψ-derivatives ∂_ψ ?

A.3.II. Milne's Dobinski formula. In the classical umbral calculus represented by the finite operator calculus of Rota the clue and source example of delta operator is

$$\Delta = exp\{D\} - I.$$

Naturally such delta operator generates Stirling numbers of the second kind via

$$k!\begin{Bmatrix} n \\ k \end{Bmatrix} = \Delta^k x^n|_{x=0}.$$

Accordingly clue and source example of delta operator of the ψ- umbral calculus would be

$$\Delta_\psi = exp\{\partial_\psi\} - I.$$

However already Milne's q-extension [12] - contrary to the above - does not rely on Jackson derivative ∂_q and it reads

$$k_q! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q = \Delta_q^k x_q^n|_{x=0},$$

where Δ_q^k - th k-th difference operator is defined inductively so that

$$\Delta_q^k = (exp\{D\} - q^{k-1}I)(exp\{D\} - q^{k-2}I)(exp\{D\} - q^{k-3}I)...(exp\{D\} - q^0 I)$$

The corresponding q-Dobinski formula (1.26 in [12]) looks " ψ-familiar" (see: (35) above):

$$B_{q,n+1} = exp_q^{-1} \sum_{1 \leq k} \frac{k_q^n}{(k-1)_q!}. \tag{36}$$

The obvious challenge is an eventual application of that type extension to other umbral calculi - including the analog of the logarithmic algebra from [73] with use of $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_\psi^{\sim}$ perhaps.

A.3.III. Normal ordering accomplishment and generating functions as coherent states. While staying with formal power series algebra or even its subalgebra of polynomials - still valuable extension have been applied as desired from at least two points of view: **a)** statistics , **b)** normal ordering task for quantum oscillator and strings.

a) As for statistics recall that q-Stirling numbers of the second kind may be treated as generating functions for various statistics counting (see: [15] and references therein). This type of role has been given by Wachs and White in [16] to p, q- Stirling number $S_{p,q}(n, k)$ which is generating function for the two different joint distribution set partitions statistics. Wachs and White in [16] also hade proposed interpretations of their p, q-analogue of Stirling numbers in terms of rook placements and restricted growth functions. From the defining recurrence (4) in [16] one sees that $S_{1,q}(n, k) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q^{\sim}$.

b) Similar two parameter r, s-Stirling numbers $S_{r,s}(n, k|q)$ arise in the normal ordering accomplishment for the expression $[(a^+)^r a^s]^n$ [6,7] where a^+ and a stay for creation and annihilation operators for q-deformed quantum oscillator which is equivalent to say that $aa^+ - qa^+a = 1$. (For example $a = \partial_q, \quad a^+ = \hat{x}$). From the recurrence (50) in [7] one sees that $S_{r,1}(n, k|q) = \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q$. This special case of Gould-Carlitz-Milne q-Stirling numbers $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q$ appears in [8]. The method to use it in order to recognize

coherent states as combinatorial objects was invented by Katriel in [6]. The authors of [9-11, 63, 64] develop the consistent scheme of applications of the properties of $S_{r,s}(n, k|q = 1) \equiv S_{r,s}(n, k)$. These include [11] closed-form expressions for $S_{r,s}(n, k)$, recursion relations, generating functions, Dobinski-type formulas. Recall that generating functions are identified with special expectation values in boson coherent states. Recall that $S_{r,s}(n, k)$ in terminology proposed in this paper are the special case of Newton-Stirling numbers of the second kind and correspondingly - Dobinski-type formulas are - see the Coherent States' Example in A.2. above. A new perspective opens while considering normal ordering task not only for quantum oscillator but also for strings which are "many, many oscillators". The first steps had been spectacular accomplished by the authors of [63]. These authors obtained not only analytical expressions but also a combinatorial interpretation of the corresponding "very much extended" Stirling and Bell numbers. Their properties are interpreted in [63] in terms of specific graphs. At the same time the authors of [63] consider an invention of a q-analog of [63] to be "an outstanding problem".

A.3.IV. From Howard via Hsu and Yu and Shiue to Remmenl and Wachs extensions. Information in brief. Howard's [75], via Hsu's and Yu's [76] and Shiue's [77] to Remmenl and Wachs [78] **sequence of extensions** starts with *degenerate* weighted Stirling numbers [75] used later on by the authors of [76, 77] to propose respectable, unified approach to generalized Stirling numbers. This sequence of extensions ends with elaborated extended rook theory [78] with its generalized Stirling numbers and (p, q)-analogues of Hsu and Shiue extensions. Recall that (p, q)-analogues of Stirling numbers were introduced by Wachs and White in [16]. In more detail. Hsu and Shiue had provided a unified scheme for many extensions of Stirling numbers of both kinds known before [77]. They introduced corresponding unified extensions under the notation:

$$\overline{S}_{n,k}^{1}(\alpha, \beta, r) \quad and \quad \overline{S}_{n,k}^{2}(\alpha, \beta, r),$$

such that $\overline{S}_{n,k}^{1}(1, 0, 0) = \begin{bmatrix} n \\ k \end{bmatrix}$ and $\overline{S}_{n,k}^{2}(1, 0, 0) = \begin{Bmatrix} n \\ k \end{Bmatrix}$ and $\overline{S}_{n,k}^{1}(\alpha, \beta, r) = \overline{S}_{n,k}^{2}(\beta, \alpha, -r)$.

Guided by Wachs and White ideas from [16] Remmel and Wachs have defined in [78] two natural p, q-analogues of Hsu and Shiue extensions

$$\overline{S}_{n,k}^{i}(\alpha, \beta, r), \quad i = 1, 2.$$

For that to do Remmel and Wachs have used what we would call the $\psi_{p,q}$ admissible sequence $\psi_{p,q} = \langle [n]_{p,q}\rangle_{n\geq 0}$ where $[\gamma]_{p,q} = \frac{p^\gamma - q^\gamma}{p-q}$ which for $\gamma \in N \cup \{0\}$ becomes the known [16] extension of Gauss extension i.e. $[n]_{p,q} = q^{n-1} + pq^{n-2} + ... + p^{n-2}q + p^{n-1}$. Factorials and $\psi_{p,q}$-binomial coefficients are then defined accordingly naturally (see: A.1. Notation.)

A.4. Extended umbral calculus and some corresponding extensions of Stirling and Bell numbers. Further examples. In this part of our presentation we just list some examples at hand where evidently ψ-extension is behind the scenario for special ψ-admissible sequence choices.

Example A.4.1. In Katriel and Kibler's celebrated paper [8] on normal ordering for deformed boson operators and operator-valued deformed Stirling numbers one uses the following ψ-admissible sequence

$$\langle \ \psi_n^{p,q} = \frac{1}{[n]!}\rangle_{n\geq 0}, \quad [n] = \frac{q^n - p^n}{q-p},$$

from Wachs and White source paper [16].

Example A.4.2. In Schork's paper [79] on fermionic relatives of Stirling and Lah numbers one uses the following ψ-admissible sequence

$$\langle \ \psi_n = \frac{1}{[n]^F!}\rangle_{n\geq 0}, \quad [n]^F = n_{q=-1} = \frac{1 - (-1)^n}{2}.$$

Example A.4.3. In Parthasarathy's paper [80] on fermionic numbers and their roles in some physical problems one uses the following ψ-admissible sequence

$$\langle \ \psi_n = \frac{1}{[n]^f!}\rangle_{n\geq 0}, \quad [n]^f == \frac{1 - (-1)^n q^n}{1 + q}.$$

Here in [80] the q-fermion numbers emerging from the q-fermion oscillator algebra are used to reproduce the q-fermionic Stirling and Bell numbers. New recurrence relations for the expansion coefficients in the 'anti-normal ordering' of the q-fermion operators are derived. Corresponding ψ-extended Dobinski formula (see: (15) in [80]) is derived.

Example A.4.4. In the paper [11] on extended Bell and Stirling numbers from hypergeometric exponentiation one uses the following ψ-admissible sequence

$$\langle \ \psi_n^L = \frac{1}{n!^{L+1}}\rangle_{n\geq 0}.$$

Here in [11] elements of ψ^L-umbral calculus are at work. Among others the corresponding ψ^L-extended Dobinski formula (see: (15) in [11]) is derived.

Example A.4.5. In the paper [81] on representations of the so called "Monomiality Principle" with Sheffer-type polynomials and boson normal ordering - just the standard ψ-admissible sequence choice

$$\langle\ \psi_n = \frac{1}{n!}\rangle_{n \geq 0}.$$

of the classical non-extended umbral calculus is naturally abounding in uncountable formal series indicators of delta operators examples; see: a) - g) page 3 in [81]. Note there also GHW - algebra formula (12). As for what the authors re-discover (?) to be the so called "monomiality principle" one should note and compare this with the source paper [56] from 1978 by George Markowsky on "Differential operators and Theory of Binomial Enumeration". In particular see the GHW - algebra in spirit Theorem 4.3 in [56] (see also [1, 33] for more on that).

A.5. Historical and bibliographical relevant remarks. To this end we shall here list few *peculiar* relevant remarks of historical and bibliographical character.

A.5.1.Remark The history of GHW algebra has its roots not later then since Graves' work [32] "On the principles which regulate the interchange of symbols in certain symbolic equations" from (**1853-1857**). See [82] by O.V. Viskov "On One Result of George Boole" (in Russian) from 1997.

A.5.2.Remark Generalizations given by formulas (3) and (4) from Cakic and Milovanovic paper [83] for another extensions of Stirling numbers of the second kind as well as their related properties are an old result published in d'Ocagne M. article in **1887** [84]. Many other later generalizations (see [83]) are consequence of Chak's work [85] and Toscano papers [86-88]. The relevant papers of importance (see: [83]) are those [89-93] and [75].

A.5.3.Remark The relevance of Schlömilch's work [94] from **1852** to N-W-C Stirling numbers is taken down here with pleasure. Another interesting paper refereeing directly to the original Dobinski's work [4] and Dobinski's point of view is the Fekete's paper [95] from 1999.

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