Carlitz q-Bernoulli Numbers and q-Stirling Numbers

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Abstract. In this paper, we consider Carlitz q-Bernoulli numbers and q-stirling numbers of the first and the second kind. From the properties of q-stirling numbers, we derive many interesting formulae associated with Carlitz q-Bernoulli numbers. Finally, we will prove

$$\beta_{n,q}=\sum_{m=0}^{n}\sum_{k=m}^{n}\frac{1}{(1-q)^{n+m-k}}\sum_{d_0+\cdots+d_k=n-k}q^{\sum_{i=0}^{k}id_i}s_{1,q}(k,m)(-1)^{n-m}\frac{m+1}{[m+1]_q},$$

where $\beta_{n,q}$ are called Carlitz q-Bernoulli numbers.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of algebraic closure of $\mathbb{Q}_p$. For d a fixed positive integer with $(p,d)=1$, let

$$\begin{aligned} X &= X_d=\varprojlim_N \mathbb{Z}/dp^N\mathbb{Z}, \qquad X_1=\mathbb{Z}_p,\\ X^* &= \bigcup_{\substack{0<a<dp\\(a,p)=1}} a+dp\mathbb{Z}_p,\\ a+dp^N\mathbb{Z}_p &= \{x\in X \mid x\equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a\in\mathbb{Z}$ lies in $0\le a<dp^N$, see [1-21]. The p-adic absolute value in $\mathbb{C}_p$ is normalized so that $|p|_p=1/p$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q\in\mathbb{C}$ or a p-adic number $q\in\mathbb{C}_p$. If $q\in\mathbb{C}_p$, then we assume $|q-1|_p<p^{-\frac{1}{p-1}}$, so that $q^x=\exp(x\log q)$ for $|x|_p\le 1$. We use the notation $[x]_q=[x:q]=\dfrac{1-q^x}{1-q}$. For $f\in C^{(1)}(\mathbb{Z}_p)=\{f\mid f'\in C(\mathbb{Z}_p)\}$, let us start with the expressions

$$\frac{1}{[p^N]_q}\sum_{0\le j<p^N}q^jf(j)=\sum_{0\le j<p^N}f(j)\mu_q(j+p^N\mathbb{Z}_p),\ \text{see [6, 8]},$$

representing q-analogue of Riemann sums for f. The p-adic q-integral of a function $f\in C^{(1)}(\mathbb{Z}_p)$ is defined by

$$\int_X f(x)d\mu_q(x)=\int_{\mathbb{Z}_p}f(x)d\mu_q(x)=\lim_{N\to\infty}\frac{1}{[p^N]_q}\sum_{x=0}^{p^N-1}f(x)q^x, \quad \text{see [8].}$$

For $f \in C^{(1)}(\mathbb{Z}_p)$, it is easy to see that,

$$|\int_{\mathbb{Z}_p} f(x) d\mu_q(x)|_p \leq p\|f\|_1, \quad \text{see } [6-14],$$

where $\|f\|_1 = \sup\left\{|f(0)|_p, \ \sup_{x\neq y}|\frac{f(x)-f(y)}{x-y}|_p\right\}$. If $f_n \to f$ in $C^{(1)}(\mathbb{Z}_p)$, namely $\|f_n - f\|_1 \to 0$, then

$$\int_{\mathbb{Z}_p} f_n(x) d\mu_q(x) \to \int_{\mathbb{Z}_p} f(x) d\mu_q(x), \text{ see } [6-10].$$

The q-analogue of binomial coefficient was known as $\begin{bmatrix} x \\ n \end{bmatrix}_q = \frac{[x]_q[x-1]_q \cdots [x-n+1]_q}{[n]_q!}$, where $[n]_q! = \prod_{i=1}^n [i]_q$, (see [1, 5, 6, 10, 11]). From this definition, we derive,

$$\begin{bmatrix} x+1 \\ n \end{bmatrix}_q = \begin{bmatrix} x \\ n-1 \end{bmatrix}_q + q^x \begin{bmatrix} x \\ n \end{bmatrix}_q = q^{x-n} \begin{bmatrix} x \\ n-1 \end{bmatrix}_q + \begin{bmatrix} x \\ n \end{bmatrix}_q, \quad \text{cf. } [6, 10].$$

Thus, we have $\int_{\mathbb{Z}_p} \begin{bmatrix} x \\ n \end{bmatrix}_q d\mu_q(x) = \frac{(-1)^n}{[n+1]_q} q^{n+1-\binom{n+1}{2}}$. If $f(x) = \sum_{k\geq 0} a_{k,q} \begin{bmatrix} x \\ k \end{bmatrix}_q$ is the q-analogue of Mahler series of strictly differentiable function f, then we see that

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \sum_{k\geq 0} a_{k,q} \frac{(-1)^k}{[k+1]_q} q^{k+1-\binom{k+1}{2}}.$$

Carlitz q-Bernoulli numbers $\beta_{k,q} (= \beta_k(q))$ can be determined inductively by

$$\beta_{0,q} = 1, \quad q(q\beta+1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing β^i by $\beta_{i,q}$, (see [2, 3, 4]). In this paper, we study the q-stirling numbers of the first and the second kind. From these q-stirling numbers, we derive some interesting q-stirling numbers identities associated with Carlitz q-Bernoulli numbers. Finally we will prove the following formula :

$$\beta_{n,q} = \sum_{m=q}^{n} \sum_{k=m}^{n} \frac{1}{(1-q)^{n+m-k}} \sum_{d_0+\cdots+d_k=n-k} q^{\sum_{i=0}^k id_i} s_{1,q}(k,m)(-1)^{n-m} \frac{m+1}{[m+1]_q},$$

where $s_{1,q}(k,m)$ is the q-stirling number of the first kind.

2. q-Stirling numbers and Carlitz q-Bernoulli numbers

For $m \in \mathbb{Z}_+$, we note that

$$\beta_{m,q} = \int_{\mathbb{Z}_p} [x]_q^m d\mu_q(x) = \int_X [x]_q^m d\mu_q(x).$$

From this formula, we derive

$$\beta_{0,q} = 1, \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing β^i by $\beta_{i,q}$. By the simple calculation of p-adic q-integral on $\mathbb{Z}_p$, we see that

$$\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{i+1}{[i+1]_q}, \tag{1}$$

where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!} = \dfrac{n(n-1)\cdots(n-i+1)}{i!}$. Let $F(t)$ be the generating function of Carlitz q-Bernoulli numbers. Then we have

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} \beta_{n,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \lim_{\rho \to \infty} \frac{1}{[p^\rho]_q} \sum_{x=0}^{p^\rho - 1} q^x e^{[x]_q t} \\ &= \sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \left\{ \sum_{k=0}^{\infty} \binom{n}{k} \frac{k+1}{[k+1]_q} (-1)^k \right\} \frac{t^n}{n!} \\ &= e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(1-q)^k} \frac{k+1}{[k+1]_q} \frac{t^k}{k!}. \end{aligned} \tag{2}$$

From (2) we note that,

$$\begin{aligned} F(t) &= e^{\frac{t}{1-q}} + e^{\frac{t}{1-q}} \sum_{k=1}^{\infty} \frac{(-1)^k}{(1-q)^{k-1}} \left(\frac{k}{1-q^{k+1}} \right) \frac{t^k}{k!} \\ &\quad + e^{\frac{t}{1-q}} \sum_{k=1}^{\infty} \frac{(-1)^k}{(1-q)^{k-1}} \left(\frac{1}{1-q^{k+1}} \right) \frac{t^k}{k!} \\ &= -t \sum_{n=0}^{\infty} q^{2n} e^{[n]_q t} + (1-q) \sum_{n=0}^{\infty} q^n e^{[n]_q t}. \end{aligned} \tag{3}$$

Therefore we obtain the following:

Lemma 1. *Let $F(t) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} [x]_q^n d\mu_q(x) \frac{t^n}{n!}$. Then we have*

$$F(t) = -t \sum_{n=0}^{\infty} q^{2n} e^{[n]_q t} + (1-q) \sum_{n=0}^{\infty} q^n e^{[n]_q t}.$$

The q-Bernoulli polynomials in the variable x in $\mathbb{C}_p$ with $|x|_p \leq 1$ are defined by

$$\beta_{n,q}(x) = \int_{\mathbb{Z}_p} [x+t]_q^n d\mu_q(t) = \int_X [x+t]_q^n d\mu_q(x). \tag{4}$$

Thus we have

$$\begin{aligned}\int_{\mathbb{Z}_p} [x+t]_q^n d\mu_q(x) &= \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \int_{\mathbb{Z}_p} [t]_q^k d\mu_q(t) \\ &= \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{kx} \beta_{k,q} = (q^x \beta + [x]_q)^n.\end{aligned}$$

From (4) we derive

$$\int_{\mathbb{Z}_p} [x+t]_q^n d\mu_q(x) = \beta_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^k q^{kx} \frac{k+1}{[k+1]_q}. \tag{5}$$

Let $F(t,x)$ be the generating function of q-Bernoulli polynomials. By (5) we see that

$$F(t,x) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!} = e^{\frac{t}{1-q}} \sum_{k=0}^{\infty} \frac{1}{(1-q)^k} q^{kx} (-1)^k \frac{k+1}{[k+1]_q} \frac{t^k}{k!}. \tag{6}$$

From (6) we note that

$$F(t,x) = -t \sum_{n=0}^{\infty} q^{2n+x} e^{[n+x]_q t} + (1-q) \sum_{n=0}^{\infty} q^n e^{[n+x]_q t}. \tag{7}$$

By (4) and (7), we easily see that

$$[m]_q^{k-1} \sum_{i=0}^{m-1} q^i \beta_{k,q^m}(\frac{x+i}{m}) = \beta_{k,q}(x), \quad m \in \mathbb{N}, k \in \mathbb{Z}_+. \tag{8}$$

If we take $x=0$ in (8), then we have

$$[n]_q \beta_{n,q} = \sum_{k=0}^{m} \binom{m}{k} \beta_{k,q^n} [n]_q^k \sum_{j=0}^{n-1} q^{j(k+1)} [j]_q^{n-k}.$$

By (2), (6) and (7), we see that

$$-\sum_{l=0}^{\infty} q^{2l+n} e^{[n+l]_q t} + \sum_{l=0}^{\infty} q^{2l} e^{[l]_q t} = \sum_{m=1}^{\infty} (m \sum_{l=0}^{n-1} q^{2l} [l]_q^{m-1}) \frac{t^{m-1}}{m!}. \tag{9}$$

Note that $\sum_{l=0}^{\infty} q^{2l+n} e^{[n+l]_q t} + \sum_{l=0}^{n} q^{2l} e^{[l]_q t} = \frac{1}{t}(F(t,n) - F(t))$. Thus, we have

$$\sum_{m=0}^{\infty} (\beta_{m,q}(n) - \beta_{m,q}) \frac{t^m}{m!} = \sum_{m=0}^{\infty} (m \sum_{l=0}^{n-1} q^{2l} [l]_q^{m-1}) \frac{t^m}{m!}. \tag{10}$$

By comparing the coefficients on both sides in (10), we see that

$$\beta_{m,q}(n) - \beta_{m,q} = m \sum_{l=0}^{n-1} q^{2l} [l]_q^{m-1}. \tag{11}$$

Therefore we obtain the following:

Proposition 2. *For $m, n \in \mathbb{N}$, we have*

$$(q-1) \sum_{l=0}^{n-1} q^l [l]_q^m + \sum_{l=0}^{n-1} q^l [l]_q^{m-1} = \frac{1}{m} \sum_{l=0}^{m-1} \binom{m}{l} [n]_q^{m-l} q^{nl} \beta_{l,q} + (q^{mn} - 1) \beta_{m,q}.$$

Now we consider the q-analogue of Jordan factor as follows:

$$[x]_{k,q} = [x]_q [x-1]_q \cdots [x-k+1]_q = \frac{(1-q^x)(1-q^{x-1}) \cdots (1-q^{x-k+1})}{(1-q)^k}.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q^{n-k+1})}{(1-q)(1-q^2) \cdots (1-q^k)}, \tag{12}$$

where $[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$. The q-binomial formulas are known as

$$\prod_{i=1}^{n} (a + bq^{i-1}) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n}{k}} a^{n-k} b^k, \tag{13}$$

and

$$\prod_{i=1}^{n} (1 - bq^{i-1})^{-1} = \sum_{k=0}^{n} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q b^k.$$

The q-Stirling numbers of the first kind $s_{1,q}(n,k)$ and the second kind $s_{2,q}(n,k)$ are defined as

$$[x]_{n,q} = q^{-\binom{n}{2}} \sum_{l=0}^{n} s_{1,q}(n,l) [x]_q^l, \quad n = 0, 1, 2, \cdots, \tag{14}$$

and

$$[x]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}} s_{2,q}(n,k) [x]_{k,q}, \quad n = 0, 1, 2, \cdots, \text{ see [2, 3, 6]}. \tag{15}$$

The values $s_{1,q}(n,1), \quad n=1,2,3,\cdots$, and $s_{2,q}(n,2), \quad n=2,3,\cdots$, may be deduced from the following recurrence relation:

$$s_{1,q}(n,k)=s_{1,q}(n-1,k-1)-[n-1]_q s_{1,q}(n-1,k), \quad \text{see [2, 3, 6]},$$

for $k=1,2,\cdots,n$, $n=1,2,\cdots$, with initial conditions $s_{1,q}(0,0)=1$, $s_{1,q}(n,k)=0$ if $k>n$. For $k=1$, it follows that

$$s_{1,q}(n,1)=-[n-1]_q s_{1,q}(n-1,1), \quad n=2,3,\cdots,$$

and since $s_{1,q}(1,1)=1$, we have $s_{1,q}(n,1)=(-1)^{n-1}[n-1]_q!, \quad n=1,2,3,\cdots$. The recurrence relation for $k=2$ reduce to $s_{1,q}(n,2)+[n-1]_q s_{1,q}(n-1,2)=(-1)^{n-2}[n-2]_q!, \quad n=3,4,\cdots$. By simple calculation, we easily see that

$$\begin{aligned}\frac{(-1)^{n+1}s_{1,q}(n+1,2)}{[n]_q!}-\frac{(-1)^n s_{1,q}(n,2)}{[n-1]_q!} &= (-1)^{n+1}\frac{s_{1,q}(n+1,2)-[n]_q s_{1,q}(n,2)}{[n]_q!}\\ &= (-1)^{n+1}\frac{(-1)^{n+1}[n-1]_q!}{[n]_q!}=\frac{1}{[n]_q}, \quad n=2,3,4,\cdots.\end{aligned}$$

Thus we have

$$\frac{(-1)^n s_{1,q}(n,2)}{[n-1]_q!}=\sum_{k=1}^{n-1}\frac{1}{[k]_q}.$$

This is equivalent to $s_{1,q}(n,2)=(-1)^n[n-1]_q!\sum_{k=1}^{n-1}\frac{1}{[k]_q}$. It is easy to see that

$$\sum_{m=1}^{n}(-1)^{m+1}q^{\binom{m+1}{2}}\begin{bmatrix}n+1\\m+1\end{bmatrix}_q\sum_{k=1}^{m}\frac{1}{[k]_q}=\sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{\begin{bmatrix}n\\k\end{bmatrix}_q}{[k]_q}.$$

From this, we derive

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{1}{[k]_q}\left(\begin{bmatrix}n\\k\end{bmatrix}_q-\begin{bmatrix}n-1\\k\end{bmatrix}_q\right) &= \sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{1}{[k]_q}\left(q^{n-k}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\right)\\ =\frac{q^n}{[n]_q}\sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q &= \frac{q^n}{[n]_q}.\end{aligned}$$

Note that $\sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q=-\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q+1=1$. Thus, we have

$$\sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{\begin{bmatrix}n\\k\end{bmatrix}_q}{[k]_q}=\sum_{k=1}^{n-1}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{\begin{bmatrix}n-1\\k\end{bmatrix}_q}{[k]_q}+\frac{q^n}{[n]_q}.$$

Continuning this process, we see that

$$\sum_{k=1}^{n}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{[k]_q}=\sum_{k=1}^{n}\frac{q^n}{[k]_q}.$$

The p-adic q-gamma function is defined as $\Gamma_{p,q}(n)=(-1)^n\prod_{\substack{1\le j<n\\(j,p)=1}}[j]_q$. For all $x\in\mathbb{Z}_p$, we have $\Gamma_{p,q}(x+1)=E_{p,q}(x)\Gamma_{p,q}(x)$, where $E_{p,q}(x)=\begin{cases}-[x]_q & \text{if } |x|_p=1\\ -1 & \text{if } |x|_p<1.\end{cases}$ Thus, we easily see that

$$\log\Gamma_{p,q}(x+1)=\log E_{p,q}(x)+\log\Gamma_{p,q}(x). \tag{16}$$

From the differentiating on both sides in (16), we derive

$$\frac{\Gamma'_{p,q}(x+1)}{\Gamma_{p,q}(x+1)}=\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)}+\frac{E'_{p,q}(x)}{E_{p,q}(x)}.$$

Continuning this process, we have

$$\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)}=\left(\sum_{j=1}^{x-1}\frac{q^j}{[j]_q}\right)\frac{\log q}{q-1}+\frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}.$$

The classical Euler constant is known as $\gamma=\frac{\Gamma'(1)}{\Gamma(1)}$. In [15], Koblitz defined the p-adic q-Euler constant as

$$\gamma_{p,q}=-\frac{\Gamma'_{p,q}(1)}{\Gamma_{p,q}(1)}.$$

Therefore, we obtain the following:

Theorem 3. *For $x\in\mathbb{Z}_p$, we have*

$$\sum_{k=1}^{x-1}(-1)^{k+1}q^{\binom{k+1}{2}}\frac{\begin{bmatrix} x-1 \\ k \end{bmatrix}_q}{[k]_q}=\frac{q-1}{\log q}\left(\frac{\Gamma'_{p,q}(x)}{\Gamma_{p,q}(x)}-\gamma_{p,q}\right).$$

From (5), (12), (14) and (15), we derive the following theorem:

Theorem 4. *For $n,k\in\mathbb{Z}_+$, we have*

$$\beta_{n,q}=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\sum_{k=0}^{l}(q-1)^k\begin{bmatrix} l \\ k \end{bmatrix}_q\sum_{m=0}^{k}s_{1,q}(k,m)\beta_{m,q},$$

where $s_{1,q}(k,m)$ is the q-Stirling number of the first kind.

By simple calculation, we easily see that

$$\begin{aligned} q^{nt} &= ([t]_q(q-1)+1)^n = \sum_{m=0}^{n} \binom{n}{m} (-1)^m (1-q)^m [t]_q^m = \sum_{k=0}^{n} (q-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q [t]_{k,q} \\ &= \sum_{k=0}^{n} (q-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{m=0}^{k} s_{1,q}(k,m)[t]_q^m = \sum_{m=0}^{n} \left(\sum_{k=m}^{n} (q-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m) \right) [t]_q^m. \end{aligned}$$

Thus we note

$$\int_{\mathbb{Z}_p} q^{nt} d\mu_q(t) = \sum_{m=0}^{n} \left(\sum_{k=m}^{n} (q-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m) \right) \beta_{m,q}. \tag{17}$$

From the definition of p-adic q-integral on $\mathbb{Z}_p$, we also derive

$$\int_{\mathbb{Z}_p} q^{nt} d\mu_q(t) = \sum_{m=0}^{n} \binom{n}{m} (q-1)^m \beta_{m,q}. \tag{18}$$

By comparing the coefficients on the both sides of (17) and (18), we see that

$$\binom{n}{m} (q-1)^m = \sum_{k=m}^{n} (q-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m).$$

Therefore we obtain the following:

Theorem 5. *For $n \in \mathbb{N}, m \in \mathbb{Z}_+$, we have*

$$\binom{n}{m} = \sum_{k=m}^{n} (q-1)^{-m+k} \begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m).$$

From Theorem 5, we can also derive the following interesting formula for q-Bernoulli numbers:

Theorem 6. *For $n \in \mathbb{Z}_+$, we have*

$$\beta_{n,q} = \frac{1}{(1-q)^n} \sum_{m=0}^{n} \left(\sum_{k=m}^{n} (q-1)^{-m+k} \begin{bmatrix} n \\ k \end{bmatrix}_q s_{1,q}(k,m) \right) (-1)^m \frac{m+1}{[m+1]_q}.$$

From the definition of q-binomial coefficient, we easily derive

$$\begin{bmatrix} x+1 \\ n \end{bmatrix}_q = \begin{bmatrix} x \\ n-1 \end{bmatrix}_q + q^x \begin{bmatrix} x \\ n \end{bmatrix}_q = q^{x-n} \begin{bmatrix} x \\ n-1 \end{bmatrix}_q + \begin{bmatrix} x \\ n \end{bmatrix}_q. \tag{19}$$

By (19), we see that

$$\int_{\mathbb{Z}_p} \begin{bmatrix} x \\ n \end{bmatrix}_q d\mu_q(x) = \frac{(-1)^n}{[n+1]_q} q^{n+1-\binom{n+1}{2}}. \tag{20}$$

From the definition of q-Stirling number of the first kind, we also note that

$$\int_{\mathbb{Z}_p} [x]_{n,q} d\mu_q(x) = [n]_q! \int_{\mathbb{Z}_p} \begin{bmatrix} x \\ n \end{bmatrix}_q d\mu_q(x) = q^{-\binom{n}{2}} \sum_{k=0}^{n} s_{1,q}(n,k)\beta_{k,q}. \tag{21}$$

By using (20), (21), we see

$$(-1)^n \frac{q[n]_q!}{[n+1]_q} = \sum_{k=0}^{n} s_{1,q}(n,k)\beta_{k,q}. \tag{22}$$

From (15) and (21), we derive

$$\beta_{n,q} = q \sum_{k=0}^{n} s_{2,q}(n,k)(-1)^k \frac{[k]_q!}{[k+1]_q}.$$

Therefore we obtain the following:

Theorem 7. *For $n \in \mathbb{Z}_+$, we have*

$$\beta_{n,q} = q \sum_{k=0}^{n} s_{2,q}(n,k)(-1)^k \frac{[k]_q!}{[k+1]_q},$$

where $s_{2,q}(n,k)$ is the q-Stirling number of the second kind.

It is easy to see that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{d_0+\cdots+d_k=n-k} q^{\sum_{i=0}^{k} i d_i}. \tag{23}$$

By Theorem 4, we have the following:

Theorem 8. *For $n \in \mathbb{Z}_+$, we have*

$$\beta_{n,q} = \sum_{m=0}^{n} \sum_{k=m}^{n} \frac{1}{(1-q)^{n+m-k}} \sum_{d_0+\cdots+d_k=n-k} q^{\sum_{i=0}^{k} i d_i} s_{1,q}(k,m)(-1)^{n-m} \frac{m+1}{[m+1]_q},$$

where $s_{1,q}(k,m)$ is the q-Stirling number of the first kind.

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