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A Unified Approach to Generalized Stirling Numbers

Leetsch C. Hsu

Institute of Mathematics, Dalian University of Technology, Dalian 116024, China

and

Peter Jau-Shyong Shiue

Department of Mathematical Sciences, University of Nevada, Las Vegas, Las Vegas, Nevada 89154-4020

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It is shown that various well-known generalizations of Stirling numbers of the first and second kinds can be unified by starting with transformations between generalized factorials involving three arbitrary parameters. Previous extensions of Stirling numbers due to Riordan, Carlitz, Howard, Charalambides-Koutras, Gould-Hopper, Tsylova, and others are included as particular cases of our unified treatment. We have also investigated some basic properties related to our general pattern. © 1998 Academic Press

1. INTRODUCTION

As may be observed, a natural approach to generalizing Stirling numbers is to define Stirling number pairs as connection coefficients of linear transformations between generalized factorials. Of course, any useful generalization should directly imply some interesting special cases that have certain applications. The whole approach adopted in this paper is entirely different from that of Hsu and Yu [13], starting with generating functions, and various new results are provided by a unified treatment.

A recent paper [24] of Théorêt investigated some generating functions for the solutions of a type of linear partial difference equation with first-degree polynomial coefficients. Theoretically, it also provides a way of

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unifying various generalized Stirling numbers, since the difference equation considered could be suitably specialized to those recurrence relations satisfied by Stirling-type numbers.

What we will present here is a kind of general pattern that implies a more transparent unification of various well-known Stirling-type numbers and their basic properties investigated by previous authors.

1.1. *Basic Definition*

Let us denote $(z \mid \alpha)_n = z(z-\alpha)\cdots(z-n\alpha+\alpha)$ for $n = 1, 2, \ldots,$ and $(z \mid \alpha)_0 = 1$, where $(z \mid \alpha)_n$ is called the generalized factorial of z with increment α, and in particular we write $(z \mid 1)_n = (z)_n$ with $(z)_0 = 1$.

Generalizing the idea involved in the previous investigations due to Carlitz [3], Howard [11], Tsylova [26], and several others, we may define a Stirling-type pair $\{S^1, S^2\} = \{S^1(n,k), S^2(n,k)\} \equiv \{S(n,k;\alpha,\beta,r), S(n,k;\beta,\alpha,-r)\}$ by the inverse relations

$$(t \mid \alpha)_n = \sum_{k=0}^{n} S^1(n,k)(t-r \mid \beta)_k \tag{1}$$

$$(t \mid \beta)_n = \sum_{k=0}^{n} S^2(n,k)(t+r \mid \alpha)_k \tag{2}$$

where $n \in N$ (set of nonnegative integers), and the parameters α, β, and r are given real or complex numbers, with $(\alpha, \beta, r) \neq (0,0,0)$.

We may call $\{S^1, S^2\}$ an $\langle \alpha, \beta, r \rangle$-pair or a $\langle \beta, \alpha, -r \rangle$-pairs as well, in which S^1 and S^2 may be called the first member and the second member of the pair, respectively.

Evidently, the classical Stirling number pair $\{s(n,k), S(n,k)\}$ is the $\langle 1,0,0 \rangle$ pair. Indeed, the two kinds of Stirling numbers may be written in the form

$$s(n,k) = S(n,k;1,0,0), \qquad S(n,k) = S(n,k;0,1,0).$$

In particular, it is obvious that the binomial coefficients are given by

$$S(n,k;0,0,1) = \binom{n}{k}.$$

Note that the number $\sigma(n,k)$ discussed and extended by Doubilet et al. [8] and by Wagner [27] may be written as $k!S(n,k;0,1,0)$.

1.2. *Special Cases*

Here several other interesting special cases may be briefly mentioned:

(i) Lah numbers

$$\frac{n!}{k!}\binom{n-1}{k-1}$$

and

$$(-1)^{n-k}\frac{n!}{k!}\binom{n-1}{k-1}$$

form the $\langle -1, 1, 0\rangle$ pair. The signless Stirling number $|s(n,k)|$ and $(-1)^{n-k}S(n,k)$ form the $\langle -1, 0, 0\rangle$ pair. See a combinatorial set-theoretic approach given by Joni et al. [14] (cf. also [22]).

(ii) Carlitz's two kinds of weighted Stirling numbers (or Kontras' noncentral Stirling numbers) just form the $\langle 1, 0, -\lambda\rangle$ pair, where $\lambda \neq 0$ (cf. [3]).

(iii) Carlitz's two kinds of degenerate Stirling numbers form the $\langle 1, \theta, 0\rangle$ pair, with $\theta \neq 0$ (cf. [2]).

(iv) Howard's weighted degenerate Stirling numbers form the $\langle 1, \theta, -\lambda\rangle$ pair (cf. [11]).

(v) Gould-Hopper's noncentral Lah numbers are basically given by the first member of the $\langle 0, 1, -a+b\rangle$ pair (cf. [9]).

(vi) Riordan's noncentral Stirling numbers form the $\langle 1, 0, b-a\rangle$ pair (cf. [20]).

(vii) The noncentral C numbers extensively studied by Charalambides and Koutras practically belong to the first member of the $\langle 1/s, 1, -a+b\rangle$ pair (cf. [4]; see also [5]).

(viii) Tsylova's numbers $A_{\alpha\beta}(r,m)$ belong to the $\langle \alpha, \beta, 0\rangle$ pair (cf. [26]).

(ix) Todorov's numbers $a_{nk}(x)$ actually belong to the first member of the $\langle 1, x, 0\rangle$ pair (cf. [25]).

(x) Ahuja-Enneking's associated Lah numbers $B(n,r,k)$ just correspond to the first member of the $\langle -1/r, 1, 0\rangle$ pair (cf. [19]).

(xi) The r-Stirling numbers of the first kind fully developed by Broder [1] actually belong to the $\langle -1, 0, r\rangle$ pair, with n and k replaced by $n-r$ and $k-r$, respectively.

Of course, the above list may not be complete. Some of the above cases will be expounded in more detail as examples in Section 2.

It is worth noticing that the standard notation $\langle \alpha, \beta, r \rangle$ with three free parameters may help us to recognize some logical implicative relations among Stirling-type numbers. For instance, one may observe at once that the numbers $a_{nk}(x)$ mentioned in case (ix) are essentially the first kind of Carlitz degenerate Stirling numbers. Furthermore, it is clear that all of the numbers discussed in cases (iii), (ix), and (x) are included in case (viii). Moreover, a comparison of cases (vii) and (iv) will reveal that the statistically useful noncentral C numbers are just the second kind of Howard's weighted degenerate Stirling numbers.

1.3. *Orthogonality Relations*

It is clear that $\{(t \mid \alpha)_n\}$ and $\{(t - r \mid \beta)_n\}$ form two different sets of bases for the linear space of polynomials, so that by substituting (1) into (2) (or (2) into (1)), one may easily get the orthogonality relations

$$\sum_{k=n}^{m} S^1(m,k)S^2(k,n) = \sum_{k=n}^{m} S^2(m,k)S^1(k,n) = \delta_{mn}, \tag{3}$$

δ_{mn} being the Kronecker symbol, viz., $\delta_{mn} = 1$ (for $m = n$), $= 0$ (otherwise). Consequently, from (3) one easily obtains the inverse relations for $n \in N$:

$$f_n = \sum_{k=0}^{n} S^1(n,k) g_k \quad \Leftrightarrow \quad g_n = \sum_{k=0}^{n} S^2(n,k) f_k. \tag{4}$$

Although we have defined two kinds of Stirling-type numbers $S^1(n,k)$ and $S^2(n,k)$, generally it suffices to consider one of them, since the parameters α, β, and r are entirely arbitrary.

In this paper we will investigate recurrence relations, generating functions, convolution formulas, and congruence properties, as well as asymptotic expansions for the numbers $S(n,k;\alpha,\beta,r)$. Moreover, Section 4 will be devoted to establishing a kind of extended Dobinski formulae for the generalized exponential polynomials

$$S_n(x) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r) x^k, \tag{5}$$

as well as for the generalized Bell numbers,

$$W_n = S_n(1) = \sum_{k=0}^{n} S(n,k;\alpha,\beta,r). \tag{6}$$

2. GENERATING FUNCTIONS

For brevity we will always use $S(n,k)$ to denote $S(n,k;\alpha,\beta,r)$, unless there is a need to indicate α, β, and r explicitly.

In the first place notice that relation (1) implies the following:

$$S(0,0)=1, \qquad S(n,n)=1, \qquad S(1,0)=r.$$

Furthermore, as a convention we assume $S(n,k)=0$ for $k>n$.

THEOREM 1. *For the numbers $S(n,k)$ defined by* (1), *we have the recurrence relations*

$$S(n+1,k)=S(n,k-1)+(k\beta-n\alpha+r)S(n,k) \tag{7}$$

where $n\geq k\geq 1$. In particular, we have

$$S(n,0)=(r\mid\alpha)_n. \tag{8}$$

Proof. In accordance with (1), we may write

$$\begin{aligned}
&\sum_{k=0}^{n+1} S(n+1,k)(t-r\mid\beta)_k \\
&\quad=(t\mid\alpha)_n(t-n\alpha) \\
&\quad=\sum_{k=0}^{n} S(n,k)(t-r\mid\beta)_k[(t-r-k\beta)+(k\beta-n\alpha+r)] \\
&\quad=\sum_{k=0}^{n} S(n,k)(t-r\mid\beta)_{k+1}+\sum_{k=0}^{n} S(n,k)(k\beta-n\alpha+r)(t-r\mid\beta)_k \\
&\quad=\sum_{k=1}^{n+1} S(n,k-1)(t-r\mid\beta)_k+\sum_{k=0}^{n} S(n,k)(k\beta-n\alpha+r)(t-r\mid\beta)_k.
\end{aligned}$$

Thus, identifying the coefficients of $(t-r\mid\beta)_k(k\geq 1)$ of the first and last expressions, we obtain (7). Furthermore, for the terms corresponding to $k=0$, we find

$$S(n+1,0)=S(n,0)(r-n\alpha), \qquad n=0,1,2,\ldots.$$

Consequently, we get

$$S(n,0)=S(0,0)r(r-\alpha)\cdots(r-n\alpha+\alpha)=(r\mid\alpha)_n. \quad \blacksquare$$

To find a vertical generating function (GF) for the sequence $\{S(n,k)\}$, we need the following:

LEMMA. *The vertical GF of* $\{S(n,k)\}$

$$y_k(t) = \sum_{n\geq 0} S(n,k)\frac{t^n}{n!} \tag{9}$$

satisfies the difference-differential equation

$$(1+\alpha t)\frac{d}{dt}y_k(t) - (k\beta + r)y_k(t) = y_{k-1}(t), \tag{10}$$

where $k = 1,2,3,\ldots,$ *and* $y_k(0) = 0$ *for* $k \geq 1$, *and*

$$y_0(t) = (1+\alpha t)^{r/\alpha}. \tag{11}$$

Proof. The property $y_k(0) = 0 (k \geq 1)$ is obvious from (9). Now making use of Theorem 1, we have

$$\sum_{n\geq 0} S(n+1,k)\frac{t^n}{n!} = \sum_{n\geq 0}(k\beta - n\alpha + r)S(n,k)\frac{t^n}{n!} + \sum_{n\geq 0} S(n,k-1)\frac{t^n}{n!}.$$

This may be rewritten in the form

$$\sum_{n\geq 1} S(n,k)\frac{t^{n-1}}{(n-1)!} + \alpha\sum_{n\geq 1} S(n,k)\frac{t^n}{(n-1)!} - (k\beta + r)y_k(t) = y_{k-1}(t),$$

which is identical to the following:

$$(1+\alpha t)\sum_{n\geq 1} S(n,k)\frac{t^{n-1}}{(n-1)!} - (k\beta + r)y_k(t) = y_{k-1}(t).$$

This is precisely equivalent to Eq. (10).

Moreover, we have

$$y_0(t) = \sum_{n\geq 0} S(n,0)\frac{t^n}{n!} = \sum_{n\geq 0}(r\mid\alpha)_n\frac{t^n}{n!} = \sum_{n\geq 0}\binom{r/\alpha}{n}(\alpha t)^n = (1+\alpha t)^{r/\alpha}.$$

Hence the lemma is proved. ∎

THEOREM 2. *The generalized Stirling numbers* $S(n,k) \equiv S(n,k;\alpha,\beta,r)$, *with* $\alpha\beta \neq 0$, *have the vertical GF*,

$$(1+\alpha t)^{r/\alpha}\left(\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right)^{k} = k!\sum_{n\geq 0} S(n,k)\frac{t^n}{n!}. \qquad (12)$$

Proof. Let the LHS of (12) be denoted by $k!\phi_k(t)$. Notice that (10) has the unique solution $y_k(t)$ under the conditions $y_k(0) = 0$ $(k \geq 1)$ and (11). Thus it suffices to show that $\phi_k(t)$ is the unique solution of (10), so that $\phi_k(t) = y_k(t)$.

Evidently, $\phi_k(0) = 0$ for $k \geq 1$ and $\phi_0(t) = (1+\alpha t)^{r/\alpha} = y_0(t)$. Moreover, using elementary differentiation and algebraic computations, one can verify that

$$(1+\alpha t)\frac{d}{dt}\phi_k(t) - (k\beta + r)\phi_k(t) = \phi_{k-1}(t), \qquad (k \geq 1).$$

Hence we conclude that $\phi_k(t) = y_k(t)$, and (12) is proved. (Here the almost routine procedure of computation is omitted.)

Note that the form of LHS of (12) has been also determined via two lemmas by Théorêt (cf. (15) of [24]).

Remark 1. Similarly, there is also a GF for the second member, $S(n,k;\beta,\alpha,-r)$. It is called a *conjugate form* of (12), which may be obtained simply by changing (α,β,r) into $(\beta,\alpha,-r)$, namely,

$$(1+\beta t)^{-r/\beta}\left(\frac{(1+\beta t)^{\alpha/\beta}-1}{\alpha}\right)^{k} = k!\sum_{n\geq 0} S(n,k;\alpha,\beta,-r)\frac{t^n}{n!}.$$

Remark 2. The condition $\alpha\beta \neq 0$ is seen necessary for the LHS of (12). However, one can still let $\alpha \to 0+$ or $\beta \to 0+$ to get suitable limits. In fact, taking $r = 0$, $\alpha = 1$ and letting $\beta \to 0+$, we easily find that (12) leads to the following:

$$(\ln(1+t))^k = k!\sum_{n\geq 0} S^1(n,k)\frac{t^n}{n!}. \qquad (12.1)$$

This is precisely the GF for the classical Stirling numbers of the first kind. Similarly, taking $r = 0$, $\beta = 1$, and $\alpha \to 0+$, we see that (12) gives the GF for the classical Stirling numbers of the second kind:

$$(e^t - 1)^k = k!\sum_{n\geq 0} S^2(n,k)\frac{t^n}{n!}. \qquad (12.2)$$

Thus, with a view to the GF of $S(n, k; \alpha, \beta, r)$ numbers, one can also write the classical pair of Stirling numbers in the form

$$s(n,k) = S(n,k;1,0+,0), \qquad S(n,k) = S(n,k;0+,1,0).$$

Briefly, it is the $\langle 1, 0+, 0\rangle$ pair.

EXAMPLE 1. Carlitz's pair of weighted Stirling numbers $\{S^1, S^2\} \equiv \{(-1)^{n+k} R_1(n,k,\lambda), R_2(n,k,\lambda)\}$ is the $\langle 1, 0+, \lambda\rangle$ pair. More precisely, their GF are given by (cf. [3])

$$(1+t)^{-\lambda}(\ln(1+t))^k = k! \sum_{n\geq k} (-1)^{n+k} R_1(n,k,\lambda)\frac{t^n}{n!} \tag{12.3}$$

$$e^{\lambda t}(e^t - 1)^k = k! \sum_{n\geq k} R_2(n,k,\lambda)\frac{t^n}{n!} \tag{12.4}$$

As is easily seen, (12.3) and (12.4) follow from (12) by letting $r = -\lambda$, $\alpha = 1$, $\beta \to 0+$; and letting $r = \lambda$, $\beta = 1$, $\alpha \to 0+$, respectively. In other words, (12.3) and (12.4) are implied by (12) and its conjugate form, respectively.

EXAMPLE 2. Carlitz's pair of degenerate Stirling numbers $\{(-1)^{n+k} \times S_1(n,k \mid \theta), S(n,k) \mid \theta\}$ is the $\langle 1, \theta, 0\rangle$ pair. Their GFs

$$\left(\frac{(1+t)^\theta - 1}{\theta}\right)^k = k! \sum_{n\geq k} (-1)^{n+k} S_1(n,k \mid \theta)\frac{t^n}{n!} \tag{12.5}$$

$$\left((1+\theta t)^\mu - 1\right)^k = k! \sum_{n\geq k} S(n,k \mid \theta)\frac{t^n}{n!} \tag{12.6}$$

with $\theta\mu = 1$ are also implied by (12) and its conjugate form (cf. [2]).

EXAMPLE 3. Howard's pair of weighted degenerate Stirling numbers $\{S^1, S^2\} \equiv \{(-1)^{n+k} S_1(n,k,(\lambda+\theta) \mid \theta), S(n,k,\lambda \mid \theta)\}$ is precisely the $\langle 1, \theta, -\lambda\rangle$ pair. In fact, their GF

$$(1+t)^{-\lambda}\left(\frac{(1+t)^\theta - 1}{\theta}\right)^k = k! \sum_{n\geq k} (-1)^{n+k} S_1(n,k,\lambda+\theta \mid \theta)\frac{t^n}{n!} \tag{12.7}$$

$$(1+\theta t)^{\mu\lambda}\left((1+\theta t)^\mu - 1\right)^k = k! \sum_{n\geq k} S(n,k,\lambda \mid \theta)\frac{t^n}{n!} \tag{12.8}$$

with $\theta\mu = 1$ may also be deduced from (12) and its conjugate form, with $\alpha = 1$, $\beta = \theta$ and $r = -\lambda$, (cf. [11]).

Remark 3. According to (3) it is seen that each pair of number sequences given in the above examples are orthogonal to each other. They also yield inverse series relations of the form (4).

EXAMPLE 4. The noncentral C numbers (also called Gould–Hopper numbers) $C(n, k, s, r)$ are defined by the relation (cf. [6])

$$(st - sa)_n = \sum_{k=0}^{n} C(n, k, s, r)(t - b)_k, \qquad (r = b - a).$$

Notice that

$$(st - sa)_n = s^n \left(t - a \middle| \frac{1}{s}\right)_n,$$

so that the above relation is precisely equivalent to the following:

$$\left(t \middle| \frac{1}{s}\right)_n = \sum_{k=0}^{n} C(n, k, s, r)(t + a - b)_k / s^n.$$

This shows that the numbers $C(n, k, s, r)/s^n$ belong to the $\langle 1/s, 1, b - a\rangle$ pair. Actually, they belong to the second kind of Howard's weighted degenerate Stirling numbers.

Consequently, by Theorem 2 we get the GF

$$(1 + t/s)^{(b-a)s}\left((1 + t/s)^s - 1\right)^k = k! \sum_{n \ge 0} C(n, k, s, r) \frac{(t/s)^n}{n!}.$$

This may be simplified to the form

$$(1 + t)^{rs}\left((1 + t)^s - 1\right)^k = k! \sum_{n \ge 0} C(n, k, s, r) \frac{t^n}{n!}. \qquad (12.9)$$

Correspondingly, there are associated numbers $C^*(n, k, s, r)$ belonging to the first kind of Howard's weighted degenerate Stirling numbers, which may be generated by the conjugate form of (12.9)

$$(1 + t)^{-r}\left[s(1 + t)^{1/s} - s\right]^k = k! \sum_{n \ge 0} C^*(n, k, s, r) \frac{t^n}{n!}. \quad (12.10)$$

Certainly, the two kinds of numbers C and C^* are orthogonal, and they yield the inverse series relations

$$f_n = \sum_{k=0}^{n} C(n, k, s, r) g_k \quad \Leftrightarrow \quad g_n = \sum_{k=0}^{n} C^*(n, k, s, r) f_k.$$

3. CONSEQUENCES OF THEOREM 2

As in the classical case, we easily get the double GF for the numbers $S(n,k) \equiv S(n,k;\alpha,\beta,r)$ from (12), namely,

$$(1+\alpha t)^{r/\alpha}\exp\left[\left((1+\alpha t)^{\beta/\alpha}-1\right)\frac{x}{\beta}\right] = \sum_{n,k\geq 0} S(n,k)\frac{t^n x^k}{n!}. \quad (13)$$

Notice that the RHS of (13) contains the exponential polynomial $S_n(x) = \sum_{k=0}^{n} S(n,k)x^k$ in the summand. We may restate (13) as a corollary of Theorem 2.

COROLLARY 1. *The sequence* $\{S_n(x)\}$ *has the following GF*:

$$(1+\alpha t)^{r/\alpha}\exp\left[\left((1+\alpha t)^{\beta/\alpha}-1\right)\frac{x}{\beta}\right] = \sum_{n\geq 0} S_n(x)\frac{t^n}{n!} \quad (14)$$

In particular, (14) *gives the GF for the generalized Bell numbers*,

$$(1+\alpha t)^{r/\alpha}\exp\left[\left((1+\alpha t)^{\beta/\alpha}-1\right)/\beta\right] = \sum_{n\geq 0} W_n\frac{t^n}{n!}, \quad (15)$$

where $W_n = S_n(1)$.

3.1. *Convolution Formulas*

Denote for brevity

$$S(n,k_i) = S(n,k_i;\alpha,\beta,r_i), \qquad (i=1,2).$$

Then, performing the product of the following two GFs:

$$(1+\alpha t)^{r_i/\alpha}\left[\frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}\right]^{k_i} = k_i!\sum_{n_i\geq 0} S(n_i,k_i)\frac{t^{n_i}}{n_i!}, \qquad (i=1,2),$$

and identifying the coefficients of t^m on the both sides, we are easily led to the following:

COROLLARY 2. *There holds the convolution formula*

$$\sum_{n=0}^{m}\binom{m}{n} S(n,k_1;\alpha,\beta,r_1)S(m-n,k_2,\alpha,\beta,r_2)$$
$$= \binom{k_1+k_2}{k_1} S(m,k_1+k_2;\alpha,\beta,r_1+r_2) \quad (16)$$

where k_1 and k_2 are nonnegative integers, and α, β, r_1, and r_2 may be any real or complex numbers.

As a simple example, one may take $k_1 = k_2 = 0$. In this case (16) implies Vandermonde's formula (recalling (8)),

$$\sum_{n=0}^{m} \binom{m}{n} (r_1 \mid \alpha)_n (r_2 \mid \alpha)_{m-n} = (r_1 + r_2 \mid \alpha)_m. \tag{17}$$

3.2. *Congruence Properties*

The convolution formula (16) may be used to investigate the congruence properties of $S(p, k)$, where p is a prime number. More precisely, we can establish the following:

THEOREM 3. *Let α, β, and r be integers. Then for any given odd prime number p, we have the congruence relations*

$$S(p, k; \alpha, \beta, r) \equiv 0 (\operatorname{mod} p), \tag{18}$$

where $1 < k < p$.

Proof. From Theorem 1, together with the relations $S(0,0) = 1$, $S(n, n) = 1$, $S(1,0) = r$, we see that $S(n, k; \alpha, \beta, r)$ are integers whenever α, β, and r are integers. Now let us make use of (16), taking $m = p$ and writing $r_1 + r_2 = r$, $k = k_1 + k_2$ with $k_1 \geq 1$ and $k_2 \geq 1$. Then from (16) we may infer that

$$\begin{aligned} \binom{k}{k_1} S(p, k; \alpha, \beta, r) &= \sum_{n=1}^{p-1} \binom{p}{n} S(n, k_1; \alpha, \beta, r_1) S(p - n, k_2; \alpha, \beta, r_2) \\ &\equiv 0 (\operatorname{mod} p). \end{aligned}$$

Since $1 < k < p$, the coefficient $\binom{k}{k_1}$ is not divisible by p, and we may conclude that $S(p, k; \alpha, \beta, r)$ should be a multiple of p. ∎

Certainly, Theorem 3 could be specialized to the cases of various well-known generalized Stirling numbers involving integer parameters (cf. [7], [10]).

4. GENERAL DOBINSKI-TYPE FORMULAS

For the classical Stirling numbers of the second kind, $S_2(n, k) = S(n, k; 0, 1, 0)$, the attractive Bell numbers B_n and the exponential polynomials $\varphi_n(x)$ are defined, respectively, by the following:

$$B_n = \sum_{k=0}^{n} S_2(n,k) \tag{19}$$

$$\varphi_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k. \tag{20}$$

It is known that the delta-operator techniques fully developed in the Binomial Enumeration Theory by Mullin and Rota [18] could be used to treat B_n, $\varphi_n(x)$, and the like very easily. For instance, the remarkable Dobinski-type formula

$$\varphi_n(x) = e^{-x} \sum_{k\geq 0} \frac{x^k k^n}{k!}. \tag{21}$$

can be obtained almost immediately by using Mullin–Rota's operator method (cf. also Roman and Rota [21], pp. 133–134).

The object of this section is to prove a general Dobinski-type formula for the generalized exponential polynomial $S_n(x)$ defined by (5). Although the general formula may also be obtainable by using Mullin-Rota's operator method, we will give it a computational proof by making use of Corollary 1 of Theorem 2.

THEOREM 4. *For the polynomial* $S_n(x) = \sum_{k=0}^{n} S(,n,k;\alpha,\beta,r)x^k$, *we have the Dobinski-type formula*

$$S_n(x) = \left(\frac{1}{e}\right)^{x/\beta} \sum_{k\geq 0} \frac{(x/\beta)^k}{k!}(k\beta + r \mid \alpha)_n. \tag{22}$$

Proof. Starting with (14), we have

$$\begin{aligned}
\sum_{n\geq 0} S_n(x)\frac{t^n}{n!} &= \left(\frac{1}{e}\right)^{x/\beta}(1+\alpha t)^{r/\alpha}\exp\left\{(1+\alpha t)^{\beta/\alpha}(x/\beta)\right\} \\
&= \left(\frac{1}{e}\right)^{x/\beta}(1+\alpha t)^{r/\alpha}\sum_{k\geq 0}\frac{(x/\beta)^k}{k!}(1+\alpha t)^{k\beta/\alpha} \\
&= \left(\frac{1}{e}\right)^{x/\beta}\sum_{k\geq 0}\frac{(x/\beta)^k}{k!}\sum_{j,l\geq 0}\binom{r/\alpha}{j}\binom{k\beta/\alpha}{l}(\alpha t)^{j+l}.
\end{aligned}$$

By identifying the coefficient of $t^n/n!$ within the first and last expressions, and using Vandermonde's formula, we find

$$S_n(x) = \left(\frac{1}{e}\right)^{x/\beta} \sum_{k\geq 0} \frac{(x/\beta)^k}{k!} \binom{r/\alpha + k\beta/\alpha}{n} \alpha^n n!$$
$$= \left(\frac{1}{e}\right)^{x/\beta} \sum_{k\geq 0} \frac{(x/\beta)^k}{k!} (k\beta + r \mid \alpha)_n. \quad \blacksquare$$

COROLLARY. *For the generalized Bell number* $W_n = S_n(1)$, *we have*

$$W_n = \sum_{k=0}^{n} S(n,k) = \left(\frac{1}{e}\right)^{1/\beta} \sum_{k\geq 0} \frac{(1/\beta)^k}{k!} (k\beta + r \mid \alpha)_n. \tag{23}$$

Evidently, Dobinski formulas (21) *and*

$$B_n = \frac{1}{e} \sum_{k\geq 0} \frac{k^n}{k!} \tag{24}$$

are particular cases of (22) *and* (23), *with* $(\alpha, \beta, r) = (0, 1, 0)$, *respectively*.

As may be observed, the formula (22) can also be used to give a closed sum formula for the following type of infinite series:

$$\psi_n(x, t, s) = \sum_{k\geq 0} \frac{x^k}{k!} \binom{kt + s}{n}, \tag{25}$$

where x, t, s are any real or complex numbers, and n is an integer ≥ 0. Actually such a type of series cannot be summed by using the hypergeometric series method.

Using (22) and making the substitutions $\beta \mapsto \alpha t$, $r \to \alpha s$, and $x \mapsto \alpha x t$, we get the following identity:

$$\sum_{k\geq 0} \frac{x^k}{k!} \binom{kt + s}{n} = \frac{e^x}{n!\alpha^n} S_n(\alpha x t). \tag{26}$$

Notice that the LHS of (26) is independent of α; we may certainly choose $\alpha = 1$ to get the sum

$$\sum_{k\geq 0} \frac{x^k}{k!} \binom{kt + s}{n} = \frac{e^x}{n!} S_n(xt), \tag{27}$$

where $S_n(xt)$ is given by

$$S_n(xt) = \sum_{k=0}^{n} S(n, k; 1, t, s)(xt)^k. \tag{28}$$

This shows that the series $\psi_n(x, t, s)$ could be summed in closed form by using generalized Stirling numbers, namely, the first kind of Howard's weighted degenerate Stirling numbers (cf. Example 3 of Section 2).

5. A KIND OF ASYMPTOTIC EXPANSION

Here we will develop a kind of asymptotic expansion for the generalized Stirling numbers $S(\lambda + n, \lambda) \equiv S(\lambda + n, \lambda; \alpha, \beta, r)$ and $S(\lambda + n, \lambda, \lambda r) \equiv S(\lambda + n, \lambda; \alpha, \beta, \lambda r)$ for large λ and n, with the condition $n = o(\lambda^{1/2})(\lambda \to \infty)$. An asymptotic formula of Tsylova [26], involving a generalization of a result by Moser and Wyman [16], is included as a special case.

5.1. *Preliminaries*

A principal tool to be employed in this section is a known result, namely an asymptotic expansion formula for the coefficients of power-type generating functions involving large parameters (cf. Hsu [12]). Such an expansion formula consists of inverse falling factorials of large numbers of the form $1/(\lambda + k)_k$, rather than those of inverse powers $1/\lambda^k$, $(k = 1, 2, \dots)$. This may be seen to be quite natural for expressing combinatorial functions like Stirling numbers.

Denote by N_+ the set of positive integers. Let $\sigma(n)$ be the set of partitions of n $(n \in N_+)$, usually represented by $1^{k_1}2^{k_2} \cdots n^{k_n}$ with $1k_1 + 2k_2 + \cdots + nk_n = n$, $k_i \geq 0$ $(i = 1, 2, \dots, n)$, and with $k = k_1 + k_2 + \cdots + k_n$ expressing the number of parts of the partition.

For given k $(1 \leq k \leq n)$, we use $\sigma(n, k)$ to denote the subset of $\sigma(n)$, which consists of partitions of n having k parts.

Let $\phi(t) = \sum_0^n a_n t^n$ be a formal power series over the complex field C, with $a_0 = \phi(0) = 1$. For every j $(0 \leq j < n)$ define

$$W(n, j) = \sum_{\sigma(n, n-j)} \frac{a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}}{k_1! k_2! \cdots k_n!}, \tag{29}$$

where the summation is taken over all such partitions $1^{k_1}2^{k_2} \cdots n^{k_n}$ of n that have $(n-j)$ parts. What we shall need is the following known result (cf. [11]):

Let $[t^n](\phi(t))^\lambda$ *denote the coefficient of* t^n *in the power series expansion of* $(\phi(t))^\lambda$. *Then for a fixed* $s \in N_+$ *and for large* λ *and* n *such that* $n = o(\lambda^{1/2})(\lambda \to \infty)$, *we have the asymptotic expansion formula*

$$\frac{1}{(\lambda)_n}[t^n](\phi(t))^\lambda = \sum_{j=0}^{s} \frac{W(n,j)}{(\lambda - n + j)_j} + o\left(\frac{W(n,s)}{(\lambda - n + s)_s}\right), \quad (30)$$

where the quantities $W(n,j)$ *are given by* (29). *In particular, when n is fixed, the remainder estimate is given by* $0(\lambda^{-s-1})$.

5.2. *Asymptotics of* $S(\lambda + n, \lambda;\ \alpha, \beta, \gamma)$ *and* $S(\lambda + n, \lambda;\ \alpha, \beta, \lambda\gamma)$

To apply (30) to the $\langle \alpha, \beta, r \rangle$ pair with $\alpha\beta \neq 0$, let us take

$$\phi(t) = (1 + \alpha t)^{r/\alpha}\left(\frac{(1+\alpha t)^{\beta/\alpha} - 1}{\beta t}\right) = \sum_{n \geq 0} \frac{S(n+1,1)}{(n+1)!} t^n, \quad (31)$$

so that $\phi(0) = 1$, where $S(n+1,1) \equiv S(n+1,1;\ \alpha, \beta, r)$. Consequently, we have

$$\begin{aligned}(\phi(t))^\lambda &= (1+\alpha t)^{\lambda r/\alpha}\left(\frac{(1+\alpha t)^{\beta/\alpha} - 1}{\beta t}\right)^\lambda \\ &= \lambda! \sum_{n \geq 0} \frac{S(\lambda + n, \lambda;\ \alpha, \beta, \lambda r)}{(\lambda + n)!} t^n. \end{aligned} \quad (32)$$

Thus, making use of (30), we obtain

$$\frac{S(\lambda + n, \lambda;\ \alpha, \beta, \lambda r)}{(\lambda)_n (\lambda + n)_n} = \sum_{j=0}^{s} \frac{W(n,j)}{(\lambda - n + j)_j} + o\left(\frac{W(n,s)}{(\lambda - n + s)_s}\right), \quad (33)$$

where $n = o(\lambda^{1/2})(\lambda \to \infty)$ and $W(n,j)(j = 0,1,2,\dots)$ are given by (29), with a_j being determined by (31), viz.,

$$a_j = [t^j]\phi(t) = S(j+1,1)/(j+1)! \quad (34)$$

More precisely, we easily compute a_j by using Vandermonde's formula as follows:

$$\begin{aligned} a_j &= [t^j](1+\alpha t)^{r/\alpha} \sum_{k\ge 1} \binom{\beta/\alpha}{k} (\alpha t)^k / (\beta t) \\ &= [t^j]\left(\frac{\alpha}{\beta}\right) \sum_{i\ge 0} \sum_{k\ge 1} \binom{r/\alpha}{i} \binom{\beta/\alpha}{k} (\alpha t)^{i+k-1} \\ &= \left(\frac{\alpha}{\beta}\right) \alpha^j \left\{ \binom{r/\alpha + \beta/\alpha}{j+1} - \binom{r/\alpha}{j+1} \right\} \\ &= \frac{1}{\beta \cdot (j+1)!} \big((\beta + r \mid \alpha)_{j+1} - (r \mid \alpha)_{j+1} \big), \qquad j = 1, 2, \ldots, n. \end{aligned}$$

Hence we may state the following:

THEOREM 5. *There holds the asymptotic expansion formula* (33) *for* $\lambda \to \infty$ *with* $n = o(\lambda^{1/2})$, *where* $W(n, j)$ *is defined by* (29), *with* a_j *being given by*

$$a_j = \big((\beta + r \mid \alpha)_{j+1} - (r \mid \alpha)_{j+1} \big) / \big((j+1)!\beta \big). \tag{35}$$

Remark 1. Notice that the formula (33) with $W(n, j)$ and a_j being defined by (29) and (35), respectively, is essentially an algebraic–analytic identity. Thus it is also applicable to the function $S(\lambda + n, \lambda;\ \alpha, \beta, r)$, provided that the quantity r contained in (33) is replaced by r/λ, i.e., the expression (35) for a_j is replaced by

$$a_j = \left[\left(\beta + \frac{r}{\lambda} \middle| \alpha \right)_{j+1} - \left(\frac{r}{\lambda} \middle| \alpha \right)_{j+1} \right] \Big/ \big((j+1)!\beta \big).$$

In particular, if λ is very large and if only a few principal terms of the asymptotic expansion for $S(\lambda + n, \lambda)$ are required, one can even use the following approximate values for a_j instead:

$$a_j = (\beta \mid \alpha)_{j+1} / \big((j+1)!\beta \big).$$

Remark 2. Starting with the generating function

$$\frac{1}{k!} \left[\frac{1 - (1 - \alpha x)^{\beta/\alpha}}{\beta} \right]^k = \sum_{m=k}^{\infty} A_{\alpha,\beta}(m, k) \frac{x^m}{m!},$$

and applying the Cauchy residue theorem, E. G. Tsylova [26] has proved the asymptotic formula (a generalization of Moser–Wyman's result): if $\alpha \neq \beta$ and if m and k tend to infinity such that $0 < m - k = o(m^{1/2})$, then

$$A_{\alpha,\beta}(m,k) = \binom{m}{k}\left(\tfrac{1}{2}(\alpha - \beta)k\right)^{m-k}(1 + o(1)).$$

Actually, this can be deduced from our general formula (33) by taking $r = 0$, $s = 1$, $\lambda = k$, and $n = m - k$.

Finally, let us give a simple example to illustrate the use of the formula (33). Assume $r = 0$ and take $s = 2$. Notice that $W(n,0)$, $W(n,1)$, and $W(n,2)$ are given by

$$W(n,0) = \frac{1}{n!}a_1^n, \qquad W(n,1) = \frac{1}{(n-2)!}a_1^{n-2}a_2,$$

$$W(n,2) = \frac{1}{(n-3)!}a_1^{n-3}a_3 + \frac{1}{2!(n-4)!}a_1^{n-4}a_2^2.$$

Clearly (33) implies the asymptotic relation

$$\frac{S(\lambda+n,\lambda;\alpha,\beta,0)}{(\lambda+n)_n} \sim (\lambda)_n W(n,0) + (\lambda)_{n-1}W(n,1) + (\lambda)_{n-2}W(n,2), \quad (36)$$

where $n = o(\lambda^{1/2})(\lambda \to \infty)$. This may be used to derive simple asymptotic expressions for Carlitz's degenerate Stirling numbers involving large parameters (cf. Example 2 of Section 2).

In particular, (36) may be applied to the two kinds of classical Stirling numbers $s(\lambda + n, \lambda) \equiv S(\lambda + n, \lambda; 1, 0+, 0)$ and $S(\lambda + n, \lambda) \equiv S(\lambda + n, \lambda; 0+, 1, 0)$. For these numbers one easily finds $a_j = (-1)^{j+1}/(j+1)$ and $a_j = 1/(j+1)!$, respectively. Thus (36) gives

$$\frac{|s(\lambda+n,\lambda)|}{(\lambda+n)_n} \sim (\lambda)_n \frac{(1/2)^{n-1}}{(n-3)!} + (\lambda)_{n-1}\frac{(1/2)^{n-2}(1/3)}{(n-2)!} + (\lambda)_{n-2}\left\{\frac{(1/2)^{n-1}}{(n-3)!} + \frac{(1/2)^{n-3}(1/3)^2}{(n-4)!}\right\},$$

$$\frac{S(\lambda+n,\lambda)}{(\lambda+n)_n} \sim (\lambda)_n \frac{(1/2)^n}{n!} + (\lambda)_{n-1}\frac{(1/2)^{n-1}(1/3)}{(n-2)!} + (\lambda)_{n-2}\left\{\frac{(1/2)^n(1/3)}{(n-3)!} + \frac{(1/2)^{n-1}(1/3)^2}{(n-4)!}\right\}.$$

Actually these special asymptotic expressions are included in the more elaborated works by Moser and Wyman [16, 17] and several others. What we have shown above is a unified way of attaining them. For more complete asymptotics of classical Stirling numbers, see Temme's recent paper [23].

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