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Note

A unified treatment of a class of combinatorial sums

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Abstract

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Here introduced is a class of combinatorial sums that can be treated by means of an embedding and inversion technique. Some classic identities and novel ones are demonstrated to be members of the class defined. Solution of Liskovets' problems is reconsidered, and an additional class of identities is formulated.

1. Introduction

The object of this paper is to develop a unified method for dealing with a wide class of combinatorial sums involving the binomial coefficient $\binom{n}{k}$ as a factor in their summands. The basic tool to be employed is the inverse series relations proved earlier by Gould and Hsu [5], namely the following proposition.

Proposition. *Let $\{a_i\}$ and $\{b_i\}$ be any two sequences of numbers such that*

$$\Phi(x, n) = \prod_{i=1}^{n} (a_i + xb_i) \neq 0 \tag{1.0}$$

for integers $x, n \geqslant 0$ *with* $\Phi(x, 0) = 1$. *Then we have the pair of reciprocal formulas:*

$$f_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Phi(k, n) g_k, \tag{1.1}$$

$$g_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a_{k+1} + k b_{k+1}) \Phi(n, k+1)^{-1} f_k. \tag{1.2}$$

This proposition suggests a fruitful concept concerning a class of identities involving binomial coefficients.

Definition. Any finite summation formula is said to belong to the class Σ if it can be expressed in either of the forms (1.1) or (1.2) in which a_i and b_i are suitable assigned numbers, and f_k and g_k may involve some independent parameters.

A good many identities appearing in Riordan's book [11] are members of the class Σ. Moreover, a rough but extensive investigation of Gould's formulary [4] reveals that almost 30% of the total 500 known identities contain $\binom{n}{k}$ as a factor in their summands. Furthermore, the majority of these identities are capable of being embedded in either of the forms (1.1) or (1.2).

2. Some remarkable examples

Here we will show that some classic identities are members of the class Σ, so that they may be readily proved anew by the embedding (and inversion) technique associated with the reciprocal relations (1.1)$\Leftrightarrow$(1.2).

Example 1 (Abel's identity). The well-known formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k} \tag{2.1}$$

is generally considered as a deep generalization of the binomial theorm (see, e.g., [1, 8]). Clearly, (2.1) may be rewritten as

$$f_n := (x, y)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \Phi(k, n) g_k, \tag{2.2}$$

with

$$g_k := (-1)^k x(x-kz)^{k-1}(y+kz)^{-k}$$

and

$$\Phi(k, n) := (y+kz)^n, \qquad a_i = y, \qquad b_i = z.$$

This shows that (2.1) can be readily embedded in (1.1).

Proof of (2.1). Inversion of (2.2) via (1.2) gives

$$(-1)^n x \frac{(x-nz)^{n-1}}{(y+nz)^n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{y+kz}{(y+nz)^{k+1}} (x+y)^k.$$

This is equivalent to the following:

$$\begin{aligned} -x(nz-x)^{n-1} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (y+kz)(y+nz)^{n-k-1}(x+y)^k \\ &= \sum_{k=0}^{n} (-1)^k \left[\binom{n}{k} y + \binom{n-1}{k-1} nz \right] (y+nz)^{n-k-1}(x+y)^k, \end{aligned}$$

the validity of which may be verified at once by use of the binomial theorem, noting that $\binom{n-1}{k-1} = 0$ for $k=0$.

Example 2 (Hagen–Rothe–Gould type identities). The convolution identity

$$\begin{aligned} &\sum_{k=0}^{n} \frac{x}{x+kz} \binom{x+kz}{k} \frac{y}{y+(n-k)z} \binom{y+(n-k)z}{n-k} \\ &\quad = \frac{x+y}{x+y+nz} \binom{x+y+nz}{n} \end{aligned} \tag{2.3}$$

has been investigated by various authors and also extended to higher dimensions by Mohanty and Handa [10]. As may be observed, (2.3) is implied by the following (with $p=1$, $q=0$ and y being replaced by $y+nz$):

$$\begin{aligned} &\sum_{k=0}^{n} \binom{x+kz}{k} \binom{y-kz}{n-k} \frac{p+qk}{(x+kz)(y-kz)} \\ &\quad = \frac{p(x+y-nz)+nxq}{x(x+y)(y-nz)} \binom{x+y}{n}. \end{aligned} \tag{2.4}$$

Using the notation $(\alpha)_k = (\alpha-1)\cdots(\alpha-k+1)$ with $(\alpha)_0 = 1$, (2.4) may be rewritten as

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} [y+k(1-z)]_n \frac{(x+kz)_k}{[y+k(1-z)]_k} \cdot \frac{p+qk}{(x+kz)(y-kz)} \\ &\quad = \frac{p(x+y-nz)+qnx}{(x+y)x(y-nz)} (x+y)_n. \end{aligned} \tag{2.5}$$

By comparing it with (1.1) we see that (2.5) can be embedded in (1.1) by defining

$$\Phi(t,n) = [y+t(1-z)]_n, \qquad a_i = y-i+1, \qquad b_i = 1-z,$$

$$g_k = (-1)^k \frac{(x+kz)_k}{[y+k(1-z)]_k} \cdot \frac{p+qk}{(x+kz)(y-kz)},$$

$$f_k = \frac{p(x+y-kz)+qkx}{(x+y)x(y-kz)}(x+y)_k.$$

Consequently both (2.3) and (2.4) belong to the class Σ.

Proof of (2.4). Making use of (1.2), one may invert (2.5) to get

$$(-1)^n \frac{(x+nz)_n}{[y+n(1-z)]_n} \cdot \frac{p+qn}{(x+nz)(y-nz)}$$
$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{y-kz}{[y+n(1-z)]_{k+1}} \cdot \frac{p(x+y-kz)+qkx}{(x+y)x(y-kz)}(x+y)_k.$$

After simplifying we get

$$(x+nz-1)_{n-1}(p+qn)$$
$$= \sum_{k=0}^{n} \binom{n}{k}(nz-y)_{n-k} \frac{p(x+y-kz)+qkx}{(x+y)x}(x+y)_k.$$

As p and q are independent parameters contained on both sides of the equation, it is clear that the equality is equivalent to the following pair of relations:

$$\frac{1}{n}\binom{x+nz-1}{n-1} = \sum_{k=0}^{n} \binom{nz-y}{n-k}\binom{x+y}{k}\frac{1}{x} - \sum_{k=1}^{n} \binom{nz-y}{n-k}\binom{x+y-1}{k-1}\frac{z}{x},$$
$$\binom{x+nz-1}{n-1} = \sum_{k=0}^{n} \binom{nz-y}{n-k}\binom{x+y-1}{k-1}.$$

These two relations follow easily from the ordinary Vandermonde convolution formula. Indeed, the right-hand side of the first relation gives

$$\frac{1}{x}\binom{nz+x}{n} - \frac{z}{x}\binom{nz+x-1}{n-1} = \frac{1}{n}\binom{nz+x-1}{n-1}.$$

This completes the proof of (2.4). $\square$

Example 3 (Liskovets' problem). In solving enumeration problems for graphs with labelled vertices it was conjectured by Liskovets [9] that there exist polynomials $\phi_\alpha(x)$ of degree α with integer coefficients such that

$$\sum_{k=1}^{n} \binom{n-1}{k-1} n^{n-k} \phi_\alpha(k)(k+\alpha)! = (2\alpha)!\, n^{n+\alpha} \tag{2.6}$$

where $\alpha \geqslant 0$ and $n \geqslant 1$ are integers. Only particular values of α, namely $\alpha = 0, 1, 2, 3, 4$, were verified in [9]. The problem of determining $\phi_\alpha(x)$ for any integer value of α was solved by Egorychev using the integral representation method and residue calculus (see [2.3]). The solution is expressed in terms of higher differences of zero, namely

$$\phi_\alpha(k) = \Delta^k O^{k+\alpha} \frac{(2\alpha)!}{(k+\alpha)!} \quad (k \geq 1,\ \alpha \geq 0). \tag{2.7}$$

Obviously (2.6) may be rewritten in the form

$$\sum_{k=0}^{n} \binom{n}{k} n^{-(k+1)} k \phi_\alpha(k)(k+\alpha)! = (2\alpha)!\, n^\alpha. \tag{2.8}$$

Thus it can be embedded in (1.2) by defining $\Phi(n, k+1) = n^{k+1}$ (with $a_i = 0$, $b_i = 1$ so that $a_{k+1} + kb_{k+1} = k$) and taking

$$f_k := \phi_\alpha(k)(k+\alpha)!\,(-1)^k, \qquad g_k := (2\alpha)!\, k^\alpha. \tag{2.9}$$

Hence (2.6) is also a member of the class Σ.

3. Confirmation of Liskovets' conjecture

The solution (2.7) can be obtained quickly by inverting (2.8) via (1.1). Indeed, with f_k and g_k being defined by (2.9) we have

$$\begin{aligned} f_n := \phi_\alpha(n)(n+\alpha)!\,(-1)^n &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \Phi(k, n) g_k \\ &= \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^n (2\alpha)!\, k^\alpha, \quad (n \geq 1). \end{aligned}$$

Thus it follows that

$$\phi_\alpha(n) = \frac{(2\alpha)!}{(n+\alpha)!} \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} k^{n+\alpha} = \frac{(2\alpha)!}{(n+\alpha)!} \Delta^n O^{n+\alpha}.$$

This procedure for obtaining the solution (2.7) is shorter and easier than that used by Egorychev (cf. [2, pp. 91–92].)

To confirm that $\phi_\alpha(n)$ is a polynomial in n of degree α with integer coefficients, let us express $\phi_\alpha(n)$ in terms of Stirling's number $S_2(n+\alpha, n)$ of the second kind, viz.

$$\phi_\alpha(n) = \frac{(2\alpha)!}{(n+\alpha)!} S_2(n+\alpha, n).$$

Notice that $S_2(n+\alpha, n)$ may be written as a polynomial in $(n+\alpha)$ of degree 2α, namely

$$S_2(n+\alpha, n) = \sum_{k=-\alpha}^{\alpha} \bar{C}_{\alpha, \alpha-k} \binom{n+\alpha}{k+\alpha},$$

where $\bar{C}_{\alpha,j}$ $(j = 0, 1, \ldots, \alpha-1)$ are integers satisfying the recurrence relations (cf. Jordan [7, §58])

$$\bar{C}_{m+1,s} = (m-s+1)\bar{C}_{m,s-1} + (2m-s+1)\bar{C}_{m,s}, \qquad \bar{C}_{m+1,2m+2} = k,$$

with $\bar{C}_{1,0}=1$, $\bar{C}_{1,s}=0\ (s\geqslant 1)$. Accordingly $\phi_\alpha(n)$ may be rewritten as

$$\phi_\alpha(n)=\sum_{k=-\alpha}^{\alpha}\bar{C}_{\alpha,\alpha-k}(2\alpha)_{\alpha-k}(n)_k.$$

Thus the conjecture is verified.

4. An additional class of identities

The following reciprocal relations

$$f_n=\sum_{k=n}^{\infty}(-1)^k\binom{k}{n}\Phi(n,k)g_k, \tag{4.1}$$

$$g_n=\sum_{k=n}^{\infty}(-1)^k\binom{k}{n}(a_{n+1}+nb_{n+1})\Phi(k,n+1)^{-1}f_k \tag{4.2}$$

are known as the rotated form of (1.1) and (1.2), where $\Phi(\,.\,,.)$ is the same as that defined by (1.0), and the sequences $\{f_k\}$ and $\{g_k\}$ are assumed to vanish ultimately (cf. [5]). Accordingly we may introduce an additional class Σ^* as in the following definition.

Definition. Identity is said to belong to the class Σ^* if it can be expressed in either of the form (4.1) or (4.2) with a_i and b_i being suitably assigned.

The pair of Moriarty identities and the corresponding inverses, namely Marcia Ascher's identity and its companion-piece (cf. [4, formulas (3.120)–(3.121), (3.177)–(3.180)]), are members of Σ^* since they can be mutually inverted by (4.1)–(4.2) with $\Phi(x,n)=1$ (i.e. $a_i=1$, $b_i=0$).

We now exhibit two other members of Σ^*.

Example 4. From Vandermonde's theorem

$$\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k}=\binom{a+b}{n},$$

we can derive

$$\sum_{k=0}^{m-n}\binom{-n}{k}\binom{m+n}{m-n-k}=\binom{m}{m-n}. \tag{4.3}$$

The latter may be rewritten as

$$\sum_{k=n}^{\infty}(-1)^k\binom{k}{n}2n\Phi(k,n+1)^{-1}\binom{m}{k}=\frac{(-1)^n}{n!}\binom{2m}{m+n}\binom{2m-1}{m}^{-1},$$

where $\Phi(k,n+1)=k(k+1)\cdots(k+n)$, and $m\geqslant 1$ is a fixed integer parameter. Thus it may be embedded in (4.2) with $a_i=i-1$, $b_i=1$. Consequently it can be inverted by (4.1) to get the classical identity due to Van Ebbenhorst Tengbergen (cf. [4, 6.50)])

$$\sum_{k=n}^{m} \binom{k+n-1}{k} \binom{2m}{m-k} \binom{k}{n} = \binom{2m-1}{m} \binom{m}{n}. \tag{4.4}$$

Of course (4.4) is a member of Σ^*. The technique illustrated in this paper may be employed to search for various combinatorial identities belonging to the union of classes $\Sigma \cup \Sigma^*$.

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