

A SUMMATION RULE USING STIRLING NUMBERS OF THE SECOND KIND

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1. A SUMMATION RULE

Recall that Stirling numbers of the second kind may be expressed as follows (cf., e.g., [1], [2]):

$$S(m, j) = \frac{1}{j!} \Delta^j 0^m = \frac{1}{j!} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} i^m,$$

where  $\Delta^j 0^m$  is the  $j^{\text{th}}$  difference of  $x^m$  at  $x = 0$  so that  $S(m, j) = 0$  for  $j > m$ ,  $S(m, 0) = 0$  for  $m \geq 1$  and  $S(0, 0) = 1$ .

**Summation Rule:** Let  $F(n, k)$  be a bivariate function defined for integers  $n, k \geq 0$ . If there can be found a summation formula or a combinatorial identity such as

$$\sum_{k=j}^n F(n, k) \binom{k}{j} = \phi(n, j) \quad (j \geq 0), \tag{1}$$

then for every given  $m \geq 0$  we have a summation formula or a combinatorial identity such as

$$\sum_{k=0}^n F(n, k) k^m = \sum_{j=0}^m \phi(n, j) j! S(m, j) \tag{2}$$

which may be called a companion formula of (1).

Generally, (2) would be practically useful when  $n$  is much bigger than  $m$ .

**Proof:** It is known that Stirling numbers of the second kind satisfy the following basic relation [which is often taken as a definition of  $S(n, k)$ ]:

$$x^m = \sum_{j=0}^m S(m, j) (x)_j, \tag{3}$$

where  $(x)_j : x(x-1)\dots(x-j+1)$  ( $j \geq 1$ ) is the falling factorial with  $(x)_0 := 1$ . Now, substituting (3) into the left-hand side of (2), changing the order of summation, and using (1), we easily obtain

$$\sum_{k=0}^n F(n, k) k^m = \sum_{j=0}^m S(m, j) \sum_{k=0}^n F(n, k) (k)_j = \sum_{j=0}^m j! S(m, j) \phi(n, j).$$

Notice that the special case for  $m = 0$  is also true. Hence, (2) holds for every  $m \geq 0$ .  $\square$

**Remark** Sometimes in applications of the rule function  $F(n, k)$  may involve some independent parameters. Moreover, for the particular case in which  $F(n, k) > 0$ , so that  $\phi(n, 0) > 0$ , the left-hand side of (2) divided by  $\phi(n, 0)$  may be considered as the  $m^{\text{th}}$  moment (about the origin) of a

discrete random variable  $X$  that may take possible values  $0, 1, 2, \dots, n$ . This means that (2) may sometimes be used for computing moments whenever  $F(n, k) / \phi(n, 0)$  just stands for probabilities ( $0 \leq k \leq n$ ), and the factorial moments  $\phi(n, j) / \phi(n, 0)$  are easily found via (1) (cf. David and Barton [3]).

2. VARIOUS EXAMPLES

For the simplest case  $F(n, k) \equiv 1$ , we have

$$\phi(n, j) \equiv \sum_{k=j}^n \binom{k}{j} = \binom{n+1}{j+1}.$$

This leads to the familiar formula

$$\sum_{k=1}^n k^m = \sum_{j=1}^m \binom{n+1}{j+1} j! S(m, j). \tag{4}$$

Actually there are many known identities of type (1) in which  $F(n, k)$  may consist of a binomial coefficient or a product of binomial coefficients. See, e.g., Egorychev [4], Gould [5], and Riordan [8]. Consequently, we may find various special summation formulas via (2). We now list a dozen formulas, as follows:

$$\sum_{k=1}^n k^m \binom{n}{k} p^k q^{n-k} = \sum_{j=1}^m \binom{n}{j} p^j j! S(m, j), \tag{5}$$

where  $p + q = 1$  and  $p > 0$ .

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} k^m = \sum_{j=0}^m 2^{n-2j-1} \binom{n-j}{j} \frac{n}{n-j} j! S(m, j), \tag{6}$$

$$\sum_{k=0}^{[n/2]} \binom{n+1}{2k+1} k^m = \sum_{j=0}^m 2^{n-2j} \binom{n-j}{j} j! S(m, j), \tag{7}$$

$$\sum_{k=0}^n \binom{n-k}{s} k^m = \sum_{j=0}^m \binom{n+1}{s+j+1} j! S(m, j), \tag{8}$$

$$\sum_{k=0}^n \binom{s+k}{s} k^m = \sum_{j=0}^m \binom{n+1}{j} \binom{n+1+s}{s} \frac{n+1-j}{s+1+j} j! S(m, j), \tag{9}$$

$$\sum_{k=0}^n (-4)^k \binom{n+k}{2k} k^m = \sum_{j=0}^m (-1)^n 2^{2j} \binom{n+j}{2j} \frac{2n+1}{2j+1} j! S(m, j), \tag{10}$$

$$\sum_{k=0}^n (-4)^k \binom{n+k}{2k} \frac{n}{n+k} k^m = \sum_{j=0}^m (-1)^n 2^{2j} \binom{n+j}{2j} \frac{n}{n+j} j! S(m, j), \tag{11}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} 2^{n-2k} k^m = \sum_{j=0}^m (-1)^j \binom{n+1}{2j+1} j! S(m, j), \tag{12}$$

$$\sum_{k=0}^n \binom{\alpha}{k} \binom{\beta}{n-k} k^m = \sum_{j=0}^m \binom{\alpha}{j} \binom{\alpha+\beta-j}{n-j} j! S(m, j), \tag{13}$$

where  $\alpha$  and  $\beta$  are real parameters.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{2n-k}{n} k^m = \sum_{j=0}^m (-1)^j \binom{n}{j}^2 j! S(m, j), \tag{14}$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} 2^{n-2k} k^m = \sum_{j=0}^m \binom{2n-2j}{n} \binom{n}{j} j! S(m, j), \tag{15}$$

$$\sum_{k=1}^n k^m H_k = \sum_{j=1}^m \binom{n+1}{j+1} \left( H_{n+1} - \frac{1}{j+1} \right) j! S(m, j), \tag{16}$$

where  $H_k := 1 + \frac{1}{2} + \dots + \frac{1}{k}$ , ( $k \geq 1$ ), are harmonic numbers.

Though most of the above formulas [except (5)] appear unfamiliar, or are difficult to find in the literature, they are actually companion formulas of some known identities. In fact, (5) is known as the  $m^{\text{th}}$  moment of the binomial distribution of a discrete random variable. Formulas (6) and (7) represent companion formulas of the pair of Moriarty identities (cf. [4, (2.73) and (2.74)]; [5, (3.120) and (3.121)]). Also, (9) and (12) are just companion formulas of the following identities:

$$\sum_{k=j}^n \binom{k+s}{s} \binom{k}{j} = \binom{n+1}{j} \binom{n+1+s}{s} \frac{n+1-j}{s+1+j}$$

and

$$\sum_{k=j}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \binom{k}{j} 2^{n-2k} = (-1)^j \binom{n+1}{2j+1}$$

due to Knuth and Marcia Ascher, respectively (cf. [5, (3.155) and (3.179)]). Moreover, (16) may be inferred from the known relation (cf., e.g., [1, pp. 98-99]).

$$\sum_{k=j}^n \binom{k}{j} H_k = \binom{n+1}{j+1} \left( H_{n+1} - \frac{1}{j+1} \right) \tag{17}$$

The verification of the rest of the formulas is left to the interested reader.

Evidently, both (8) and (9) imply (4) with  $s = 0$ , and (13) yields the Vandermonde convolution identity when  $m = 0$ . Moreover, it is easily found that (16) leads to an asymptotic relation, for  $n \rightarrow \infty$ , of the following,

$$\sum_{k=1}^n k^m H_k \sim \frac{n^{m+1}}{m+1} \left( \log n + \gamma - \frac{1}{m+1} \right),$$

where  $\gamma := \lim_n (H_n - \log n) = 0.5772\dots$  is Euler's constant.

3. AN EXTENSION OF THE SUMMATION RULE

In what follows, we will adopt the notations:

$$(x|h)_n := x(x-h)(x-2h)\cdots(x-nh+h), (x|h)_0 = 1,$$

$$\binom{x}{n}_h := (x|h)_n / n!, \binom{x}{n}_1 = \binom{x}{n} = (x)_n / n!, \binom{x}{n}_0 = x^n / n!.$$

Here  $\binom{x}{n}_h$  is known as the generalized binomial coefficient (cf. Jordan [7, ch. 2, §22]). Now, suppose that  $\alpha$  and  $\beta$  are two distinct real numbers. Consider the following pair of expressions for polynomials  $(x|\alpha)_n$  and  $(x|\beta)_n$ :

$$(x|\alpha)_n = \sum_{k=0}^n S_\alpha(n, k|\beta)(x|\beta)_k, \tag{18}$$

$$(x|\beta)_n = \sum_{k=0}^n S_\beta(n, k|\alpha)(x|\alpha)_k. \tag{19}$$

The coefficients  $S_\alpha(n, k|\beta)$  and  $S_\beta(n, k|\alpha)$  involved in (18) and (19) are uniquely determined, and they may be called a pair of symmetrically generalized Stirling numbers associated with the number pair  $(\alpha, \beta)$ . Consequently, the ordinary Stirling numbers of the first and second kinds are associated with the number pair  $(1, 0)$ , and are usually denoted by the following:

$$S_1(n, k) \equiv s(n, k) := S_1(n, k|0), S_2(n, k) \equiv S(n, k) := S_0(n, k|1).$$

Certainly, all the well-known properties enjoyed by the ordinary Stirling numbers, e.g., recurrence relations, orthogonality relations, and inversion formulas, etc., can be readily extended to these generalized Stirling numbers. For example, a simple recurrence relation may be deduced from (19), namely

$$S_\beta(n, k|\alpha) = S_\beta(n-1, k-1|\alpha) + (k\alpha - n\beta + \beta)S_\beta(n-1, k|\alpha), (k \geq 1). \tag{20}$$

Recall that there is a general form of Newton's expansion for a polynomial  $f(x)$  of degree  $n$ , viz.,

$$f(x) = \sum_{k=0}^n \frac{(x|\alpha)_k}{k! \alpha^k} \Delta_\alpha^k f(0), \tag{21}$$

where  $\Delta_\alpha^k f(0)$  is the  $k^{\text{th}}$  difference (with increment  $\alpha$ ) of  $f(x)$  at  $x = 0$ . Thus, comparing (21) with (19) and (18), we find (with  $\alpha\beta \neq 0$ ),

$$S_\alpha(n, k|\alpha) = \frac{1}{k! \alpha^k} \Delta_\alpha^k (x|\beta)_n \Big|_{x=0}, \tag{22}$$

$$S_\alpha(n, k|\beta) = \frac{1}{k! \beta^k} \Delta_\beta^k (x|\alpha)_n \Big|_{x=0}. \tag{23}$$

Here, it is easily observed that  $S_\beta(n, k|\alpha) = 0$  for  $k > n$ , and  $S_\beta(0, 0|\alpha) = S_\beta(n, n|\alpha) = 1$ . Moreover, notice that for  $\beta = 0$  (23) should be replaced by

$$S_\alpha(n, k|0) = \frac{1}{k!} \lim_{\beta \rightarrow 0} \frac{1}{\beta^k} \Delta_\beta^k(x|\alpha)_n \Big|_{x=0} = \frac{1}{k!} \left( \frac{d}{dx} \right)^k (x|\alpha)_n \Big|_{x=0}$$

**Extended Summation Rule:** Let  $F(n, k)$  be defined for integers  $n, k \geq 0$ . If there can be found a summation formula such as

$$\sum_{k=0}^n F(n, k) \binom{k}{j}_\alpha = G(n, j), \quad (j \geq 0), \tag{24}$$

then for every  $m \geq 0$  we have a summation formula of the form

$$\sum_{k=0}^n F(n, k) \binom{k}{m}_\beta = \sum_{j=0}^m G(n, j) \frac{j!}{m!} S_\beta(m, j|\alpha). \tag{25}$$

Also, suppose that the following series is convergent to  $g(j)$  for every  $j \geq 0$ :

$$\sum_{k=0}^\infty f(k) \binom{k}{j}_\alpha = g(j). \tag{26}$$

Then we have a summation formula, as follows:

$$\sum_{k=0}^\infty f(k) \binom{k}{m}_\beta = \sum_{j=0}^m g(j) \frac{j!}{m!} S_\beta(m, j|\alpha). \tag{27}$$

**Proof:** Notice that (19) implies

$$\binom{x}{m}_\beta = \frac{1}{m!} \sum_{j=0}^m j! S_\beta(m, j|\alpha) \binom{x}{j}_\alpha. \tag{28}$$

Thus, both of the implications (24)  $\Rightarrow$  (25) and (26)  $\Rightarrow$  (27) can be verified in a manner similar to that used to prove (1)  $\Rightarrow$  (2). In fact, the verification of (27) can be accomplished by substituting (28) into the left-hand side of (27) and by using (26), in which the change of order of summation is justified by the convergence of the series (26). Moreover, it is evident that

$$S_\alpha(n, k|\alpha) = \begin{cases} 1 & \text{for } k = n, \\ 0 & \text{for } k < n, \end{cases}$$

so that (25) and (27) will transform back to (24) and (26), respectively, when  $\beta = \alpha$ . Hence, (27) holds for every real number  $\beta$ .  $\square$

**Examples:** For the case  $\alpha = 1$ , we may write

$$S_\beta(m, j|1) = \frac{1}{j!} \Delta^j(x|\beta)_m \Big|_{x=0}. \tag{29}$$

In particular, we have

$$S_0(m, j|1) = S(m, j), \quad S_{-1}(m, j|1) = \frac{m!}{j!} \binom{m-1}{j-1},$$

where  $S_{-1}(m, j|1)(-1)^m$  is known as Lah's number.

Making use of the rule (24)  $\Rightarrow$  (25) (with  $\alpha = 1$ ), it is readily seen that each of the formulas from (5) through (16) may be generalized to the form in which  $k^m$  is replaced by  $\binom{k}{m}_\beta$  and  $S(m, j)$  by the following:  $S_\beta(m, j|1)/m!$ . Thus, for instance, (13) and (16) may be replaced, respectively, by:

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} \binom{k}{m}_\beta = \sum_{j=0}^m \binom{x+y-j}{n-j} \frac{(x)_j}{m!} S_\beta(m, j|1), \tag{30}$$

$$\sum_{k=1}^n \binom{k}{m}_\beta H_k = \sum_{j=1}^m \binom{n+1}{j+1} \left( H_{n+1} - \frac{1}{j+1} \right) \frac{j!}{m!} S_\beta(m, j|1). \tag{31}$$

In particular, for  $\beta = 0, 1, -1$ , we have  $\binom{k}{m}_0 = k^m/m!$ ,  $\binom{k}{m}_1 = \binom{k}{m}$ , and  $\binom{k}{m}_{-1} = \binom{k+m-1}{m}$ , so that either (30) or (31) may yield at least three special identities of some interest. Indeed, (31) implies (16), (17), and the identity

$$\sum_{k=1}^n \binom{k+m-1}{m} H_k = \sum_{j=1}^m \binom{n+1}{j+1} \binom{m-1}{j-1} \left( H_{n+1} - \frac{1}{j+1} \right).$$

Moreover, as a simple consequence of (30), one may take  $x = y = n$  and  $\beta = 0$  to get

$$\sum_{k=0}^n \binom{n}{k}^2 k^m = \sum_{j=0}^m \binom{2n-j}{n} (n)_j S(m, j).$$

This is an example mentioned in Comtet [2, ch. 5, p. 225].

To indicate an application of the rule (26)  $\Rightarrow$  (27), let us consider the simple example with  $f(k) = q^k$ :

$$\sum_{k=0}^{\infty} \binom{k}{j} q^k = q^j (1-q)^{-j-1}, \quad (|q| < 1).$$

Consequently, we obtain

$$\sum_{k=0}^{\infty} \binom{k}{m}_\beta q^k = \sum_{j=0}^m \frac{q^j \cdot j!}{(1-q)^{j+1} m!} S_\beta(m, j|1). \tag{32}$$

This may be used to evaluate an infinite series involving both generalized binomial coefficients and Fibonacci numbers. Denote  $a = \frac{1}{2}(1 + \sqrt{5})$ ,  $b = \frac{1}{2}(1 - \sqrt{5})$ , and let  $\rho > a$ . Then the following series,

$$S = \sum_{k=0}^{\infty} \binom{k}{m}_\beta \rho^{-k} F_k,$$

is obviously convergent for every  $m \geq 0$ , where  $f_k = (a^{k+1} - b^{k+1})/\sqrt{5}$ . Certainly one may compute the series by means of (32) as follows:

$$S = \frac{\rho}{\sqrt{5}} \sum_{k=0}^{\infty} \binom{k}{m}_\beta [(a/\rho)^{k+1} - (b/\rho)^{k+1}] = \frac{\rho}{\sqrt{5}} \sum_{j=0}^m \left[ \left( \frac{a}{\rho-a} \right)^{j+1} - \left( \frac{b}{\rho-b} \right)^{j+1} \right] \frac{j!}{m!} S_\beta(m, j|1).$$

In particular, we have

$$\sum_{k=0}^{\infty} k^m \rho^{-k} F_k = \frac{\rho}{\sqrt{5}} \sum_{j=0}^m \left[ \left( \frac{a}{\rho-a} \right)^{j+1} - \left( \frac{b}{\rho-b} \right)^{j+1} \right] j! S(m, j).$$

Finally, it may be worthy of mention that, for the case  $\alpha = 1$ , relation (26), apart from the factor  $(-1)^j$  just stands for the  $\delta^*$ -transformation of the given sequence  $\{f(k)\}$ , which is connected with quasi-Hausdorff transformations (cf. Hardy [6, §11.19]). Moreover, it may be remarked that the rule (24)  $\Rightarrow$  (25) can still be generalized. Let the functions  $h(x, m)$  and  $g(x, j)$  be related by

$$h(x, m) = \sum_{j=0}^m t(m, j) g(x, j), \tag{33}$$

where the  $t(m, j)$  are complex numbers. Define

$$\sum_{k=0}^n F(n, k) g(k, j) = \phi(n, j). \tag{34}$$

Then we have

$$\sum_{k=0}^n F(n, k) h(k, m) = \sum_{j=0}^m \phi(n, j) t(m, j). \tag{35}$$

This extended rule (34)  $\Rightarrow$  (35) may even be used to obtain some interesting formulas involving Comtet's generalized Stirling numbers whose definitions may be found in [9]. However, we will omit the details here.

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