REFERENCE

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# ASSOCIATED STIRLING NUMBERS

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## 1. INTRODUCTION

For $r \geq 0$, define the integers $s_r(n, k)$ and $S_r(n, k)$ by means of

$$(1.1) \qquad \left(\log(1 - x)^{-1} - \sum_{i=1}^{r} x^i/i\right)^k = \left(\sum_{j=r+1}^{\infty} x^j/j\right)^k = k! \sum_{n=(r+1)k}^{\infty} s_r(n, k)x^n/n!,$$

$$(1.2) \qquad \left(e^x - \sum_{i=0}^{r} x^i/i!\right)^k = \left(\sum_{j=r+1}^{\infty} x^j/j!\right)^k = k! \sum_{n=(r+1)k}^{\infty} S_r(n, k)x^n/n!.$$

We will call $s_r(n, k)$ the $r$-associated Stirling number of the first kind, and $S_r(n, k)$ the $r$-associated Stirling number of the second kind. The terminology and notation are suggested by Comtet [6, pp. 221, 257]. When $r = 0$, we have $s_0(n, k) = (-1)^{n+k}s(n, k)$, where $s(n, k)$ is the Stirling number of the first kind, and $S_0(n, k) = S(n, k)$ is the Stirling number of the second kind. (In Comtet's notation this is true when $r = 1$.) If we define the polynomials $s_{r,n}(y)$ and $S_{r,n}(y)$ by means of

$$(1.3) \qquad \exp\left(y \sum_{j=r+1}^{\infty} x^j/j\right) = \sum_{n=0}^{\infty} s_{r,n}(y)x^n/n!,$$

$$(1.4) \qquad \exp\left(y \sum_{j=r+1}^{\infty} x^j/j!\right) = \sum_{n=0}^{\infty} S_{r,n}(y)x^n/n!,$$

it follows immediately that

$$(1.5) \qquad s_{r,n}(y) = \sum_{j=0}^{[n/r+1]} s_r(n, j)y^j,$$

and

$$(1.6) \qquad S_{r,n}(y) = \sum_{j=0}^{[n/r+1]} S_r(n, j)y^j.$$

Since the $r$-associated Stirling numbers of the second kind have appeared in two recent papers [7] and [9], it may be of interest to examine their combinatorial significance, their history, and their basic properties. We do this in §2, §3, and §4 for both the numbers of the first and second kind. Another purpose of this paper is to show how all the results of two recently published articles concerned with Stirling and Bell numbers, [7] and [16] can be generalized by the use of (1.2), (1.4), and (1.6). This is done in §5 and §6. To the writer's knowledge, the $r$-associated Stirling numbers of the first kind have not been studied before. Since most of their properties and formulas are analogous to those of the numbers of the second kind, it seems appropriate to include them in this paper.

## 2. COMBINATORIAL SIGNIFICANCE

Let $a_1$, $a_2$, $a_3$, ... be any strictly increasing sequence of positive integers. It follows from [12, Ch. 4] that the numbers $t(n, k)$ and $T(n, k)$ defined by means of

(2.1) $$\left(\sum_{j=1}^{\infty} x^{a_j}/a_j\right)^k = k!\sum_{n=0}^{\infty} t(n, k)x^n/n!,$$

and

(2.2) $$\left(\sum_{j=1}^{\infty} x^{a_j}/(a_j)!\right)^k = k!\sum_{n=0}^{\infty} T(n, k)x^n/n!$$

have the following combinatorial interpretation: $t(n,k)$ is the number of permutations of 1, 2, ..., $n$ having exactly $k$ cycles such that the number of elements in each cycle is equal to one of the $a_i$; $T(n, k)$ is the number of set partitions of 1, 2, ..., $n$ consisting of exactly $k$ blocks (subsets) such that the number of elements in each block is equal to one of the $a_i$. Furthermore, if we define $t_n(y)$ and $T_n(y)$ by means of

(2.3) $$\exp\left(y\sum_{k=1}^{\infty} x^{a_k}/a_k\right) = \sum_{n=0}^{\infty} t_n(y)x^n/n!,$$

and

(2.4) $$\exp\left(y\sum_{k=1}^{\infty} x^{a_k}/(a_k)!\right) = \sum_{n=0}^{\infty} T_n(y)x^n/n!,$$

it follows that

(2.5) $$t_n(y) = \sum_{j=0}^{n} t(n, j)y^j,$$

and

(2.6) $$T_n(y) = \sum_{j=0}^{n} T(n, j)y^j.$$

Thus $t_n(1)$ is the number of permutations of 1, 2, ..., $n$ such that the number of elements in each cycle is equal to one of the $a_i$, and $T_n(1)$ is the number of set partitions of 1, 2, ..., $n$ such that the number of elements in each block is equal to one of the $a_i$.

As Riordan [12, p. 74] points out, the presence or absence of cycles (or blocks) of various lengths can easily be included in the generating functions (2.1) and (2.2), though the mathematics required to obtain numerical results may be very elaborate. There are many examples in the problems of [12, pp. 80-89]. Other interesting examples can be found in [1] and [3].

It is clear, then, that the $r$-associated Stirling numbers have the following interpretations:

The number $s_r(n, k)$ is equal to the number of permutations of 1, 2, ..., $n$ having exactly $k$ cycles such that each cycle has at least $r+1$ elements. It is understood that in any cycle the smallest element is written first. The number $s_{r,n}(1)$ is equal to the number of permutations of 1, 2, ..., $n$ such that each cycle has at least $r+1$ elements. If we give a permutation with exactly $j$ cycles a "weight" of $y^j$, then $s_{r,n}(y)$ is the sum of the weights of all the permutations of 1, 2, ..., $n$ such that each cycle has at least $r+1$ elements.

The number $S_r(n, k)$ is equal to the number of set partitions of 1, 2, ..., $n$ consisting of exactly $k$ blocks such that each block contains at least $r+1$

The number $S_{r,n}(1)$ is equal to the number of set partitions of 1, 2, ..., $n$ such that each block has at least $r + 1$ elements. If we give a set partition with exactly $j$ blocks a weight of $y^j$, then $S_{r,n}(y)$ is the sum of the weights of all the set partitions of 1, 2, ..., $n$ such that each block has at least $r + 1$ elements.

## 3. HISTORY OF THE $r$-ASSOCIATED STIRLING NUMBERS

The Stirling numbers of the first kind, $s(n, k)$, and of the second kind, $S(n, k)$, were evidently first introduced in 1730 by James Stirling [13, pp. 8, 11]. They are usually defined in the following way:

$$(x)_n = x(x-1) \ldots (x-n+1) = \sum_{j=0}^{n} s(n, j)x^j; \tag{3.1}$$

$$x^n = \sum_{j=0}^{n} S(n, j)(x)_j. \tag{3.2}$$

It is not the purpose of this paper to review the history or well-known properties of the Stirling numbers; there are many good references, including [6, Ch. 5], [10, Ch. 4], and [12, pp. 32-38 and Ch. 4]. We are using the notation of Riordan [12] for the Stirling numbers of the first and second kind.

The numbers $s_1(n, k)$ and $S_1(n, k)$ were introduced in 1933-34 by Jordan [11] and Ward [17]. Using different notations, these authors defined $s_1(n, j)$ and $S_1(n, j)$ by means of

$$s(n, n-k) = (-1)^k \sum_{j=0}^{k} s_1(2k-j, k-j)\binom{n}{2k-j}, \tag{3.3}$$

$$S(n, n-k) = \sum_{j=0}^{k} S_1(2k-j, k-j)\binom{n}{2k-j}. \tag{3.4}$$

The purpose of these definitions was to prove that $s(n, n-k)$ and $S(n, n-k)$ are both polynomials in $n$ of degree $2k$, and also to show how $s(n, n-k)$ and $S(n, n-k)$ can be written as linear combinations of binomial coefficients. Formulas (3.3) and (3.4) can also be useful in determining $s(n, n-k)$ and $S(n, n-k)$ when $n$ is large and $k$ is small. The generating functions (1.1) and (1.2) were not given in [11] or [17]. This approach to $s_1(n, k)$ and $S_1(n, k)$ is also discussed in [11, Ch. 4]. In [12, Ch. 4], the generating functions are given, and the combinatorial interpretations are thoroughly discussed. It is also shown that

$$(-1)^{n+k}s(n, k) = \sum_{j=0}^{k} \binom{n}{j} s_1(n-j, k-j). \tag{3.5}$$

$$S(n, k) = \sum_{j=0}^{k} \binom{n}{j} S_1(n-j, k-n), \tag{3.6}$$

$$s_1(n+1, k) = ns_1(n, k) + ns_1(n-1, k-1), \tag{3.7}$$

$$S_1(n+1, k) = kS_1(n, k) + nS_1(n-1, k-1). \tag{3.8}$$

Applications for $s_1(n, k)$ and $S_1(n, k)$ have been found; see [1], [4], and [5], for example. A good discussion of these numbers can also be found in [2].

The $r$-associated Stirling number of the second kind, for arbitrary $r$, was apparently first defined and used by Tate and Goen [15] in 1958. They made the following definition:

(3.9)
$$G_r(n,\ k) = (-1)^k n! \sum \frac{(-1)^{k_1}(k_1)^A}{k_1!k_2!\ \ldots\ k_{r+2}!A!Q},$$

where

$$A = A(k_1,\ \ldots,\ k_{r+2}) = n - \sum_{i=0}^{r} ik_{i+2},$$

$$Q = Q(k_1,\ \ldots,\ k_{r+2}) = \prod_{i=0}^{r} (i!)^{k_{i+2}},$$

and the sum is over all $k_1,\ k_2,\ \ldots,\ k_{r+2}$ such that $k_1 + k_2 + \cdots + k_{r+2} = k$, and $0 \le k_i \le k$. For $r = 0$, (3.9) reduces to the familiar formula for $S(n,\ k)$:

(3.10)
$$G_0(n,\ k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n = S(n,\ k).$$

Now by induction we can show that $G_r(n,\ k) = S_r(n,\ k)$. It is true for $r = 0$; assume it is true for a fixed $r$. Then, by (1.2), we have

$$\sum_{n=(r+2)k}^{\infty} k!S_{r+1}(n,\ k)x^n/n! = \left(\sum_{j=r+1}^{\infty} x^j/j! - x^{r+1}/(r+1)!\right)^k$$
$$= \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}[(r+1)!]^{i-k}x^{(r+1)(k-i)}\left(\sum_{j=r+1}^{\infty} x^j/j!\right)^i$$
$$= \sum_{i=0}^{k}\sum_{m=0}^{\infty}\binom{k}{i}(-1)^{k-i}[(r+1)!]^{i-k}i!(m!)^{-1}G_r(m,\ i)x^{m+(r+1)(k-i)}.$$

By using (3.9) to rewrite $G_r(m,\ i)$ and then comparing coefficients of $x^n$, we have $G_{r+1}(n,\ k) = S_{r+1}(n,\ k)$. For example, we have

(3.11)
$$S_1(n,\ k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}\sum_{m=0}^{j}\binom{j}{m}(n)_m(k-j)^{n-m}.$$

A formula equivalent to (3.11) was also proved by Carlitz [2].

The $r$-associated Stirling numbers of the second kind have appeared in problems in [6, pp. 221-222] and [12, p. 102]. Recently, Enneking and Ahuja [7] have used these numbers to extend earlier results of Uppuluri and Carpenter [16] concerning the Bell numbers. In another recent paper the writer [9] has shown the relationship of $S_r(n,\ k)$ to the numbers $A_{r,n}$ defined by

(3.12)
$$(x^r/r!)\left(\sum_{j=r}^{\infty}x^j/j!\right)^{-1} = \sum_{n=0}^{\infty}A_{r,n}x^n/n!.$$

The relationship is

(3.13)
$$A_{r,n} = \sum_{j=1}^{n}(-r!)\ j!n!S_r(n+rj,\ j)/(n+rj)!.$$

The number $A_{1,n}$ is the $n$th Bernoulli number.

Evidently, the $r$-associated Stirling numbers of the first kind have not been studied, though they do appear in a problem in [6, pp. 256-257].

## 4. BASIC FORMULAS

In [7] and [9] formulas for $S_r(n,\ k)$ and the polynomials defined by (1.4) and (1.6) were derived. The notation for $S_r(n,\ k)$ is $d_r(n,\ k)$ in [7] and

$b(r;\, n,\, k)$ in [9]. In this section we are concerned mainly with the analogous formulas for the $r$-associated numbers of the first kind. The following formulas have been proved:

(4.1) $$S_r(n+1,\, k) = kS_r(n,\, k) = \binom{n}{r}S_r(n-r,\, k-1),$$
with $S_r(0,\, 0) = 1$,

(4.2) $$S_r(n,\, k) = \sum \frac{n!}{k!u_1!u_2!\ \dots\ u_k!},$$

the sum over all compositions (ordered partitions) $u_1 + u_2 + \cdots + u_k = n$, each $u_i \geq r + 1$,

(4.3) $$S_{r,n}(y) = \sum_{i=0}^{n} \frac{n!(r!)^{-i}(-y)^i}{i!(n-ri)!} S_{r-1,\, n-ri}(y),$$

(4.4) $$S_{r-1,n}(y) = \sum_{i=0}^{n} \frac{n!(r!)^{-i}y^i}{i!(n-ri)!} S_{r,\, n-ri}(y),$$

(4.5) $$S_{r-1}(n,\, k) = \sum_{j=0}^{k} \frac{n!(r!)^{-j}}{j!(n-rj)!} S_r(n-jr,\, k-j),$$

(4.6) $$S_r(n,\, k) = \sum_{j=0}^{k} \frac{(-1)^j n!(r!)^{-j}}{j!(n-jr)!} S_{r-1}(n-jr,\, k-j),$$

(4.7) $$S_{r,\, n+1}(y) = y\sum_{i=0}^{n-r}\binom{n}{i} S_{r,\, i}(y).$$

It should be noted that there are misprints in formulas (5.14) and (5.16) of [9], which correspond to (4.7) and (4.4), respectively, in this paper. Also, in the table following (5.11) in [9], the value of $g(6,\, 2)$ is 10, not 0. We also note that the Tate-Goen formula (3.9) can be proved inductively by means of (4.6).

We now look at the analogous formulas for $s_r(n,\, k)$. First, we have the recurrence

(4.8) $$s_r(n+1,\, k) = ns_r(n,\, k) + (n)_r s_r(n-r,\, k-1),$$

where $(n)_r = n(n-1)\ \dots\ (n-r+1)$ and $s_r(0,\, 0) = 1$, $s_r(n,\, 0) = 0$ if $n \neq 0$. We shall use a combinatorial argument to prove (4.8). In the permutations of $n + 1$ elements which have $k$ cycles, each cycle containing at least $r + 1$ elements, enumerated by $s_r(n+1,\, k)$, element $n + 1$ is in some $r + 1$ cycle or it is not. If it is not, it is inserted into one of the $k$ cycles of $n$ elements enumerated by $s_r(n,\, k)$, and this can be done in $n$ ways. If it is, there are $\binom{n}{r}$ ways to choose the other $r$ elements of the $r + 1$ cycle, and since the smallest element must be first, there are $r!$ ways the elements can be arranged in the cycle. Note that $r!\binom{n}{r} = (n)_r$. There are then $n - r$ elements left to be arranged in $k - 1$ cycles.

By comparing coefficients of $x^n$ on both sides of (1.1), we have

(4.9) $$s_r(n,\, k) = \frac{n!}{k!u_1u_2\ \dots\ u_k},$$

where the sum is over all compositions $u_1 + u_2 + \cdots + u_k = n$, each $u_i \geq r + 1$. This generalizes the formula for $s(n,\, k)$ given in [10, p. 146, formula (5)].

Formulas analogous to (4.3)-(4.7) can be derived. From (1.3), we have

$$\sum_{n=0}^{\infty} s_{r,n}(y)x^n/n! = \exp\left(y\sum_{j=r}^{\infty} x^j/j\right)\exp(-yx^r/r)$$
$$= \sum_{n=0}^{\infty} s_{r-1,n}(y)x^n/n! \sum_{j=0}^{\infty}(-y)^j x^{rj} r^{-j}/j!.$$

Comparing coefficients of $x^n$, we have

(4.10) $$s_{r,n}(y) = \sum_{j=0}^{[n/r]} \binom{n}{rj} f_r(rj)(-y)^j s_{r-1,n-rj}(y),$$

when $f_r(0) = 1$ and for $j > 0$,

(4.11) $$f_r(rj) = (rj)!/r(2r)(3r)\ \ldots\ (jr),$$

that is, $f_r(rj)$ is the same as $(rj)!$ with every $r$th term divided out. With a similar argument, we have

(4.12) $$s_{r-1,n}(y) = \sum_{j=0}^{[n/r]} \binom{n}{rj} f_r(rj) y^j s_{r,n-rj}(y).$$

It follows from (1.5), (4.10), and (4.12) that

(4.13) $$s_r(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n}{rj} f_r(rj) s_{r-1}(n - rj, k - j),$$

and

(4.14) $$s_{r-1}(n, k) = \sum_{j=0}^{k} \binom{n}{rj} f_r(rj) s_r(n - rj, k - j).$$

Equation (4.14) generalizes (3.5) and shows how to write $s_{r-1}(n, k)$ as a linear combination of binomial coefficients. In fact it is not difficult to see from (4.14) and (4.5) that, for $k > 0$ and fixed $r$,

(4.15) $$r^m s_{r-1}(rm + k, m) = (rm + k)(rm + k - 1)\ \ldots\ mR_k(m),$$

and

(4.16) $$(r!)^m S_{r-1}(rm + k, m) = (rm + k)(rm + k - 1)\ \ldots\ mQ_k(m),$$

where $R_k(m)$ and $Q_k(m)$ are polynomials in $m$ of degree $k - 1$. By differentiating (1.3) with respect to $x$ and comparing coefficients of $x^n$, we have

(4.17) $$s_{r,n+1}(y) = y\sum_{i=0}^{n-r} (n)_i s_{r,i}(y).$$

If we define the numbers $d_{r,n}$ by means of

(4.18) $$(x^r/r)\left(\sum_{j=r}^{\infty} x^j/j\right)^{-1} = \sum_{n=0}^{\infty} d_{r,n}x^n,$$

then it follows from [9, formulas 4.11 and 4.12] that

(4.19) $$d_{r,n} = \sum_{j=1}^{n} (-1)^j [f_r(rj)(n + rj)_n]^{-1} s_r(n + rj, j),$$

(4.20) $$d_{r,n} = \sum_{j=1}^{n} (-1)^j \binom{n+1}{j+1} [f_r(rj)(n + rj)_n]^{-1} s_{r-1}(n + rj, j).$$

When $r = 1$ in (4.16), we have

$$-x[\ln(1-x)]^{-1} = \sum_{n=0}^{\infty} d_{1,n}x^n, \tag{4.21}$$

so that $d_{1,n} = (-1)^n b_n$, where $b_n$ is the Bernoulli number of the second kind [10, pp. 265-287]. Thus, by (4.17) and (4.18), we have

$$b_n = \sum_{j=1}^{n} (-1)^{n+j}[(n+j)_n]^{-1}s_1(n+j, j), \tag{4.22}$$

$$b_n = \sum_{j=1}^{n} (-1)^j \binom{n+1}{j+1}[(n+j)_n]^{-1}s(n+j, j). \tag{4.23}$$

It can also be proved [see 10, p. 267] that

$$n!b_n = \sum_{k=0}^{n} s(n, k)/(k+1). \tag{4.24}$$

We can compare formulas (4.22), (4.23), and (4.24) to similar formulas involving the ordinary Bernoulli numbers and the Stirling numbers of the second kind [10, pp. 182, 219, and 599].

## 5. GENERALIZATION OF THE PAPER BY UPPULURI AND CARPENTER

In [16] Uppuluri and Carpenter defined a sequence $C_0$, $C_1$, $C_2$ ... by means of

$$\exp(1 - e^x) = \sum_{j=0}^{\infty} C_j x^j/j!, \tag{5.1}$$

and they derived some formulas involving the $C_j$ and Bell numbers $B_1$, $B_2$, ..., defined by

$$B_n = \sum_{j=1}^{n} S(n, j). \tag{5.2}$$

In this section, we show how all the results of [16] can be extended by using (1.4) and (1.6). In Propositions 5.1-5.10, which correspond to Propositions 1-10 in [16], we use the notation

$$S_{0,n}(y) = S_n(y), \tag{5.3}$$

so clearly $B_n = S_n(1)$ and $C_n = S_n(-1)$. We omit any proof which is obvious or which is analogous to the corresponding proof in [16].

*Proposition 5.1*: $S_k(y) = e^{-y}\sum_{m=0}^{\infty} y^m m^k/m!$, $k = 0, 1, 2, \ldots$ .

*Proposition 5.2*: Equation (1.6) of this paper.

*Proposition 5.3*: Equation (4.7) of this paper.

*Proposition 5.4*: $\Delta^n S_1(y) = \sum_{j=0}^{n} (-1)^{n-j}\binom{n}{j}S_{j+1}(y) = yS_n(y)$.

Using Proposition 5.4 and $S_1(y) = y$, we can compute $S_2(y)$, ..., $S_n(y)$ for small values of $n$. For example, $\Delta S_1(y) = yS_1(y) = y^2$, so

$$S_2(y) = S_1(y) + \Delta S_1(y) = y + y^2,$$

and

$$S_3(y) = S_2(y) + \Delta S_2(y) = S_2(y) + \Delta^2 S_1(y) + \Delta S_1(y)$$
$$= (y + y^2) + (y^2 + y^3) + y^2 = y + 3y^2 + y^3.$$

*Proposition 5.5:* $\sum_{k=0}^{n} \binom{n}{k} S_k(y) S_{n-k}(-y) = 0,\ n = 1, 2, \ldots,$ and $S_0(y) = 1$.

*Proposition 5.6:* $\sum_{j=0}^{n} \binom{n}{j} S_j(-y) S_{n+1-j}(y) = y,\ n = 0, 1, 2, \ldots .$

*Proposition 5.7:* Same as Proposition 5.6.

*Proposition 5.8:* Let $a_i = S_i(y)/i!$. Then

(a)
$$S_n(-y) = (-1)^n n! \begin{vmatrix} a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_n & a_{n-1} & & \cdots & & a_1 \end{vmatrix}$$
$$= (-1)^n n! g_n,$$

(b)
$$(-1)^n S_n(-y) = n! \sum_{k=0}^{n-1} (-1)^k g_{n-k-1} S_{k+1}(y)/(k+1)!.$$

*Proposition 5.9:*
$$S_{n+1}(-y) = (-1)^{n+1} \begin{vmatrix} y & 1 & 0 & 0 & \cdots & 0 \\ y & y & 1 & 0 & \cdots & 0 \\ y & 2y & y & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \binom{n}{0} & \binom{n}{1}y & & \cdots & & \binom{n}{n}y \end{vmatrix}$$

In Proposition 5.9, the element in the $i$th row, $j$th column, for $j \le i$, is $\binom{i-1}{j-i} y$.

*Proposition 5.10:*
$$S_{n+1}(-y) = (-1)^{n+1} \begin{vmatrix} y & 1 & 0 & 0 & \cdots & 0 \\ y & y & 2 & 0 & \cdots & 0 \\ y/2 & y & y & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ y/n! & y/(n-1)! & & \cdots & & y/0! \end{vmatrix}$$

In Proposition 5.10, the element in the $i$th row, $j$th column, for $j \le i$, is $y/(i-j)!$.

The proof of Proposition 10 in [16] is not given. A reference is given to a formula of Ginsburg [8] for the Bell numbers, but unfortunately Ginsburg's proof is obscure. Proposition 5.10 is easily proved, however, by multiplying the $k+1$st row of the determinant in Proposition 5.9 by $1/k!$ and the $k+1$st column by $k!$ $(k = 1, 2, \ldots, n)$.

The motivation given in [16] for studying the numbers $C_j$ defined by (5.1) is the following: Define $B_n^{(k)}$ by

(5.3)
$$B_n^{(k)} = \sum_{j=1}^{n} j^k S(n, j).$$

Then

$$B_n^{(0)} = B_n,$$
$$B_n^{(1)} = B_{n+1} - B_n,$$
$$B_n^{(2)} = B_{n+2} - 2B_{n+1},$$

and these equations lead to a search for a general expression for $B_n^{(k)}$ in terms of the Bell numbers $B_n, B_{n+1}, \ldots, B_{n+k}$. It is stated, though not actually proved, that

(5.4)
$$B_n^{(k)} = \sum_{j=0}^{k} \binom{k}{j} C_j B_{n+k-j}.$$

We now generalize this result by defining $S_n^{(k)}(y)$ by

(5.5)
$$S_n^{(k)}(y) = \sum_{j=1}^{n} j^k S(n, j) y^j,$$

and showing that

(5.6)
$$S_n^{(k)}(y) = \sum_{j=0}^{k} \binom{k}{j} S_j(-y) S_{n+k-j}(y).$$

For example,

$$S_n^{(1)}(y) = S_{n+1}(y) - yS_n(y),$$
$$S_n^{(2)}(y) = S_{n+2}(y) - 2yS_{n+1}(y) + (y^2 - y)S_n(y).$$

To prove (5.6), we start with (1.4) with $r = 0$. Differentiating $n$ times with respect to $x$, we have

(5.7)
$$D^{(n)} \exp y(e^x - 1) = \sum_{j=0}^{\infty} S_{n+j}(y) x^j / j!.$$

Now consider the numbers $q_n^{(m)}(y)$ defined by

(5.8)
$$(\exp y(1 - e^x)) D^{(n)} \exp y(e^x - 1) = \sum_{m=0}^{\infty} q_n^{(m)}(y) x^m / m!.$$

It follows from (1.4), (5.7), and (5.8) that

$$q_n^{(k)}(y) = \sum_{j=0}^{k} S_j(-y) S_{n+k-j}(y) \binom{k}{j}.$$

Now we show by induction that

(5.9)
$$q_n^{(k)}(y) = S_n^{(k)}(y).$$

For $n = 1$, we have, from (5.8),

$$ye^x = \sum_{m=0}^{\infty} q_1^{(m)}(y) x^m / m!,$$

so

$$q^{(k)}(y) = y = S_1^{(k)}(y).$$

Assume (5.9) holds for a fixed $n$, and also assume

$$D^{(n)} \exp y(e^x - 1) = (\exp y(e^x - 1)) \sum_{i=1}^{n} e^{xi} S(n, i) y^i.$$

Then we have

$$(5.10)\quad D^{(n+1)}\exp y(e^x-1) = (\exp y(e^x-1))\sum_{i=1}^{n} e^{xi}(iS(n,\ i) + S(n,\ i-1))y^i$$
$$= (\exp y(e^x-1))\sum_{i=1}^{n+1} e^{xi}S(n+1,\ i)y^i.$$

Multiplying both sides of (5.10) by $\exp y(1-e^x)$ and comparing coefficients of $x$, we see that $q_{n+1}^{(k)}(y) = S_{n+1}^{(k)}(y)$.

## 6. GENERALIZATION OF THE PAPER BY ENNEKING AND AHUJA

In [7] Enneking and Ahuja defined a generalized Bell number by

$$(6.1)\quad B_r(n) = \sum_{j=0}^{n} S_r(n,\ j),$$

and they were able to generalize some of the formulas in [16]. Note that

$$B_r(n) = S_{r,n}(1).$$

By considering $S_{r,n}(y)$, we can extend each of the twelve properties in [7]; Properties 6.1-6.12 in this paper correspond to Properties 1-12 in [7]. We omit any proof which is obvious or is analogous to the corresponding proof in [7].

*Property 6.1*: Equation (1.4) of this paper.

*Property 6.2*: Equation (4.7) of this paper.

*Property 6.3*: Equation (4.3) of this paper.

*Property 6.4*: $S_{1,n}(y) = e^{-y}\sum_{m=0}^{\infty} y^m(m-y)^n/m!.$

*Proof*: We have, from (1.4),

$$\sum_{n=0}^{\infty} S_{1,n}(y)x^n/n! = e^{-y}e^{-xy}\exp(ye^x) = e^{-y}\left(\sum_{i=0}^{\infty}(-y)^i x^i/i!\right)\left(\sum_{m=0}^{\infty} y^m e^{xm}/m!\right)$$
$$= e^{-y}\left(\sum_{m=0}^{\infty} y^m/m! \sum_{j=0}^{\infty}(xm)^j/j!\right)\left(\sum_{i=0}^{\infty}(-y)^i x^i/i!\right)$$
$$= e^{-y}\sum_{m=0}^{\infty} y^m/m! \sum_{n=0}^{\infty}(m-y)^n x^n/n!,$$

and Property 6.4 is proved when we compare coefficients of $x^n$.

For Property 6.5, we need the following definition of $H_r(x)$:

$$(6.2)\quad \exp y(e^x - 1 - x - \cdots - x^r/r!) = (\exp y(e^x-1))H_r(x),$$

where

$$(6.3)\quad H_r(x) = \sum_{i=0}^{\infty} h_{r,i}(y)x^i/i!,\ r \geq 1.$$

Throughout the remainder of this paper we will also continue to use the notation of (5.3).

*Property 6.5*: $S_{r,n}(y) = \sum_{i=0}^{n}\binom{n}{i}h_{r,i}(y)S_{n-i}(y)$, where

$$h_{r,n+1}(y) = -y\sum_{j=0}^{r-1}\binom{n}{j}h_{r,n-j}(y),\ h_{0,n}(y) = 0 \text{ for } r \geq 0,\ h_{r,0}(y) = 1.$$

To generalize (5.5), we make the following definition:

$$S_{r,n}^{(k)}(y) = \sum_{m=1}^{n} m^k S_r(n, m)y^m. \tag{6.4}$$

*Property 6.6*: $S_{r,n}^{(k+1)}(y) = S_{r,n+1}^{(k)}(y) - y\binom{n}{r}\sum_{j=0}^{k}\binom{k}{j}S_{r,n-r}^{(j)}(y).$

*Property 6.7*: $S_n^{(k+1)}(y) = S_{n+1}^{(k)}(y) - y\sum_{j=0}^{k}\binom{k}{j}S_n^{(j)}(y).$

Now we want to generalize (5.6); that is, we want to express $S_{r,n}^{(k)}(y)$ in terms of the $S_{r,n}(y)$. For example,

$$S_{r,n}^{(1)}(y) = S_{r,n+1}(y) - \binom{n}{r}yS_{r,n-r}(y),$$

$$S_{r,n}^{(2)}(y) = S_{r,n+2}(y) - y\left[\binom{n+1}{r} + \binom{n}{r}\right]S_{r,n+1-r}(y) + y^2\binom{n-r}{r}\binom{n}{r}S_{r,n-2r}(y) - y\binom{n}{r}S_{r,n-r}(y).$$

*Property 6.8*: $S_{r,n}^{(k)}(y) = \sum_{i=0}^{k}\sum_{j=0}^{i} a_{ij}(n, k, r)S_{r,n+k-i-jr}(y)$, where

$$a_{0,0}(n, k, r) = 1,\ a_{ij}(n, k, r) = 0 \text{ if } j = 0 \text{ and } i > 0,$$

and

$$a_{ij}(n, k+1, r) = a_{ij}(n+1, k, r) - y\binom{n}{r}\sum_{m=k-i+j}^{k} a_{i+m-k-1,j-1}(n-r, m, r).$$

When $r = 0$, we have

$$\sum_{j=0}^{i} a_{ij}(n, k, 0) = \binom{k}{i}S_i(-y), \tag{6.5}$$

independent of $n$ for $i = 1, 2, \ldots, k$. Letting $r = 0$ in Property 6.8, letting $i = k + 1$, and summing on $j$, we have

$$S_{k+1}(-y) = -y\sum_{m=0}^{k}\binom{k}{m}S_m(-y),$$

which agrees with Proposition 5.3.

Now let

$$W_{r,n}^{(k)}(y) = \sum_{j=0}^{n}(j)_k S_r(n, j)y^j. \tag{6.6}$$

We shall use the notation $W_{0,n}^{(k)}(y) = W_n^{(k)}(y)$.

*Property 6.9*: $W_{r,n}^{(k+1)}(y) = W_{r,n+1}^{(k)}(y) - kW_{r,n}^{(k)}(y) - \binom{n}{r}y\left(kW_{r,n-r}^{(k+1)}(y) + W_{r,n-r}^{(k)}(y)\right).$

*Property 6.10*: $W_n^{(k+1)}(y) = W_{n+1}^{(k)}(y) - (k+y)W_n^{(k)}(y) - ykW_n^{(k-1)}(y)$.

*Property 6.11*: $W_n^{(k)}(y) = \sum_{i=0}^{k} w(k, i, y)S_{n+k-i}(y)$,

where the $w(k, i, y)$ satisfy $w(k, 0, y) = 1$ and

$$w(k+1, i, y) = w(k, i, y) - (k+y)w(k, i-1, y) - ykw(k-1, i-2, y).$$

For example,

$$W_n^{(1)}(y) = S_{n+1}(y) - yS_n(y),$$
$$W_n^{(2)}(y) = S_{n+2}(y) - (2y+1)S_{n+1}(y) + y^2S_n(y).$$

It is noted in [7], without proof, that for $y = 1$ the $w(k, i, y)$ are the coefficients of a special case of the Poisson-Charlier polynomials $P_n(x)$ [14, p. 34]. These polynomials can be defined by

$$P_k(x) = \sum_{i=0}^{k} p(k, i, u)x^{k-i}, \tag{6.7}$$

$$p(k, i, u) = \sum_{j=0}^{i} (-1)^j \binom{k}{j} u^{j-k} s(k-j, k-i). \tag{6.8}$$

(This definition is slightly different from the one given by Szegö [14].) We now show that when $u = y$,

$$w(k, i, y) = y^k p(k, i, y). \tag{6.9}$$

We prove (6.9) by showing that $y^k p(k, i, y)$ satisfies the same recurrence as $w(k, i, y)$. For convenience, in the proof we use the notation $p(k, i) = y^k p(k, i, y)$. Then we have $p(k, 0) = 1$ and

$$\begin{aligned} p(k+1, i) &= \sum_{j=0}^{i} (-1)^j \binom{k+1}{j} s(k+1-j, k+1-i)y^j \\ &= \sum_{j=0}^{i} (-1)^j \left[\binom{k}{j} + \binom{k}{j-1}\right][s(k-j, k-i) - (k-j)s(k-j, k+1-i)]y^j \\ &= p(k, i) - \sum_{j=0}^{i-1} (-1)^j \binom{k}{j} s(k-j, k+1-i)y^{j+1} \\ &\quad - k\sum_{j=0}^{i} (-1)^j \binom{k-1}{j} s(k-j, k+1-i)y^j. \end{aligned}$$

Replacing $\binom{k-1}{j}$ by $\binom{k}{j} - \binom{k-1}{j-1}$, we have

$$p(k+1, i) = p(k, i) - yp(k, i-1) - kp(k, i-1) - ykp(k-1, i-2).$$

This completes the proof of (6.9).

Now we want to express $W_{r,n}^{(k)}(y)$ in terms of the $S_{r,j}(y)$. For example

$$W_{r,n}^{(1)}(y) = S_{r,n+1}(y) - \binom{n}{r} yS_{r,n-r}(y),$$
$$W_{r,n}^{(2)}(y) = S_{r,n+2}(y) - S_{r,n+1}(y) - y\left[\binom{n+1}{r} + \binom{n}{r}\right]S_{r,n-r+1}(y) + \binom{n}{r}\binom{n-r}{r}y^2S_{r,n-2r}(y).$$

*Property 6.12*: $W_{r,n}^{(k)}(y) = \sum_{i=0}^{k}\sum_{j=0}^{i} b_{ij}(n, k, r)S_{r,\,n+k-i-jr}(y)$,

where the $b_{ij}(n, k, r)$ satisfy $b_{0,0}(n, k, r) = 1$, $b_{kj}(n, k, r) = 0$ for $j = 0, \ldots, k - 1$, and

$$b_{ij}(n, k+1, r) = b_{ij}(n+1, k, r) - kb_{i-1,j}(n, k, r) - \binom{n}{r}y[b_{i-1,j-1}(n-r, k, r) + kb_{i-2,j-1}(n-r, k-1, r)].$$

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