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Research Paper

A new explicit formula for the Bernoulli and Genocchi numbers in terms of the Stirling numbers

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Abstract

In the paper, the authors concisely review some explicit formulas and establish a new explicit formula for the Bernoulli and Genocchi numbers in terms of the Stirling numbers of the second kind.

Keywords: *explicit formula; Bernoulli number; Genocchi number; Stirling number of the second kind*
MSC: *Primary 11B68; Secondary 11B73*

1. Introduction and main results

It is well known that the Bernoulli numbers B_n for $n \ge 0$ may be defined by the power series expansion

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}, \quad |x| < 2\pi, \tag{1}$$

that Euler polynomials $E_n(x)$ are defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{2}$$

that the Genocchi numbers G_n for $n \in \mathbb{N}$ are given by the generating function

$$\frac{2t}{e^t+1} = \sum_{n=1}^{\infty} G_n \frac{t^n}{n!}, \tag{3}$$

and that the Stirling numbers of the second kind which may be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!}, \quad k\in\mathbb{N} \tag{4}$$

and may be computed by

$$S(k,m)=\frac{1}{m!}\sum_{\ell=1}^{m}(-1)^{m-\ell}\binom{m}{\ell}\ell^k, \quad 1\le m\le k. \tag{5}$$

By the way, the Stirling number of the second kind $S(n,k)$ may be interpreted combinatorially as the number of ways of partitioning a set of n elements into k nonempty subsets.

The Bernoulli numbers B_n for $n\in\{0\}\cup\mathbb{N}$ satisfy

$$B_0=1, \quad B_1=-\frac{1}{2}, \quad B_{2n+2}\ne 0, \quad B_{2n+3}=0. \tag{6}$$

For $n\in\mathbb{N}$, the Genocchi numbers meet $G_{2n+1}=0$. The first few Genocchi numbers G_n are listed in Table 1.1. The

Table 1.1: The first few Genocchi numbers G_n

n	1	2	4	6	8	10	12	14	16	18
G_n	1	-1	1	-3	17	-155	2073	-38227	929569	-28820618

Genocchi numbers G_{2n} may be represented in terms of the Bernoulli numbers B_{2n} and Euler polynomials $E_{2n-1}(0)$ as

$$G_{2n}=2(1-2^{2n})B_{2n}=2nE_{2n-1}(0), \quad n\in\mathbb{N}. \tag{7}$$

See [1, p. 49]. As a result, we have

$$G_n=2(1-2^n)B_n, \quad n\in\mathbb{N}. \tag{8}$$

The first formula for the Bernoulli numbers B_n listed in [2] is

$$B_n=\sum_{k=0}^{n}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n, \quad n\ge 0, \tag{9}$$

which is a special case of the general formula [13, (2.5)]. The formula (9) is equivalent to

$$B_n=\sum_{k=0}^{n}(-1)^k\frac{k!}{k+1}S(n,k), \quad n\in\{0\}\cup\mathbb{N}, \tag{10}$$

which was listed in [3, p. 536] and [4, p. 560]. Recently, four alternative proofs of the formula (10) were provided in [7, 16]. A generalization of the formula (10) was supplied in [6]. In all, we may collect at least seven alternative proofs for the formula (9) or (10) in [2, 4, 7, 13, 14, 16] and closely related references therein.

In [2, p. 48, (11)], it was deduced that

$$B_n=\sum_{j=0}^{n}(-1)^j\binom{n+1}{j+1}\frac{n!}{(n+j)!}\sum_{k=0}^{j}(-1)^{j-k}\binom{j}{k}k^{n+j}, \quad n\ge 0, \tag{11}$$

which may be rearranged as

$$B_n=\sum_{i=0}^{n}(-1)^i\frac{\binom{n+1}{i+1}}{\binom{n+i}{i}}S(n+i,i), \quad n\ge 0. \tag{12}$$

The formula (12) was rediscovered in the paper [8]. On 21 January 2014, the authors searched out that the formula (12) was also derived in [12, p. 59] and [17, p. 140].

In [11, p. 1128, Corollary], among other things, it was found that

$$B_{2k} = \frac{1}{2} - \frac{1}{2k+1} - 2k\sum_{i=1}^{k-1} \frac{A_{2(k-i)}}{2(k-i)+1} \tag{13}$$

for $k \in \mathbb{N}$, where A_m is defined by

$$\sum_{m=1}^{n} m^k = \sum_{m=0}^{k+1} A_m n^m.$$

In [15, Theorem 1.4], among other things, it was presented that

$$B_{2k} = \frac{(-1)^{k-1}k}{2^{2(k-1)}(2^{2k}-1)} \sum_{i=0}^{k-1}\sum_{\ell=0}^{k-i-1} (-1)^{i+\ell}\binom{2k}{\ell}(k-i-\ell)^{2k-1}, \quad k \in \mathbb{N}. \tag{14}$$

In [10, Theorem 3.1], it was obtained that

$$B_{2k} = 1 + \sum_{m=1}^{2k-1} \frac{S(2k+1,m+1)S(2k,2k-m)}{\binom{2k}{m}} - \frac{2k}{2k+1}\sum_{m=1}^{2k} \frac{S(2k,m)S(2k+1,2k-m+1)}{\binom{2k}{m-1}}, \quad k \in \mathbb{N}. \tag{15}$$

The aim of this paper is to find the following new explicit formula for the Bernoulli numbers B_k, or say, the Genocchi numbers G_k, in terms of the Stirling numbers of the second kind $S(k,m)$.

Theorem 1.1 *For all $k \in \mathbb{N}$, the Genocchi numbers G_k may be computed by*

$$G_k = 2(1-2^k)B_k = (-1)^k k \sum_{m=1}^{k} (-1)^m \frac{(m-1)!}{2^{m-1}} S(k,m). \tag{16}$$

2. Proof of Theorem 1.1

Differentiating on both sides of the equation (3) and employing Leibniz identity for differentiation give

$$\left(\frac{2t}{e^t+1}\right)^{(k)} = 2\left[t\left(\frac{1}{e^t+1}\right)^{(k)} + k\left(\frac{1}{e^t+1}\right)^{(k-1)}\right] = \sum_{n=k}^{\infty} G_n \frac{t^{n-k}}{(n-k)!}.$$

In [9, Theorem 2.1] and [18, Theorem 3.1], it was obtained that, when $\lambda > 0$ and $t \ne -\frac{\ln\lambda}{\alpha}$ or when $\lambda < 0$ and $t \in \mathbb{R}$,

$$\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^{(k)} = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)!S(k+1,m)\left(\frac{1}{\lambda e^{\alpha t}-1}\right)^m. \tag{17}$$

Specially, when $\lambda = -1$ and $\alpha = 1$, the identity (17) becomes

$$\left(\frac{1}{e^t+1}\right)^{(k)} = (-1)^{k+1} \sum_{m=1}^{k+1} (-1)^m (m-1)!S(k+1,m)\left(\frac{1}{e^t+1}\right)^m. \tag{18}$$

Consequently, it follows that

$$G_k = \lim_{t\to 0}\sum_{n=k}^{\infty} G_n \frac{t^{n-k}}{(n-k)!} = 2k\lim_{t\to 0}\left(\frac{1}{e^t+1}\right)^{(k-1)}$$

$$= 2k(-1)^k \sum_{m=1}^{k} (-1)^m (m-1)!S(k,m) \lim_{t\to 0}\left(\frac{1}{e^t+1}\right)^m$$

$$= (-1)^k k \sum_{m=1}^{k} (-1)^m \frac{(m-1)!}{2^{m-1}} S(k,m).$$

The proof of Theorem 1.1 is complete.

Remark 2.1 *This paper is a slightly modified version of the preprint [5].*

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