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GENERATION OF STIRLING NUMBERS BY MEANS OF SPECIAL PARTITIONS OF NUMBERS

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1. INTRODUCTION

Stirling numbers of the First and Second Kinds appear as numerical coefficients in expressions relating factorials of variables to powers of the variable and vice versa. Riordan [1] investigates the properties of Stirling numbers in great detail, particularly with respect to recurrence formulas and relationships to other special numbers.

In the series expansions on certain functions of logarithms, Adams [2] develops and tabulates coefficients which run through positive and negative indices. A rearrangement of Adams' table for positive indices together with an appropriate alternation of sign yield Stirling numbers of the First Kind while a different rearrangement for negative indices yields Stirling numbers of the Second Kind.

An excellent summary of the properties of Stirling numbers including recursion and closed form expressions for finding Stirling numbers is presented in a recent Bureau of Standards publication [3]. In this regard, it is interesting to note that members of special partitions of numbers described in the April, 1964, issue of this Journal [4] can also be used to develop Stirling numbers. A discussion of this latter method follows.

2. DESCRIPTION OF COEFFICIENTS

Riordan uses the notation $S(n,k)$ and $s(n,k)$ for Stirling numbers of the Second and First Kinds, respectively, where the integers n and k are positive. Stirling numbers of the First Kind, the sum of whose n and k is odd, are negative. Adams chooses C_k^n, where n is a negative or positive integer and k is zero or a positive integer. Although none of Adams' C's are negative, a negative value for n identifies a C equal to a First Kind Stirling number, neglecting sign. For convenience of manipulation, the obviously subscripted (R for Riordan, A for Adams) indicates n_R, n_A, k_R, k_A replace the n and k's. By direct comparison, it can be seen that

(1) $$k_R = -n_A$$

(2) $$n_R = k_R + k_A$$

(applies for Second Kind only),

(3) $$n_R = n_A$$

(4) $$n_R = k_R + k_A$$

(applies for First Kind only).

The above equations lead to

(5) $$S(n_R, k_R) = C^{-k_R}_{n_R - k_R},$$

(6) $$C^{n_A}_{k_A} = S(k_A - n_A, -n_A),$$

(7) $$s(n_R, k_R) = (-1)^{n_R + k_R} \cdot C^{n_R}_{n_R - k_R},$$

(8) $$C^{n_A}_{k_A} = (-1)^{2n_A - k_A} \cdot s(n_A, n_A - k_A).$$

Tabulations of a few Stirling numbers are given below.

Table 1 $S(n_R, k_R)$

$n_R \backslash k_R$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

Table 2 $s(n_R, k_R)$

$n_R \backslash k_R$	1	2	3	4	5
1	1				
2	-1	1			
3	2	-3	1		
4	-6	11	-6	1	
5	24	-50	35	-10	1

In Adams' table, vertical entries for positive n_A are, with appropriate signs, First Kind Stirling numbers, and 45-degree, negative slope, diagonal entries for negative n_A are Second Kind Stirling numbers.

3. GENERATION OF SECOND KIND STIRLING NUMBERS

The negative n_A section of Adams' table suggests a numerical procedure by which Second Kind Stirling numbers can be generated simultaneously with the generation of members of the special partitions described in [4]. For example, in Table 3 consider a few column entries from Adams' table for $n_A = -4$. Differences between the entries are included

Table 3

$k_A \backslash n_A$	-4	Differences
0	1	1
1	10	9
2	65	55
3	350	285

If the differences were known the table entries could be found easily. The differences, however, do not stem from simple recursion formulas. If the manner in which successive sets of Second Kind Stirling numbers are found is investigated, it is seen that the differences are sums of products whose range is controlled by n_A and k_A. As an example from Table 3 ($n_A = -4$, $k_A = 3$) the products can be set up and sums formed vertically and horizontally as is shown in (9).

$$
(9)\qquad
\begin{array}{ccccccccc}
1 & & 2 & & 2\times 2 & & 2\times 2\times 2 & & \\
 & & 3 & & 2\times 3 & & 2\times 2\times 3 & & \\
 & & 4 & & 2\times 4 & & 2\times 2\times 4 & & \\
 & & & & 3\times 3 & & 2\times 3\times 3 & & \\
 & & & & 3\times 4 & & 2\times 3\times 4 & & \\
 & & & & 4\times 4 & & 2\times 4\times 4 & & \text{Vertical Sums} \downarrow \\
 & & & & & & 3\times 3\times 3 & & \\
 & & & & & & 3\times 3\times 4 & & \\
 & & & & & & 3\times 4\times 4 & & \\
 & & & & & & 4\times 4\times 4 & & \\
\overline{1} & + & \overline{9} & + & \overline{55} & + & \overline{285} & = & 350 \\
\end{array}
$$

$\xrightarrow{\hspace{4em}}$ Horizontal Sums

The significant fact demonstrated by (9) is that exclusive of the initial "one," the multiplication signs, and the resultant summations, the array presented by (9) is identically that found in the development of the partition set

$$PV(\geq 2, \leq 12 \,|\, \geq 1, \leq 3 \,|\, \geq 2, \leq 4)$$

according to the methods described in [4]. For the purposes of this paper, the PV set designation implies that the set of partitions is arranged in columns, each column consisting of partitions having exactly as many members as the column number. Thus, the set designation

$$\{1, PV(\geq 2, \leq 12 \,|\, \geq 1, \leq 3 \,|\, \geq 2, \leq 4)\}$$

includes an initial "one" and the properly arranged partitions.

In general, the set

$$\{1,\ PV(\geq 2, \leq -n_A k_A \,|\, \geq 1, \leq k_A \,|\, \geq 2, \leq -n_A)\}$$

when interpreted as in (9) yields Adams'

$$C^{n_A}_{k_A}$$

for negative n_A. Through use of (1) and (2), it is seen that the Second Kind Stirling number $S(n_R, k_R)$ can be found from the set

$$\{1, PV(\geq 2, \leq k_R(n_R - k_R) \,|\, \geq 1, \leq n_R - k_R \,|\, \geq 2, \leq k_R)\}\ .$$

The method suggested above leads directly to

$$C_{k_A}^{n_A}$$

or $S(n_R, k_R)$. An ALGOL language computer program for obtaining the partitions described in [3] was developed as a result of student projects under the author's direction. It is obvious that only a slight modification of this program would be required to generate and store products (as the corresponding partition is formed) needed to obtain C's or S's directly as exemplified by (9).

4. GENERATION OF FIRST KIND STIRLING NUMBERS

Adams lists the following formulas for finding

$$C_0^{n_A},\ C_1^{n_A},\ \text{and } C_2^{n_A}\ .$$

The sum forms are applicable for n_A positive, but the product forms apply for n_A positive or negative.

$$C_0^{n_A} = 1\ , \tag{10}$$

$$C_1^{n_A} = 1 + 2 + 3 + \cdots + (n_A - 1) = \frac{n_A(n_A - 1)}{2} \tag{11}$$

$$\begin{aligned} C_2^{n_A} &= 1 \times 2 + 1 \times 3 + 1 \times 4 + \cdots + 1 \times (n_A - 1) \\ &\quad + 2 \times 3 + 2 \times 4 + \cdots + 2 \times (n_A - 1) \\ &\quad + 3 \times 4 + \cdots + 3 \times (n_A - 1) \\ &\quad + \cdots\cdots\cdots\cdots\cdots \\ &\quad + (n_A - 2)(n_A - 1) \\ &= \frac{n_A(n_A - 1)(n_A - 2)(3n_A - 1)}{24} \end{aligned} \tag{12}$$

Although Adams gives no formula for $k_A > 2$, (10), (11), and (12) suggest that tabulations of sums of products might be useful for an extension beyond k_A =

2. This is indeed the case as can be demonstrated in an example in which $n_A = 5$. Tabulations corresponding to the known formulas (10), (11), and (12) are listed below. For reasons given later, crossed-out dummy entries are included.

(13)

1	1	2	2	~~2 x 2~~	Vertical Sums ↓
		3	3	2 x 3	
		4	4	2 x 4	
				~~3 x 3~~	
				3 x 4	
				~~4 x 4~~	
$1 = C_0^5$	$1 + 9 = 10 = C_1^5$		$9 + 26 = 35 = C_2^5$		

Horizontal Sums →

Consider the possible extensions beyond (13) for $k_A = 3$ and $k_A = 4$ shown in (14).

(14)

~~2x2~~	~~2x2x2~~	~~2x2x2~~	Vertical Sums ↓
2x3	~~2x2x3~~	~~2x2x3~~	
2x4	~~2x2x4~~	~~2x2x4~~	
~~3x3~~	~~2x3x3~~	~~2x3x3~~	
3x4	2x3x4	2x3x4	
~~4x4~~	~~2x4x4~~	~~2x4x4~~	
	~~3x3x3~~	~~3x3x3~~	
	~~3x3x4~~	~~3x3x4~~	
	~~3x4x4~~	~~3x4x4~~	
	~~4x4x4~~	~~4x4x4~~	
$26 + 24 = 50 = C_3^5$		$24 = C_4^5$	

Horizontal Sums →

Again, note that the crossed-out entries do not contribute to a sum. The extensions exemplified by (14) yield the correct C_3^5 and C_4^5.

It is seen that exclusive of the initial "ones" (where present), the multiplication signs, the crossed-out lines, and the resultant summations, the tabulations of (13) and (14) are each a partition set of the type described earlier. Further, it is seen that only those entries with repeating members are crossed out. The success of (13) and (14) is not accidental. An investigation of the breakdown of First Kind Stirling numbers reveals that the pattern of (13) and (14) is general.

Exclusion of the crossed-out entries changes a partition set to one with non-repeating members. For identification, the designation changes to P_uV. One way of obtaining P_uV sets would be to generate PV sets and ignore repeating member partitions. This process is, of course, inefficient and can be circumvented as will be shown later.

For the example given, the following implications can be expressed:

$$
\begin{aligned}
&\{1\} \longrightarrow C_0^5 = 1 \\
&\{1,\ P_uV(\geq 2, \leq 4 | \geq 1, \leq 1 | \geq 2, \leq 4)\} \longrightarrow C_1^5 = 10 \\
(15)\qquad &\{1,\ P_uV(\geq 2, \leq 8 | \geq 1, \leq 2 | \geq 2, \leq 4)\} \longrightarrow C_2^5 = 35 \\
&\{0,\ P_uV(\geq 4, \leq 12 | \geq 2, \leq 3 | \geq 2, \leq 4)\} \longrightarrow C_3^5 = 50 \\
&\{0,\ P_uV(\geq 6, \leq 12 | \geq 3, \leq 3 | \geq 2, \leq 4)\} \longrightarrow C_4^5 = 24 \ .
\end{aligned}
$$

For the general case, the implication is that

$$
(16)\quad \left\{ \left[\frac{k_A + 3}{2k_A + 2}\right], P_uV\left(\geq 2\left(k_A - 1 + \left[\frac{k_A + 3}{2k_A + 2}\right]\right), \leq \left(k_A - \left[\frac{k_A}{n_A - 1}\right]\right)\cdot(n_A - 1) \middle| \geq k_A - 1 + \left[\frac{k_A + 3}{2k_A + 2}\right], \leq k_A - \left[\frac{k_A}{n_A - 1}\right] - \left[\frac{k_A}{n_A}\right] \middle| \leq 2, \leq (n_A - 1) \right) \right\} \longrightarrow C_{k_A}^{n_A}, \quad n_A > 0 \ .
$$

It can be observed from (16) that*

$$
C_{k_A}^{n_A}
$$

does not exist for $k_A \geq n_A$. The corresponding expression for Stirling numbers of the First Kind is found through application of (7) to (16) as

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*Brackets [ ] except where obviously used for references are used in the customary manner with real numbers to indicate the greatest integer less than or equal to the number bracketed. See Uspensky and Heaslet [5].

$$\left\{\left[\frac{n_R - k_R + 3}{2n_R - 2k_R + 2}\right], P_uV\left(\geq 2\left(n_R - k_R - 1 + \left[\frac{n_R - k_R + 3}{2n_R - 2k_R + 2}\right]\right), \leq \left(n_R - k_R -\right.\right.\right.$$

$$\left.\left.- \left[\frac{n_R - k_R}{n_R - 1}\right]\right)(n_R - 1) \,\middle|\, \geq n_R - k_R - 1 + \left[\frac{n_R - k_R + 3}{2n_R - 2k_R + 2}\right], \leq n_R - k_R - \right. \tag{17}$$

$$\left.\left. - \left[\frac{n_R - k_R}{n_R - 1}\right] - \left[\frac{n_R - k_R}{n_R}\right] \,\middle|\, \geq 2, \leq n_R - 1\right)\right\} \longrightarrow (-1)^{n_R + k_R} s(n_R, k_R) .$$

5. REDUCTION OF P_uV TO SIMPLER PV FORMS

As was indicated earlier, one way of obtaining the P_uV partitions is first to generate PV partitions and then to retain non-repeating member partitions. The repeating member partitions serve only as devices for successive generation of partitions. Equations (13) and (14) illustrate graphically the wastefulness of such a procedure. It is possible to generate simpler PV partitions which easily can be modified to yield the desired P_uV partitions. The method of doing this is described below. While this method applies particularly for the partitions of this paper and is not intended to be general, it has the computational feature of generating exactly as many PV partitions as are needed for conversion to P_uV partitions — no more!

A P_uV partition applicable for this paper can be expressed as

$$P_uV(\geq 2c, \leq ab \,|\, \geq c, \leq b \,|\, \geq 2, \leq a) \tag{18}$$

whether either $b = c$ alone or $b = c$ and $b = c + 1$, depending on whether the set (1 or 0, P_uV has one or two columns pf partitions. (See (15) for example). Assume that $b = c$. If the PV designation applied for (18), the largest (and last) b-member partition would total ab and would appear as b a's, $(a, a, \cdots, a)$. The u subscript, however, would not permit this partition, the closest approach being

$$(a - b + 1, \ a - b + 2, \ \cdots, \ a) \ .$$

However, $(a - b + 1, \ a - b + 2, \cdots, a)$ can be formed by member addition of

$$(a - b + 1,\ a - b + 1, \cdots, a - b + 1) \text{ and } (0, 1, 2, \cdots, b - 1)\ .$$

For a given b,

$$(a - b + 1,\ a - b + 1, \cdots, a - b + 1)$$

is an acceptable last partition in a one-partition column PV set and has a greatest member $a - b + 1$ and the sum $ab - b(b - 1)$. The lower limits of the new PV designation remain the same as in (18). Thus, a member-by-member addition of $(0, 1, 2, \cdots, b - 1)$ to the members of

(19) $$PV(\geq 2, \leq ab - b(b - 1) \,|\, \geq b, \leq b \,|\, \geq 2, \leq 2 - b + 1)$$

produces the desired form of (18) where $b = c$. For the case of two columns of partitions (i.e., $b = c$, $b = c + 1$),

(20) $$PV(\geq 2c, \leq ac - c(c - 1) \,|\, \geq c, \leq c \,|\, \geq 2, \leq a - c + 1)$$

is augmented by $(0, 1, 2, \cdots, c - 1)$ and

(21) $$PV(\geq 2(c + 1) \leq a(c + 1) - c(c + 1) \,|\, \geq c + 1, \leq c + 1 \,|\, \geq 2, \leq a - c)$$

is augmented by $(0, 1, 2, \cdots, c)$. An example for $a = 4$, $b = 3$, $c = 2$ follows.

(22)

$PV(\geq 4, \leq 6 \,|\, \geq 2, \leq 2 \,|\, \geq 2, \leq 3)$

2, 2
2, 3 + (0, 1) ⟶ (to P_uV column: 2, 3; 2, 4; 3, 4)
3, 3

$P_uV(\geq 4, \leq 12 \,|\, \geq 2, \leq 3 \,|\, \geq 2, \leq 4)$

2, 3	2, 3, 4
2, 4	
3, 4	

$PV(\geq 6, \leq 6 \,|\, \geq 3, \leq 3 \,|\, \geq 2, \leq 2)$

2, 2, 2 + (0, 1, 2) ⟶ (to 2, 3, 4)

Comparison of (22) with (14) shows the reduction in computation.

REFERENCES

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3. National Bureau of Standards, Handbook of Mathematical Functions, AMS 55, U. S. Government Printing Office, Washington, D. C., 1964, pp. 824-825.

4. D. C. Fielder, "Partition Enumeration by Means of Simpler Partitions," The Fibonacci Quarterly, Vol. 2, No. 2, April, 1964, pp. 115-118.

5. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill Book Co., New York, New York, 1939, pp. 94-99.

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ERRATA FOR FACTORIZATION OF 2 X 2 INTEGRAL MATRICES WITH DETERMINANT ±1

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Please make the following corrections to "Factorization of 2x2 Matrices with Determinant ±1," by Gene B. Gale, appearing in the February 1968 issue, Fibonacci Quarterly, pp. 3 — 22.

Page	Line	Reads	Should Read
5	6	$d < 0$	$d > 0$
5	-8	$c \leq d$	$c < d$
8	-3	$\|ru - \|st\|\|$	$\|\|ru\| - \|st\|\|$
9	5	$ad - bc \geq ad - cd = (a - c)d \geq 0$	$ad - bc > ad - cd = (a - c)d > 0$
9	-1	$\begin{pmatrix} a & r+1 \\ c & w \end{pmatrix}$	$\begin{pmatrix} a & r+1 \\ c & d \end{pmatrix}$
9	4	$cd \geq 0$	$c, d \geq 0$
{11, 12}	{-5, 3}	N	n
12	-6	$ar = (a - 1)(r - 1)$	$ar - (a - 1)(r - 1)$
15	6	$d(rF_k + sF_{k-1})$	$d\|(rF_k + sF_{k-1})$
16	-4	A_2B	A, B
17	-9	$\left\|\frac{ab - bc}{bd}\right\|$	$\left\|\frac{ad - bc}{bd}\right\|$

Continued on p. 112