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A Combinatorial Approach for q-Analogue of r-Stirling Numbers

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Abstract

We define a q-analogue of r-Stirling numbers of the second kind using their combinatorial interpretation in terms of set partition. Some properties are obtained including recurrence relation, explicit formula and certain symmetric formula. Moreover, a q-analogue of r-Stirling numbers of the first kind is introduced to obtain a q-analogue of the orthogonality and inverse relations of the two kinds of r-Stirling numbers.

Keywords: Stirling numbers; r-Stirling Numbers; q-binomial coefficients; q-factorial
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1 Introduction

Several generalizations of Stirling numbers have appeared in the literature. Almost all the generalizations of Stirling numbers have been listed in [1]. One of these is the r-Stirling numbers of the first and second kind in [2] which are defined, respectively, as follows

$\begin{bmatrix} n \\ k \end{bmatrix}_r$:= number of permutations of the set $\{1, 2, \cdots, n\}$ into k nonempty disjoint cycles, such that the numbers $1, 2, \cdots, r$ are in distinct cycles.

$\begin{Bmatrix} n \\ k \end{Bmatrix}_r$:= number of partitions of the set $\{1, 2, \cdots, n\}$ into k nonempty disjoint classes (or blocks), such that the numbers $1, 2, \cdots, r$ are in distinct classes (or blocks).

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Detailed discussion on r-Stirling numbers and some related works can be found in [2, 3, 4]. Recently, the r-Stirling numbers of the second kind have been generalized further in [5] by replacing the condition

the numbers $1, 2, \cdots, r$ are in distinct classes (or blocks)

with the condition

for given subsets $R_1, \ldots, R_r$ of $\{1, 2, \ldots, n\}$ where $|R_i| = r_i$ and $R_i \cap R_j = \emptyset$, for all $i, j = 1, \ldots, r$ $i \neq j$, the elements of each subsets R_i, $i = 1, \ldots, r$ are in distinct classes (or blocks).

This generalization of r-Stirling numbers of the second kind is called the $(r_1, \ldots, r_r)$-Stirling numbers of the second kind.

On the other hand, Certain generalization of Stirling numbers has been defined in [6] by considering the normal ordering of powers $(VU)^n$ of the noncommuting variables U and V satisfying $UV = VU + hV^s$ where $h \in \mathbb{C} - \{0\}$ and $s \in \mathbb{N}_0$. More precisely,

$$(VU)^n = \sum_{k=1}^{n} \mathfrak{S}_{s;h}(n,k) V^{s(nk)+k} U^k$$

where $\mathfrak{S}_{s;h}(n,k)$ denotes their generalized Stirling numbers. In [7], the numbers $\mathfrak{S}_{s;h}(n,k)$ were expressed in terms of the unified generalization of Stirling numbers in [1]. This result was used to derive more properties for $\mathfrak{S}_{s;h}(n,k)$. Further investigation of these numbers has been done in [8] by considering the particular case $s = 2$ corresponding to the meromorphic Weyl algebra.

One of the outgrowths in generalizing Stirling numbers is the introduction of their q-analogues. The study of q-analogue has become more popular nowadays due to its application in physics and other areas in mathematics, particularly, in the study of fractals, dynamical system, quantum groups, q-deformed superalgebras, fermionic oscillator, creation-annihilation principle and Ising model. There are two main classification of q-analogues: the combinatorial q-analogues and the q-analogues extended by F.H. Jackson [9]. This present study can be classified as part of combinatorial q-analogues.

A q-analogue of a number, polynomial, theorem, identity or expression is a generalization involving a new parameter q such that when $q \to 1$, it gives back the original number, polynomial, theorem, identity or expression. For instance, a given polynomial $a_k(q)$ is a q-analogue of an integer a_k if

$$\lim_{q \to 1} a_k(q) = a_k.$$

Hence, the polynomials

$$[n]_q = \frac{q^n - 1}{q - 1}, \quad [n]_q! = \prod_{i=1}^{n} [i]_q, \quad \binom{n}{k}_q = \prod_{i=1}^{k} \frac{q^{n-i+1} - 1}{q^i - 1}$$

are the q-analogues of the integers n, $n!$, and $\binom{n}{k}$. It is important to note that a q-analogue of a number, polynomial, theorem, identity or expression is not unique. For example, a q-analogue of the classical Stirling numbers has been defined by some authors in different manner (cf [10, 11]). In 1992, a new q-analogue of Stirling numbers has been defined by Cigler in [12] using the concept of set partitions (see also [13]). This is closely related to the q-Stirling numbers defined in [14] in three different ways using generating functions. This work of Cigler motivates the present authors to define a q-analogue of r-Stirling numbers of the second kind using their combinatorial interpretation in terms of set partitions. Moreover, a q-analogue of r-Stirling numbers of the first kind is defined by means of certain generating function, which, consequently, gives the orthogonality and inverse relations of the q-analogue of both kinds of r-Stirling numbers.

2 A *q*-Analogue of *r*-Stirling Numbers of the Second Kind

The classical Stirling numbers of the second kind $S(n,k)$ were defined in [15] as the cardinality of set B of partitions of $\{0,1,2,\cdots,n-1\}$ into k nonempty disjoint subsets. Based on this definition, a q-analogue of $S(n,k)$ was defined in [12] to be the following sum

$$\sum_{\pi\in B} w(\pi), \quad w(\pi) = q^{\sum_{i\in B_0} i}$$

where B_0 is a subset in partition π which contains 0.

On the other hand, the above definition of r-Stirling numbers of the second kind $\left\{ n \atop k \right\}_r$ can be restated as follows:

$\left\{ n \atop k \right\}_r$:= number of partitions π of $\{0,1,...,n-1\}$ into k nonempty subsets $B_0, B_1, ..., B_{k-1}$ such that the first r elements are in distinct subsets.

In this section, a q-analogue of r-Stirling numbers of the second kind will be defined parallel to the work of Cigler. First, we choose B_0 so that the number $0 \in B_0$. Then, let us define the following notations:

- the weight of partition π
$$w(\pi) = q^{s(B_0)}, \;\; s(B_0) = \sum_{i\in B_0} i.$$
- the weight of each set of partitions A
$$w(A) := \sum_{\pi\in A} w(\pi)$$
- $A_{n,k,r}$:= the set of all partitions of $0,1,...,n-1$ into k nonempty parts such that the first r elements are in distinct partitions.

Now, we have the following definition:

Definition 2.1. A q-analogue $\left\{ n \atop k \right\}_{q,r}$ of r-Stirling number of the second kind is defined by

$$\left\{ n \atop k \right\}_{q,r} := w(A_{n,k,r}) \quad n,k \geq 1, \;\; n \geq k \geq r$$

where $\left\{ 0 \atop k \right\}_{q,r} := \delta_{0k}$ and $\left\{ n \atop 0 \right\}_{q,r} := \delta_{0n}, \quad n,k \geq 0$

Remark 2.1. We choose the above weight function so that, when $q = 1$,

$$\left\{ n \atop k \right\}_{1,r} = |A_{n,k,r}| = \left\{ n \atop k \right\}_r .$$

Moreover, the above weight function is a kind of variation of the weight function corresponding to the q-Stirling numbers of the second kind in [11] resulting to a new q-analogue of second kind Stirling-type numbers. One may also try to define a q-analogue of r-Stirling numbers of the second kind using the weight function in terms of non-inversion numbers.

When $n = 4, k = 3$ and $r = 2$, we have the following partitions of $\{0, 1, 2, 3\}$:

$$A_{4,3,2} = \{\{0\}\{1\}\{2,3\}\}, \{\{0,2\}\{1\}\{3\}\}, \{\{0\}\{1,2\}\{3\}\}, \{\{0,3\}\{1\}\{2\}\}, \{\{0\}\{1,3\}\{2\}\}.$$

Then

$$\left\{ {4 \atop 3} \right\}_{q,2} = q^0 + q^{0+2} + q^0 + q^{0+3} + q^0 = 3 + q^2 + q^3.$$

To compute quickly the first values of the q-analogue, let us consider the following recurrence relation:

Theorem 2.1. *The number* $\left\{ {n \atop k} \right\}_{q,r}$ *satisfy the following recurrence relation*

$$\left\{ {n+1 \atop k} \right\}_{q,r} = \left\{ {n \atop k-1} \right\}_{q,r} + (k - 1 + q^n) \left\{ {n \atop k} \right\}_{q,r}$$

where $n \geq k \geq r \geq 0$.

Proof. We write $A_{n+1,k,r} = C_1 \cup C_2 \cup C_3$ such that

- C_1 is the set of all $\pi \in A_{n+1,k,r}$ such that $\{n\}$ is one of the nonempty parts of π.
- C_2 is the set of all π such that $n \in B_i$, $i \neq 0$, and $B_i \backslash \{n\} \neq \phi$.
- C_3 is the set of all π such that $n \in B_0$.

Then we have

$$w(C_1) = \left\{ {n \atop k-1} \right\}_{q,r}, \ w(C_2) = (k-1) \left\{ {n \atop k} \right\}_{q,r}, \text{ and } w(C_3) = q^n \left\{ {n \atop k} \right\}_{q,r}.$$

□

Using this recurrence relation, we can generate the first values of the q-analogue.

The next theorem contains an explicit formula for $\left\{ {n \atop k} \right\}_{q,r}$, which is analogous to certain identity in [2]. But before that, let us consider first the following lemma.

Lemma 2.2.

$$\sum_{r \leq j_1 < j_2 < \cdots < j_i \leq n} q^{j_1 + j_2 + \cdots + j_i} = \binom{n - r + 1}{i}_q q^{\frac{i(i-1+2r)}{2}}.$$

Proof. Note that from [16]

$$(a + x)(a + qx) \cdots (a + q^{n-r}x) = \sum_{i=0}^{n-r+1} \binom{n-r+1}{i}_q q^{\binom{i}{2}} x^i a^{n-r+1-i}$$

Replacing x by $q^r x$, we have

$$\begin{aligned}(a+q^r x)(a+q^{r+1}x)\cdots(a+q^n x) &= \sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\binom{i}{2}} q^{ri} x^i a^{n-r+1-i} \\ &= \sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\frac{i(i-1)}{2}} q^{ri} x^i a^{n-r+1-i} \\ &= \sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}} x^i a^{n-r+1-i}\end{aligned}$$

And comparing the coefficients of x^i at $a=1$ gives

$$\sum_{i=0}^{n-r+1}\left(\sum_{r\le j_1<j_2<\cdots<j_i\le n} q^{j_1+j_2+\cdots+j_i}\right)x^i = \sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}} x^i$$

$$\sum_{r\le j_1<j_2<\cdots<j_i\le n} q^{j_1+j_2+\cdots+j_i} = \binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}}$$

□

Writing $\pi \in A_{n+1,k+1}$ in the form

$$\pi = \{0, j_1, j_2, \cdots, j_i\}/B_1/\cdots/B_k$$

where $j_l \neq 1, 2, \ldots, r-1$, we get therefore

$$\begin{aligned}\left\{ {n+1 \atop k+1} \right\}_{q,r} &= w(A_{n+1,k+1}) = \sum_{\pi\in A_{n+1,k+1}} w(\pi) \\ &= \sum_{i=0}^{n}\sum_{r\le j_1<\cdots<j_i\le n} q^{j_1+\cdots+j_i}\left\{ {n-i \atop k} \right\}_{r-1}\end{aligned}$$

Thus, using Lemma 2.2, we obtain the following explicit formula.

Theorem 2.3. *The explicit formula for* $\left\{ {n+1 \atop k+1} \right\}_{q,r}$ *is given by*

$$\left\{ {n+1 \atop k+1} \right\}_{q,r} = \sum_{i=0}^{n}\binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}}\left\{ {n-i \atop k} \right\}_{r-1}. \tag{2.1}$$

Remark 2.2. Equation (2.1) is a q-analogue of the identity in [2], which is given by

$$\left\{ {n \atop m} \right\}_r = \sum_k \binom{n-r}{k}\left\{ {n-p-k \atop m-p} \right\}_{r-p} p^k,$$

when $p=1$.

Remark 2.3. The r-Stirling numbers of the second kind in [2] satisfy the following exponential generating function

$$\sum_{k\geq 0}\left\{{k+r \atop m+r}\right\}_r \frac{z^k}{k!}=\frac{1}{m!}e^{rz}(e^z-1)^m.$$

Using the Binomial Theorem and the expansion of exponential function, this can be expressed further as

$$\sum_{k\geq 0}\left\{{k+r \atop m+r}\right\}_r \frac{z^k}{k!}=\sum_{k\geq 0}\left\{\frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(r+j)^k\right\}\frac{z^k}{k!}.$$

This implies that

$$\left\{{k+r \atop m+r}\right\}_r=\frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(r+j)^k. \tag{2.2}$$

This formula can also be obtained via (r,β)-Stirling numbers $\left\langle {k \atop m}\right\rangle_{\beta,r}$ in [17] by taking $\beta=1$. That is,

$$\left\{{k+r \atop m+r}\right\}_r=\frac{1}{m!}\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(r+j)^k=\left\langle {k \atop m}\right\rangle_{1,r}. \tag{2.3}$$

Thus, using (2.2), the explicit formula in (2.1) can be rewritten as

$$\begin{aligned}\left\{{n+1 \atop k+1}\right\}_{q,r} &= \sum_{i=0}^{n}\sum_{j=0}^{k-r+1}\frac{(-1)^{k-r+1-j}\binom{k-r+1}{j}\binom{n-r+1}{i}_q q^{\frac{i(i-1+2r)}{2}}(r-1+j)^{n-r+1-i}}{(k-r+1)!}\\ &= \frac{1}{(k-r+1)!}\sum_{j=0}^{k-r+1}(-1)^{k-r+1-j}\binom{k-r+1}{j}(r-1+j)^{n-r+1}\times\\ &\quad\times\left\{\sum_{i=0}^{n-r+1}\binom{n-r+1}{i}_q q^{\frac{i(i-1)}{2}}\left(\frac{q^r}{r-1+j}\right)^i\right\}.\end{aligned}$$

Applying a q-identity in [15], which is given by

$$\sum_{i=0}^{n}\binom{n}{i}_q q^{\frac{i(i-1)}{2}}x^i=\prod_{i=0}^{n-1}\left(1+xq^i\right),$$

we obtain

$$\left\{{n+1 \atop k+1}\right\}_{q,r}=\frac{1}{(k-r+1)!}\sum_{j=0}^{k-r+1}(-1)^{k-r+1-j}\binom{k-r+1}{j}\prod_{i=0}^{n-r}(r-1+j+q^{r+i}). \tag{2.4}$$

This identity is a kind of q-analogue of that identity in (2.3) since, when $q=1$, (2.4) reduces immediately to (2.3).

The next theorem contains a symmetric formula for $\left\{{n \atop k}\right\}_{q,r}$ which is analogous to the horizontal generating function of Stirling numbers of the second kind.

Theorem 2.4. *A q-analogue of r-Stirling numbers of the second kind satisties the following relation*

$$\sum_{k=0}^{n-1}\left\{{n \atop k+1}\right\}_{q,r}(x-r+1)^{\underline{k-r+1}}=(x+q^r)(x+q^{r+1})\cdots(x+q^{n-1}).$$

Proof. From the well-known formula

$$\frac{\Delta^k}{k!}(x+r)^n|_{x=0} = \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r ,$$

we get

$$\begin{aligned} \begin{Bmatrix} n \\ k+1 \end{Bmatrix}_{q,r} &= \sum_{i=0}^{n-1} \binom{n-r}{i}_q q^{\frac{i(i-1+2r)}{2}} \begin{Bmatrix} n-i-1 \\ k \end{Bmatrix}_{r-1} \\ &= \frac{\Delta^{k-r+1}}{(k-r+1)!} \sum_{i=0}^{n-1} \binom{n-r}{i}_q q^{\frac{i(i-1+2r)}{2}} (x+r-1)^{n-r-i}|_{x=0} \\ &= \frac{\Delta^{k-r+1}}{(k-r+1)!} \sum_{i=0}^{n-r} \binom{n-r}{i}_q q^{\frac{i(i-1+2r)}{2}} (x+r-1)^{n-i-r}|_{x=0}. \end{aligned}$$

It is known that, for a positive integer n, a real number $q \neq 1$, and an indeterminate z, we have

$$\prod_{i=1}^{n}(a+q^{i-1}z) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{2}} z^k a^{n-k}.$$

With $z = q^r$ and $a = x + r$, we obtain

$$\begin{Bmatrix} n \\ k+1 \end{Bmatrix}_{q,r} = \frac{\Delta^{k-r+1}}{(k-r+1)!}(q^r+x+r-1)(q^{r+1}+x+r-1)\cdots(q^{n-1}+x+r-1)|_{x=0}.$$

The well-known formula for higher order difference operator yields

$$\sum_{k=0}^{n-1} \begin{Bmatrix} n \\ k+1 \end{Bmatrix}_{q,r} (x-r+1)^{\underline{k-r+1}} = \sum_{k=0}^{n-1} \left\{ \frac{\Delta^{k-r+1}}{(k-r+1)!} \prod_{j=r}^{n-1}(q^j+x+r-1)|_{x=0} \right\} (x-r+1)^{\underline{k-r+1}}$$

$$\begin{aligned} &= \sum_{k=0}^{n-1} \frac{1}{(k-r+1)!} \left\{ \sum_{j=0}^{k-r+1} (-1)^{k-r+1-j} \binom{k-r+1}{j} \times . \right. \\ &\quad \left. . \prod_{l=r}^{n-1}(q^l+r+j-1) \right\} (x-r+1)^{\underline{k-r+1}} \\ &= \sum_{k=0}^{n-1} \frac{1}{(k-r+1)!} \left\{ \sum_{j=0}^{k-r+1} (-1)^{k-r+1-j} \binom{k-r+1}{j} \times . \right. \\ &\quad \left. . \left\{ \sum_{i=0}^{n-r} \sum_{r\le i_1<i_2<\cdots<i_i\le n-1} q^{i_1+i_2+\cdots+i_i}(r+j-1)^{n-r-i} \right\} \right\} (x-r+1)^{\underline{k-r+1}} \end{aligned}$$

$$\begin{aligned} &= \sum_{k=0}^{n-1} \frac{1}{(k-r+1)!} \sum_{i=0}^{n-r} \sum_{r\le i_1<\cdots<i_i\le n-1} q^{i_1+\cdots+i_i} \times \\ &\quad \left\{ \sum_{j=0}^{k-r+1} (-1)^{k-r+1-j} \binom{k-r+1}{j} (r+j-1)^{n-r-i} \right\} (x-r+1)^{\underline{k-r+1}}. \end{aligned}$$

Using the explicit formula for (r,β)-Stirling numbers in (2.3) which also appears in [19], we have

$$\sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_{q,r} (x-r+1)^{\underline{k-r+1}} = \sum_{i=0}^{n-r} \sum_{r\le i_1<i_2<\cdots<i_i\le n-1} q^{i_1+i_2+\cdots+i_i} \left\{ \sum_{k=0}^{n-1} \left\langle {n-r-i \atop k-r+1} \right\rangle_{1,r-1} (x-r+1)^{\underline{k-r+1}} \right\}.$$

A relation in [19] implies that

$$\begin{aligned}\sum_{k=0}^{n-1} \left\{ {n \atop k+1} \right\}_{q,r} (x-r+1)^{\underline{k-r+1}} &= \sum_{i=0}^{n-r} \sum_{r\le i_1<i_2<\cdots<i_i\le n-1} q^{i_1+i_2+\cdots+i_i} x^{n-r-i} \\ &= (x+q^r)(x+q^{r+1})\cdots(x+q^{n-1}).\end{aligned}$$

□

For example, when $n=4$ and $r=2$, we have

$$\begin{aligned}\sum_{k=0}^{3} \left\{ {4 \atop k+1} \right\}_{q,2} (x-1)^{\underline{k-1}} &= \left\{ {4 \atop 2} \right\}_{q,2} + \left\{ {4 \atop 3} \right\}_{q,2} (x-1) + \left\{ {4 \atop 4} \right\}_{q,2} (x-1)(x-2) \\ &= (1+q^2+q^3+q^5) + (3+q^2+q^3)(x-1) + (x-1)(x-2) \\ &= x^2+q^2x+q^3x+q^5 = (x+q^2)(x+q^3).\end{aligned}$$

It is worth mentioning that certain generalization of Bell numbers, called r-Bell numbers, has been investigated in [18] resulting to several interesting properties of these numbers. These numbers were first defined in [19] as the sum of r-Stirling numbers of the second kind. It is then interesting to define a q-analogue of r-Bell numbers in terms of the above q-analogue of r-Stirling numbers of the second kind and establish some properties analogous to those obtained in [18] for r-Bell numbers.

3 A *q*-Analogue of *r*-Stirling Numbers of the First Kind

It is known that the classical Stirling numbers satisfy the following inverse relation

$$f_n = \sum_{k=0}^{n} \left[{n \atop k} \right] g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\} f_k. \tag{3.1}$$

This inverse relation can be obtained using the following generating functions

$$\begin{aligned} x^{\underline{n}} &= \sum_{k=0}^{n} \left[{n \atop k} \right] x^k \\ x^n &= \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}. \end{aligned}$$

This motivates the authors to define a q-analogue of r-Stirling numbers of the first kind as follows:

Definition 3.1. A q-analogue of r-Stirling number of the first kind is defined by

$$(x-r+1)^{\underline{n-r}} = \sum_{k=0}^{n} \left[{n \atop k} \right]_{q,r} (-1)^{n-k} (x+q^r)(x+q^{r+1})\dots(x+q^{k-1}) \tag{3.2}$$

with $r \leq k-1$. By convention, $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r} = 1$ when $r = k$ and $n \geq k$, $\begin{bmatrix} n \\ 0 \end{bmatrix}_{q,r} = 1$ when $n = 0$, $\begin{bmatrix} n \\ 0 \end{bmatrix}_{q,r} = 0$ when $n > 0$ and $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r} = 0$ when $n < k$ or $n, k < 0$.

Using the relation in Theorem 2.4, we have

$$\begin{aligned}(x-r+1)^{\underline{n-r}} &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} (-1)^{n-k} \sum_{m=1}^{k} \begin{Bmatrix} k \\ m \end{Bmatrix}_{q,r} (x-r+1)^{\underline{m-r}} \\ &= \sum_{m=1}^{n} \left\{ \sum_{k=m}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} \begin{Bmatrix} k \\ m \end{Bmatrix}_{q,r} \right\} (x-r+1)^{\underline{m-r}}.\end{aligned}$$

Comparing the coefficients of $(x-r+1)^{\underline{n-r}}$, we obtain

$$\sum_{k=m}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} \begin{Bmatrix} k \\ m \end{Bmatrix}_{q,r} = \delta_{mn}$$

where δ_{mn} is the Kronecker delta. On the other hand, the relation in Theorem 2.4 can be written as

$$\begin{aligned}(x+q^r)\cdots(x+q^{n-1}) &= \sum_{k=1}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} \sum_{m=0}^{k} \begin{bmatrix} k \\ m \end{bmatrix}_{q,r} (-1)^{k-m} (x+q^r)\dots(x+q^{m-1}) \\ &= \sum_{m=1}^{k} \left\{ \sum_{k=m}^{n} (-1)^{k-m} \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} \begin{bmatrix} k \\ m \end{bmatrix}_{q,r} \right\} (x+q^r)\dots(x+q^{m-1}).\end{aligned}$$

Thus, we can state formally these results in the following theorem.

Theorem 3.1. *The q-analogue of r-Stirling numbers of the first kind satisties the following orthogonality relations*

$$\begin{aligned}\sum_{k=m}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} \begin{Bmatrix} k \\ m \end{Bmatrix}_{q,r} &= \delta_{mn} \\ \sum_{k=m}^{n} (-1)^{k-m} \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} \begin{bmatrix} k \\ m \end{bmatrix}_{q,r} &= \delta_{mn}.\end{aligned}$$

Remark 3.1. This theorem immediately implies that

$$\left((-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix}_{q,r} \right)_{0 \leq i,j \leq n} \left(\begin{Bmatrix} i \\ j \end{Bmatrix}_{q,r} \right)^{T}_{0 \leq i,j \leq n} = I_{n+1}$$

where I_{n+1} is the identity matrix of order $n+1$. That is,

$$\left((-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix}_{q,r} \right)^{-1}_{0 \leq i,j \leq n} = \left(\begin{Bmatrix} i \\ j \end{Bmatrix}_{q,r} \right)^{T}_{0 \leq i,j \leq n}$$

and

$$\det \left[\left((-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix}_{q,r} \right)_{0 \leq i,j \leq n} \left(\begin{Bmatrix} i \\ j \end{Bmatrix}_{q,r} \right)^{T}_{0 \leq i,j \leq n} \right] = 1.$$

As a direct consequence of this theorem, we have the following inverse relations of q-analogue of r-Stirling numbers.

Theorem 3.2. *The q-analogue of r-Stirling numbers of the first kind satisties the following inverse relations*

$$f_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} g_k \iff g_n = \sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} f_k$$
$$f_k = \sum_{n=0}^{\infty} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} g_n \iff g_k = \sum_{n=0}^{\infty} \begin{Bmatrix} n \\ k \end{Bmatrix}_{q,r} f_n.$$

For quick computation of the first values of $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}$, we need the following triangular recurrence relation.

Theorem 3.3. *The q-analogue of r-Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}$ satisfies*

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_{q,r} = \begin{bmatrix} n \\ k-1 \end{bmatrix}_{q,r} + (n-1+q^k) \begin{bmatrix} n \\ k \end{bmatrix}_{q,r}. \tag{3.3}$$

Proof. Equation (3.2) implies that

$$\sum_{k=0}^{n+1} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{q,r} (-1)^{n+1-k}(x+q^r)(x+q^{r+1})\ldots(x+q^{k-1}) = (x-r+1-n+r)(x-r+1)^{\underline{n-r}}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} (-1)^{n-k}(x+q^k-q^k+1-n)(x+q^r)(x+q^{r+1})\ldots(x+q^{k-1})$$

$$\begin{aligned} = & \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} (-1)^{n-k}(x+q^r)(x+q^{r+1})\ldots(x+q^{k-1})(x+q^k) \\ & + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} (-1)^{n-k}(-q^k+1-n)(x+q^r)(x+q^{r+1})\ldots(x+q^{k-1}) \\ = & \sum_{k=0}^{n+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{q,r} (-1)^{n+1-k}(x+q^r)(x+q^{r+1})\ldots(x+q^{k-1}) \\ & + \sum_{k=0}^{n} (q^k-1+n) \begin{bmatrix} n \\ k \end{bmatrix}_{q,r} (-1)^{n+1-k}(x+q^r)(x+q^{r+1})\ldots(x+q^{k-1}) \end{aligned}$$

By comparing the coefficients, we obtain the desired recurrence relation. □

We observe that the q-Stirling numbers $\begin{bmatrix} n \\ k \end{bmatrix}_q^*$ in [12] satisfy the relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^* = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^* + (n-1+q^k) \begin{bmatrix} n \\ k \end{bmatrix}_q^*$$

which is analogous to the recurrence relation in Theorem 3.3. This recurrence relation has been used to give combinatorial interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q^*$ in terms of the weight of permutations in $\{1, 2, \ldots, n\}$ with k nonempty cycles. Hence, we can also use the recurrence relation in Theorem 3.3 to give combinatorial interpretation for $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}$ by following the same argument in constructing the combinatorial interpretation of $\begin{bmatrix} n \\ k \end{bmatrix}_q^*$.

To sketch the construction, first, we let $\mathcal{P}_n$ be the set of all permutations of $\{1, 2, \ldots, n\}$, $\mathcal{P}_{n,r}$ be the set of all permutations of $\{1, 2, \ldots, n\}$ such that elements $1, 2, \ldots, r$ are in different cycles and $w(\pi)$ be the weight of $\pi \in \mathcal{P}_n$. As defined in [12], the decomposition into nonempty cycles $C_0, C_1, \ldots, C_{k-1}$ of a permutation $\pi \in \mathcal{P}_n$ is called a *natural decomposition* if the ordering is according to decreasing largest elements of the cycles, the natural ordering. Since $max(C_0) = n$, the natural decomposition of C_0 is given by $\{n\}, C_{01}, C_{02}, \ldots, C_{0i}$. Also, in [12], for $\pi = [C_{01}|C_{02}|\ldots|C_{0i}|n]C_1|C_2|\ldots|C_{k-1} \in \mathcal{P}_n$, we define

$$w(\pi) := q^{j_1+j_2+\ldots+j_i}$$

where $j_l = m$ if C_{0l} lies between C_{m-1} and C_m in the natural ordering of cycles and $j_l = k$ if $max(C_{0l}) < max(C_{k-1})$. Then the q-analogue $\begin{bmatrix} n \\ k \end{bmatrix}_{q,r}$ of r-Stirling numbers of the first kind can be interpreted as the sum of the weights of all permutations $\pi \in \mathcal{P}_{n,r}$ such that the natural decomposition has exactly k cycles.

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Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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