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Some Theorems on the q-Analogue of the Generalized Stirling Numbers

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Abstract. In this paper, we establish more properties for the q-analogue of the unified generalization of Stirling numbers including the vertical and horizontal recurrence relations, and the rational generating function. This generating function plays an important role in deriving one of the explicit formulas in symmetric function form which will be used in giving combinatorial interpretations of the q-analogue in the context of 0-1 tableau. Moreover, using the combinatorics of 0-1 tableaux, we obtain certain generalization of Carlitz identity.

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1. Introduction

The unified generalization of Stirling numbers given by Hsu and Shuie [16] is a Stirling-type number pair

$$\{S^1(n,k), S^2(n,k)\} \equiv \{S(n,k;\alpha,\beta,\gamma), S(n,k;\beta,\alpha,-\gamma)\}$$

defined by the inverse relation

$$(t|\alpha)_n = \sum_{k=0}^{n} S^1(n,k)(t-\gamma|\beta)_k$$
$$(t|\beta)_n = \sum_{k=0}^{n} S^2(n,k)(t+\gamma|\alpha)_k$$

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where $n \in N$ (the set of nonnegative integers), α, β, and γ may be real or complex numbers with $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, and

$$(t|\alpha)_n = t(t-\alpha)(t-2\alpha)\ldots(t-(n-1)\alpha)$$

the *generalized factorial of* t *of degree* n *with increment* α. For simplicity, we can use $S(n,k;\alpha,\beta,\gamma)$ to denote the *generalized Stirling numbers*. All Stirling-type number pairs may be expressed in terms of $S(n,k;\alpha,\beta,\gamma)$ with suitable choices of α, β, and γ (c.f. [1, 2, 4–7, 9, 15–17, 19, 20]). Other properties and combinatorial applications were thoroughly discussed in [10] and [12]

The *exponential-type Stirling number pair* was defined by Corcino *et al.* [11] by means of the following relations

$$[t|a]_n = \sum_{k=0}^{n} S^1[n,k][t-c|b]_k \tag{1.1}$$

$$[t|b]_n = \sum_{k=0}^{n} S^2[n,k][t+c|a]_k \tag{1.2}$$

where a, b, and c may be real or complex parameters with

$$[t|a]_n = \prod_{j=0}^{n-1}(t-a^j), \quad [t|a]_0 = 1, \quad [t|a]_1 = t-1.$$

More precisely, $S^1[n,k]$ and $S^2[n,k]$ may be denoted by

$$S^1[n,k] = S[n,k;a,b,c], \quad S^2[n,k] = S[n,k;b,a,-c].$$

Several properties of these numbers were obtained including orthogonality and inverse relations, triangular recurrence relation, horizontal recurrence relation, and explicit formula.

It is known that a given polynomial $a_k(q)$ is a q-analogue of an integer a_k if

$$\lim_{q\to1} a_k(q) = a_k.$$

For example, the polynomials

$$[n]_q = \frac{q^n-1}{q-1}, \quad q \neq 1$$

$$[n]_q! = [n]_q[n-1]_q[n-2]_q \ldots [2]_q[1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{n-i+1}-1}{q^i-1} = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

are the q-analogues of the integers n, $n!$ and $\binom{n}{k}$, respectively, since

$$\lim_{q\to1} [n]_q = n, \quad \lim_{q\to1} [n]_q! = n! \quad \text{and} \quad \lim_{q\to1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

Throughout this paper we use $[n]$ to denote $[n]_q$.

The q-analogue $\{\sigma^1[n,k], \sigma^2[n,k]\}$ of the pair of generalized Stirling numbers was then defined in terms of the exponential-type Stirling numbers as follows

$$\sigma^1[n,k] \equiv \sigma^1[n,k;\alpha,\beta,\gamma]_q := S[n,k;q^\alpha,q^\beta,q^\gamma-1](q-1)^{k-n},$$

$$\sigma^2[n,k] \equiv \sigma^2[n,k;\alpha,\beta,\gamma]_q := S[n,k;q^\beta,q^\alpha,1-q^\gamma](q-1)^{k-n}.$$

It was verified in [11] that the following limit relations hold

$$\lim_{q\to 1} \sigma^1[n,k;\alpha,\beta,\gamma]_q = S^1(n,k)$$
$$\lim_{q\to 1} \sigma^2[n,k;\alpha,\beta,\gamma]_q = S^2(n,k).$$

These justify the fact that $\{\sigma^1[n,k],\sigma^2[n,k]\}$ is a proper q-analogue of the pair of generalized Stirling numbers $\{S^1(n,k),S^2(n,k)\}$. To obtain properties for these proper q-analogue, we simply multiply both sides of each identity for the exponential-type Stirling numbers with an appropriate power of $q-1$. For instance, the explicit formula for $\sigma^1[n,k]$ which is given by

$$(1.3)\qquad \sigma^1[n,k;\alpha,\beta,\gamma]_q = \left(\prod_{i=1}^{k}[i\beta]\right)^{-1} \sum_{j=0}^{k}(-1)^{k-j} b^{<k|j>} \begin{bmatrix} k \\ j \end{bmatrix}_b [[j\beta]+[\gamma]|[\alpha]]_n\,,$$

with $\langle k|j\rangle = \binom{j+1}{2} - kj$, $q \neq 1$ and

$$[[j\beta]+[\gamma]|[\alpha]]_n = \prod_{l=0}^{n-1}([j\beta]+[\gamma]-[l\alpha]),$$

can easily be obtained by multiplying both sides of the explicit formula of the exponential-type Stirling numbers in [11]

$$(1.4)\qquad S[n,k;a,b,c] = \prod_{i=1}^{k}(b^i-1)^{-1} \sum_{j=0}^{k}(-1)^{k-j} b^{<k|j>} \begin{bmatrix} k \\ j \end{bmatrix}_b [b^j + c|a]_n, \qquad b \neq 1$$

by $(q-1)^{k-n}$ with $a = q^\alpha$, $b = q^\beta$, and $c = q^\gamma - 1$. Note that when $\alpha = 0, \beta = 1, \gamma = 0$, (1.3) yields

$$(1.5)\qquad \sigma^1[n,k;0,1,0]_q = \frac{1}{[k]!}\sum_{j=0}^{k}(-1)^{k-j} q^{<k|j>} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]^n = S_q[n,k]$$

which is the ordinary q-Stirling numbers of the second kind due to Carlitz [3]. This further gives the explicit form of the second kind of Stirling numbers as a limiting case:

$$\lim_{q\to 1} \sigma[n,k;0,1,0]_q = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n = S(n,k;0,1,0) = S(n,k).$$

Moreover, the triangular recurrence relation

$$(1.6)\qquad \sigma^1[n,k] = \sigma^1[n-1,k-1] + ([k\beta]-[(n-1)\alpha]+[\gamma])\sigma^1[n-1,k]$$

can also be obtained by multiplying both sides of the triangular recurrence relation of the exponential-type Stirling numbers in [11]

$$S[n,k;a,b,c] = S[n-1,k-1;a,b,c] + (b^k - a^{n-1} + c)S[n-1,k;a,b,c]$$

by $(q-1)^{k-n}$ with $a = q^\alpha$, $b = q^\beta$, and $c = q^\gamma - 1$. When $\alpha = -1$, $\beta = 0$ and $\gamma = 0$, (1.6) yields

$$\sigma^1[n,k;1,0,0] = \sigma^1[n-1,k-1;1,0,0] - [n-1]\sigma^1[n-1,k;1,0,0]$$

which gives the triangular recurrence relation of the q-Stirling numbers of the first kind $c_q[n,k]$ (see [14, 18])

$$c_q[n,k] = c_q[n-1,k-1] + [n-1]c_q[n-1,k]$$

with $c_q[n,k] = (-1)^{n-k}\sigma^1[n,k;1,0,0]$, and when $q \to 1$, this further gives the triangular recurrence relation of the Stirling numbers of the first kind

$$s(n,k) = s(n-1,k-1) + (n-1)s(n-1,k).$$

On the other hand, when $\alpha = 0$, $\beta = 1$, $\gamma = 0$, (1.6) yields the triangular recurrence relation of the q-Stirling numbers of the second kind $S_q[n,k]$ (see [3, 14, 18])

$$S_q[n,k] = S_q[n-1,k-1] + [k]S_q[n-1,k]$$

and when $q \to 1$, this further gives the triangular recurrence relation of the Stirling numbers of the second kind

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

The theory of q-analogues of generalized Stirling numbers needs to be enriched so that some possible combinatorial interpretations and applications can easily be drawn. In this paper, we establish another two types of recurrence relations - the vertical and horizontal recurrence relations, and obtain rational generating function for $S^1[n,k]$ when $a = 1$ which will be used to derive an explicit formula in a homogeneous symmetric function form of degree n. Moreover, using (1.1), we obtain another explicit formula for $S^1[n,k]$ when $b = 1$ in an elementary symmetric function form of degree n. These two explicit formulas are necessary in giving combinatorial interpretations for $\sigma^1[n,k;0,\beta,\gamma]_q$ and $\sigma^1[n,k;\alpha,0,\gamma]_q$ in the context of 0-1 tableau. Moreover, using the combinatorics of 0-1 tableaux, we obtain certain formula which is a kind of generalization of Carlitz identity (used to express the q-binomial coefficients in terms of q-Stirling numbers [3]).

2. Recurrence relations and explicit formulas

We know that recurrence relations help us generate quickly the first values of a number. There are three different types of recurrence relations that we can use to construct the table of values of a number: the triangular recurrence relation, the horizontal recurrence relation, and the vertical recurrence relation. In [11], the first two types have already been given to the q-analogue $\sigma^1[n,k]$. Here, we obtain another form of horizontal recurrence relation and a vertical recurrence relation using the triangular recurrence relation for $\sigma^1[n,k]$.

Theorem 2.1. *For nonnegative integers n and k, and complex numbers α, β, and γ, the q-analogues $\sigma^1[n+1,k+1]$ of the generalized Stirling numbers satisfy the following vertical and horizontal recurrence relations*

(i)

$$\sigma^1[n+1,k+1] = \sum_{j=k}^{n} \frac{[[\beta(k+1)] + [\gamma]|[\alpha]]_{n+1}}{[[\beta(k+1)] + [\gamma]|[\alpha]]_{j+1}} \sigma^1[j,k]$$

(ii)

$$\sigma^1[n,k] = \sum_{j=0}^{n-k} (-1)^j \frac{\{[\gamma]-[n\alpha]|[\beta]\}_{k+j+1}}{\{[\gamma]-[n\alpha]|[\beta]\}_{k+1}} \sigma^1[n+1,k+j+1]$$

where

$$\{[t]|[a]\}_n = \prod_{j=0}^{n-1}([t]+[ja])$$

with initial values $\sigma^1[0,0]=1$ *and* $\sigma^1[k,k]=1,\ k\geq 1.$

Proof. By repeated application of the triangular recurrence relation in (1.6), we have

$$\begin{aligned}
&\sigma^1[n+1,k+1]\\
&= \sigma^1[n,k] + ([(k+1)\beta]-[n\alpha]+[\gamma])\sigma^1[n-1,k]\\
&\quad + ([(k+1)\beta]-[n\alpha]+[\gamma])([(k+1)\beta]-[(n-1)\alpha]+[\gamma])\sigma^1[n-2,k]\\
&\quad + \ldots + ([(k+1)\beta]-[n\alpha]+[\gamma])\cdot([(k+1)\beta]-[(n-1)\alpha]+[\gamma])\\
&\quad \cdot \ldots \cdot ([(k+1)\beta]-[(k+2)\alpha]+[\gamma])\cdot \sigma^1[k+2,k+1].
\end{aligned}$$

Again, by applying the triangular recurrence relation to $\sigma^1[k+2,k+1]$ with $\sigma^1[k+1,k+1]=\sigma^1[k,k]$, consequently we obtain the vertical recurrence relation (i). For (ii), we evaluate its right-hand side (RHS) using (1.6) and obtain

$$\begin{aligned}
RHS &= \sum_{j=0}^{n-k}(-1)^j \frac{\{[\gamma]-[n\alpha]|[\beta]\}_{k+j+2}}{\{[\gamma]-[n\alpha]|[\beta]\}_{k+1}}\sigma^1[n,k+j+1]\\
&\quad + \sum_{j=0}^{n-k}(-1)^j \frac{\{[\gamma]-[n\alpha]|[\beta]\}_{k+j+1}}{\{[\gamma]-[n\alpha]|[\beta]\}_{k+1}}\sigma^1[n,k+j].
\end{aligned}$$

Reindexing the first sum, we get

$$\begin{aligned}
RHS &= \sum_{j=1}^{n-k}(-1)^{j-1} \frac{\{[\gamma]-[n\alpha]|[\beta]\}_{k+j+1}}{\{[\gamma]-[n\alpha]|[\beta]\}_{k+1}}\sigma^1[n,k+j]\\
&\quad + \sum_{j=1}^{n-k}(-1)^j \frac{\{[\gamma]-[n\alpha]|[\beta]\}_{k+j+1}}{\{[\gamma]-[n\alpha]|[\beta]\}_{k+1}}\sigma^1[n,k+j] \quad + \quad \sigma^1[n,k]\\
&= \sigma^1[n,k].
\end{aligned}$$

∎

We notice that the recurrence relations in Theorems 2.1 are analogous to the Chu Shih-Chieh's identity of the binomial coefficients also known as the Hockey Stick identity [8]. We may easily remember these recurrence relations with the help of the patterns as shown in the following table.

n / k	$\dots$	k	$k+1$	$k+2$	$\dots$	$n+1$
$\vdots$		$\downarrow \quad \downarrow$ $\downarrow \quad \downarrow$				
k		$\sigma^1[k,k]$ $\downarrow \quad \downarrow$ $\downarrow \quad \downarrow$				
$k+1$		$\sigma^1[k+1,k]$ $\downarrow \quad \downarrow$				
$\vdots$		$\downarrow \quad \vdots \quad \downarrow$ $\downarrow \quad \downarrow$				
n		$\boldsymbol{\sigma^1[n,k]}$ $\searrow \quad \nwarrow$				
$n+1$		$\searrow$	$\longleftarrow\longleftarrow\longleftarrow\longleftarrow$ $\boldsymbol{\sigma^1[n+1,k+1]}$ $\longleftarrow\longleftarrow\longleftarrow\longleftarrow$	$\longleftarrow\longleftarrow\longleftarrow\longleftarrow$ $\sigma^1[n+1,k+2]$ $\longleftarrow\longleftarrow\longleftarrow\longleftarrow$	$\longleftarrow$ $\dots$ $\longleftarrow$	$\longleftarrow\longleftarrow\longleftarrow\longleftarrow$ $\sigma^1[n+1,n+1]$ $\longleftarrow\longleftarrow\longleftarrow\longleftarrow$

The entries in the table that are involved in computing the value of $\sigma^1[n+1,k+1]$ and $\sigma^1[n,k]$ using the vertical and horizontal recurrence relations in Theorem 2.1, clearly, form a hockey stick as illustrated by the arrows.

It has been mentioned in the introduction that both exponential-type $S^1[n,k]$ and q-analogue $\sigma^1[n,k]$ of generalized Stirling numbers have explicit formulas. These formulas are quite complicated in form that their combinatorial interpretations may not be easy to establish. Based on the work of Medicis and Leroux [13], we can possibly obtain a combinatorial interpretation for $\sigma^1[n,k]$ if we can express it in a symmetric function form of degree n. In this section, we can establish two such explicit formulas with restriction $\alpha = 0$ for the first and $\beta = 0$ for the second.

To derive the first formula, let us consider first the rational generating functions for $S[n,k;1,b,c]$.

Suppose that

$$\frac{1}{\prod_{j=0}^{k}(1-(b^j+c-1)t)} = \sum_{j=0}^{k} \frac{A_j}{1-(b^j+c-1)t}. \tag{2.1}$$

Then, we have

$$\sum_{i=0}^{k} A_i \prod_{j_1=0}^{i-1}\big(1-(b^{j_1}+c-1)t\big) \prod_{j_2=i+1}^{k}\big(1-(b^{j_2}+c-1)t\big) = 1.$$

For each $j \in \{0,1,2,\dots,k\}$, we set $t = 1/(b^j+c-1)$ to obtain

$$\sum_{i=0}^{k} A_i \left\{\prod_{j_1=0}^{i-1}\left(1-\frac{b^{j_1}+c-1}{b^j+c-1}\right)\right\}\left\{\prod_{j_2=i+1}^{k}\left(1-\frac{b^{j_2}+c-1}{b^j+c-1}\right)\right\} = 1.$$

Clearly, there is only one term in the expansion which is not equal to 0, for each $j \in \{0,1,2,\dots,k\}$. That is, when $i=j$, we have

$$A_j \left\{\prod_{j_1=0}^{j-1}\left(1-\frac{b^{j_1}+c-1}{b^j+c-1}\right)\right\}\left\{\prod_{j_2=j+1}^{k}\left(1-\frac{b^{j_2}+c-1}{b^j+c-1}\right)\right\} = 1.$$

Thus,

$$\begin{aligned} A_j &= \frac{(b^j+c-1)^k}{\left\{\prod_{j_1=0}^{j-1} b^{j_1}(b^{j-j_1}-1)\right\}\left\{(-1)^{k-j}\prod_{j_2=j+1}^{k} b^j(b^{j_2-j}-1)\right\}} \\ &= \frac{(-1)^{k-j}(b^j+c-1)^k}{\prod_{j_1=1}^{j}\left\{\frac{b^{j_1-1}(b^{j-j_1+1}-1)(b^{j_1}-1)}{b^{j_1}-1}\right\}\prod_{j_2=j+1}^{k}\left\{\frac{b^j(b^{j_2-j}-1)(b^{j_2}-1)}{b^{j_2}-1}\right\}} \\ &= \frac{(-1)^{k-j}(b^j+c-1)^k\prod_{j_1=1}^{k}(b^{j_1}-1)}{\left\{\prod_{j_1=1}^{k}(b^{j_1}-1)\right\} b^{j(j-1)/2}b^{(k-j)j}\left\{\prod_{j_1=1}^{j}(b^{j_1}-1)\prod_{j_1=1}^{k-j}(b^{j_1}-1)\right\}}. \end{aligned}$$

By making use of the Gaussian polynomial or q-binomial coefficient (with $q\neq 1$) defined by

$$\begin{bmatrix} k \\ j \end{bmatrix}_q = \prod_{i=1}^{j}\frac{q^{k-i+1}-1}{q^i-1}, \quad \begin{bmatrix} k \\ 0 \end{bmatrix}_q = 1,$$

we have

$$A_j = \left(\prod_{j_1=1}^{k}(b^{j_1}-1)^{-1}\right)(-1)^{k-j}b^{\langle k|j\rangle}\begin{bmatrix} k \\ j \end{bmatrix}_b (b^j+c-1)^k.$$

Thus, using equation (2.2), we get

$$\begin{aligned} &\frac{t^k}{\prod_{j=0}^{k}(1-(b^j+c-1)t)} \\ &= \sum_{j=0}^{k}\frac{\left(\prod_{j_1=1}^{k}(b^{j_1}-1)^{-1}\right)(-1)^{k-j}b^{\langle k|j\rangle}\begin{bmatrix} k \\ j \end{bmatrix}_b (b^j+c-1)^k t^k}{1-(b^j+c-1)t} \\ &= \sum_{j=0}^{k}\left(\prod_{j_1=1}^{k}(b^{j_1}-1)^{-1}\right)(-1)^{k-j}b^{\langle k|j\rangle}\begin{bmatrix} k \\ j \end{bmatrix}_b (b^j+c-1)^k t^k\sum_{\nu\geq 0}[(b^j+c-1)t]^\nu \\ &= \sum_{\nu\geq 0}\left(\prod_{j_1=1}^{k}(b^{j_1}-1)^{-1}\right)\sum_{j=0}^{k}(-1)^{k-j}b^{\langle k|j\rangle}\begin{bmatrix} k \\ j \end{bmatrix}_b (b^j+c-1)^{k+\nu}t^{k+\nu} \\ &= \sum_{n\geq k}\left\{\left(\prod_{j_1=1}^{k}(b^{j_1}-1)^{-1}\right)\sum_{j=0}^{k}(-1)^{k-j}b^{\langle k|j\rangle}\begin{bmatrix} k \\ j \end{bmatrix}_b (b^j+c-1)^n\right\}t^n. \end{aligned}$$

Note that when $a=1$, formula (1.4) reduces to

$$S[n,k;1,b,c] = \left(\prod_{j_1=1}^{k}(b^{j_1}-1)^{-1}\right)\sum_{j=0}^{k}(-1)^{k-j}b^{\langle k|j\rangle}\begin{bmatrix} k \\ j \end{bmatrix}_b (b^j+c-1)^n.$$

This completes the proof of the following theorem.

Theorem 2.2. *For nonnegative integers n and k, and complex numbers b and c with $b \neq 1$, the rational generating function for $S[n,k;1,b,c]$ is given by*

$$\sum_{n\geq k} S[n,k;1,b,c]t^n = \frac{t^k}{\prod_{j=0}^{k}(1-(b^j+c-1)t)}. \tag{2.2}$$

Theorem 2.2 plays an important role in proving the following corollary.

Corollary 2.1. *The numbers $S[n,k;1,b,c]$ have the following explicit formula*

$$S[n,k;1,b,c] = \sum_{c_0+c_1+\ldots+c_k=n-k} \prod_{j=0}^{k}(b^j+c-1)^{c_j}. \tag{2.3}$$

Proof. The rational generating function in Theorem 2.2 can be written as

$$\sum_{n\geq k} S[n,k;1,b,c]t^{n-k} = \prod_{j=0}^{k}\sum_{c_j\geq 0}(b^j+c-1)^{c_j}t^{c_j}.$$

From the identity in [9], we obtain

$$\begin{aligned}\sum_{n\geq k} S[n,k;1,b,c]t^{n-k} &= \sum_{c_0+c_1+\ldots+c_k\geq 0}\left\{\prod_{j=0}^{k}(b^j+c-1)^{c_j}\right\}t^{c_0+c_1+\ldots+c_k}\\ &= \sum_{n\geq k}\left\{\sum_{c_0+c_1+\ldots+c_k=n-k}\prod_{j=0}^{k}(b^j+c-1)^{c_j}\right\}t^{n-k}.\end{aligned}$$

Comparing the coefficients of t^{n-k} completes the proof of the corollary. ∎

We observe that the summand at the right-hand side of (2.4) has exactly $n-k$ factors of the form b^j+c-1 such that, for each j, the factor b^j+c-1 is repeated c_j times in the expansion. With this observation, we can easily verify the following corollary.

Corollary 2.2. *An alternative form of the explicit formula in* (2.4) *is given by*

$$S[n,k;1,b,c] = \sum_{0\leq j_1\leq j_2\leq\ldots\leq j_{n-k}\leq k}\prod_{i=1}^{n-k}(b^{j_i}+c-1). \tag{2.4}$$

Again, by multiplying both sides of (2.5) by $(q-1)^{k-n}$ with $b=q^\beta$ and $c=q^\gamma-1$, we can easily obtain the following theorem. For brevity, we use the notation

$$\sigma[n,k]_q^{\beta,\gamma} = \sigma^1[n,k;0,\beta,\gamma]_q.$$

Theorem 2.3. *For nonnegative integers n and k, and complex numbers β and γ, the explicit formula for the numbers $\sigma[n,k]_q^{\beta,\gamma}$ in a homogeneous symmetric function form of degree n is given by*

$$\sigma[n,k]_q^{\beta,\gamma} = \sum_{0\leq j_1\leq j_2\leq\ldots\leq j_{n-k}\leq k}\prod_{i=1}^{n-k}([j_i\beta]+[\gamma]).$$

As mentioned in Section 1, the q-Stirling numbers of the second kind introduced by Carlitz [3] may be expressed in terms of $\sigma^1[n,k]$ when $\beta = 1$ and $\gamma = 0$. Hence, using Theorem 3, we present an explicit formula for q-Stirling numbers of the second kind [3, 14, 18] as follows

$$\sigma[n,k]_q^{1,0} = \sum_{0\le j_1\le j_2\le\ldots\le j_{n-k}\le k} \prod_{i=1}^{n-k} [j_i] = S_q[n,k].$$

Clearly, this implies the explicit formula of the second kind of Stirling numbers [7] as a limiting case:

$$\lim_{q\to 1} \sigma[n,k]_q^{1,0} = \sum_{0\le j_1\le j_2\le\ldots\le j_{n-k}\le k} j_1 j_2 \ldots j_{n-k} = S(n,k;0,1,0) = S(n,k).$$

Now, if we replace t with $t+c+1$ and take $b = 1$, (1.1) yields

$$[t+c+1|a]_n = \sum_{k=0}^{n} S[n,k;a,1,c]t^k.$$

Note that for $n = 1,\ 2$, we have

$$\begin{aligned}
[t+c+1|a]_1 &= t + (-1)(a^0 - c - 1)\\
[t+c+1|a]_2 &= t^2 + (-1)\{(a^0 - c - 1) + (a^1 - c - 1)\}t\\
&\quad + (-1)^2(a^0 - c - 1)(a^1 - c - 1).
\end{aligned}$$

Hence, by applying induction on n

$$[t+c+1|a]_n = \sum_{k=0}^{n} \left\{ (-1)^{n-k} \sum_{0\le j_1<j_2<\ldots<j_{n-k}\le n-1} \prod_{i=1}^{n-k} (a^{j_i} - c - 1) \right\} t^k.$$

Thus,

$$\sum_{k=0}^{n} S[n,k;a,1,c]t^k = \sum_{k=0}^{n} \left\{ (-1)^{n-k} \sum_{0\le j_1<j_2<\ldots<j_{n-k}\le n-1} \prod_{i=1}^{n-k} (a^{j_i} - c - 1) \right\} t^k.$$

Comparing the coefficients of t^k, we obtain

$$(-1)^{n-k} S[n,k;a,1,c] = \sum_{0\le j_1<j_2<\ldots<j_{n-k}\le n-1} \prod_{i=1}^{n-k} (a^{j_i} - c - 1).$$

Furthermore, multiplying both sides of the preceding equation with $(q-1)^{k-n}$ and taking $a = q^\alpha$ and $c = q^\gamma - 1$, we get

$$(-1)^{n-k}\sigma^1[n,k;\alpha,0,\gamma] = \sum_{0\le j_1<j_2<\ldots<j_{n-k}\le n-1} \prod_{i=1}^{n-k} \left(\frac{q^{j_i\alpha} - 1}{q-1} - \frac{q^\gamma - 1}{q-1} \right).$$

Using the notation

$$\widehat{\sigma}[n,k]_q^{\alpha,\gamma} = (-1)^{n-k}\sigma^1[n,k;\alpha,0,\gamma],$$

we obtain the following theorem.

Theorem 2.4. *For nonnegative integers n and k, and complex numbers α and γ, the explicit formula for the numbers $\widehat{\sigma}[n,k]_q^{\alpha,\gamma}$ in an elementary symmetric function form of degree n is given by*

$$\widehat{\sigma}[n,k]_q^{\alpha,\gamma} = \sum_{0\leq j_1<j_2<\ldots<j_{n-k}\leq n-1} \prod_{i=1}^{n-k} ([j_i\alpha]-[\gamma]).$$

Letting $\alpha = 1$ and $\gamma = 0$, Theorem 2.4 gives

$$\widehat{\sigma}[n,k]_q^{1,0} = \sum_{1\leq j_1<j_2<\ldots<j_{n-k}\leq n-1} \prod_{i=1}^{n-k} [j_i] = c_q[n,k]$$

the q-Stirling numbers of the first kind [14, 18].

3. Combinatorial interpretation in the context of 0-1 tableau

As defined in [13], a 0-1 *tableau* is a pair $\phi = (\lambda, f)$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a partition of an integer m and $f = (f_{ij})_{1\leq j\leq\lambda_i}$ is a filling of the cells of corresponding Ferrers diagram of the shape λ with 0's and 1's, such that there is exactly one '1' in each column. For example, using the partition $\lambda = (5,3,3,2,1)$ we can construct 60 distinct 0-1 tableaux. Figure 1 below shows one of the tableaux with $f_{14} = f_{15} = f_{23} = f_{31} = f_{42} = 1$, $f_{ij} = 0$ elsewhere such that $1 \leq j \leq \lambda_i$.

$$\begin{array}{|c|c|c|c|c|}
\hline
0 & 0 & 0 & 1 & 1 \\
\hline
0 & 0 & 1 \\
\cline{1-3}
1 & 0 & 0 \\
\cline{1-3}
0 & 1 \\
\cline{1-2}
0 \\
\cline{1-1}
\end{array}$$

Figure 1. The 0-1 tableau $\varphi = (\lambda, f)$.

Again in [13], an *A-tableau* is defined as a list Φ of column c of a Ferrers diagram of a partition λ (by decreasing order of length) such that the length $|c|$ are part of the sequence $A = (a_i)_{i\geq 0}$, a strictly increasing sequence of nonnegative integers.

Let ω be a function from the set of nonnegative integers N to a ring K. Suppose Φ is an A-tableau with r columns of lengths $|c|$. Then we set

$$\omega_A(\Phi) = \prod_{c\in\Phi} \omega(|c|).$$

Note that as mentioned in [13] Φ might contain a finite number of columns whose lengths are zero since $0 \in A = \{0,1,2,\ldots,k\}$ and if $\omega(0) \neq 0$. Now, we define $T^A(x,y)$ to be the set of A-tableaux with $A = \{0,1,2,\ldots,x\}$ and exactly y columns (with some columns possibly of zero length). We can now state the following theorem.

Theorem 3.1. *Let $\omega : N \to K$ denote a function from N to a ring K (column weights according to length) which is defined by $\omega(|\overline{c}|) = [|\overline{c}|\beta] + [\gamma]$ where β and γ*

are complex numbers, and $|\overline{c}|$ *is the length of column* $\overline{c}$ *of an* A*-tableau in* $T^A(k, n-k)$. *Then*

$$\sigma[n,k]_q^{\beta,\gamma} = \sum_{\Phi\in T^A(k,n-k)} \prod_{\overline{c}\in\Phi} \omega(|\overline{c}|).$$

Proof. Let Φ be an A-tableau in $T^A(k, n-k)$. Then Φ has exactly $n-k$ columns, say $\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_{n-k}$ whose lengths are $j_1, j_2, \ldots, j_{n-k}$, respectively. Now for each column $\overline{c}_i \in \Phi$, $i = 1, 2, \ldots, n-k$, we have $|\overline{c}_i| = j_i$ and $\omega(|\overline{c}_i|) = [|\overline{c}_i|\beta] + [\gamma]$. Then,

$$\prod_{\overline{c}\in\Phi} \omega(|\overline{c}|) = \prod_{i=1}^{n-k} \omega(|\overline{c}_i|) = \prod_{i=1}^{n-k} ([j_i\beta] + [\gamma]).$$

Using the fact that an A-tableau $\Phi \in T^A(k, n-k)$ can be represented by a multiset whose entries are the column lengths of Φ, we have

$$\sum_{\Phi\in T^A(k,n-k)} \prod_{\overline{c}\in\Phi} \omega(|\overline{c}|) = \sum_{0\le j_1\le j_2\le\ldots\le j_{n-k}} \prod_{i=1}^{n-k} ([j_i\beta] + [\gamma]).$$

Applying Theorem 2.3, we obtain the desired result. ∎

Before considering the next theorem, let us define first $Td^A(x, y)$ to be the subset of $T^A(x, y)$ containing all A-tableaux with columns of distinct lengths.

Theorem 3.2. *Let* $\omega : N \to K$ *be the column weight according to length which is defined by* $\omega(|c|) = [|\overline{c}|\alpha] - [\gamma]$ *where* α *and* γ *are complex numbers, and* $|\overline{c}|$ *is the length of column* $\overline{c}$ *of an* A*-tableau in* $Td^A(n-1, n-k)$. *Then*

$$\widehat{\sigma}[n,k]_q^{\alpha,\gamma} = \sum_{\Phi\in Td^A(n-1,n-k)} \prod_{\overline{c}\in\Phi} \omega(|\overline{c}|).$$

Proof. Following the same argument as in the proof of Theorem 3.1, we can easily complete the proof of the theorem. ∎

4. A generalization of Carlitz identity

In this section, we are going to derive recurrence formula using the combinatorics of 0-1 tableaux which is a kind of generalization of Carlitz identity [3] given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{j=k}^{n} \binom{n}{j} (q-1)^{j-k} S_q[j,k].$$

Suppose that for some numbers c_1 and c_2 we have $|\overline{c}| = c_1 + c_2$. Then, Corollary 2.2 yields

$$S[n,k;1,b,c_1+c_2] = \sum_{0\le j_1\le j_2\le\ldots\le j_{n-k}\le k} \prod_{i=1}^{n-k} (b^{j_i} + c_1 + c_2 - 1).$$

That is, for any $\Phi \in T^A(k, n-k)$,

$$\omega_A(\Phi) = \prod_{\overline{c}\in\Phi} (c_1 + b^{|\overline{c}|} + c_2 - 1),$$

where $|\overline{c}| \in \{0, 1, 2, \ldots, k\}$. Note that the weight of each column of Φ can be considered as a finite sum with additive constant c_1, that is, for each $\overline{c} \in \Phi$, we can write

$$\omega(|\overline{c}|) = c_1 + \omega^*(|\overline{c}|) \tag{4.1}$$

where $\omega^*(|\overline{c}|) = c_2 + b^{|\overline{c}|} - 1$. The following theorem determines how an additive constant affects the recurrence formula for $S[n, k; 1, b, c]$.

Theorem 4.1. *For nonnegative integers n and k, and complex numbers b and c with $b \neq 1$, the exponential-type Stirling numbers $S[n, k; 1, b, c]$ satisfy the following identity*

$$S[n, k; 1, b, c] = \sum_{j=k}^{n} \binom{n}{j} c_1^{n-j} S[j, k; 1, b, c_2]$$

where $c = c_1 + c_2$ for some numbers c_1 and c_2.

Proof. From Theorem 3.1,

$$S[n, k; 1, b, c] = \sum_{\Phi \in T^A(k, n-k)} \omega_A(\Phi)$$

where

$$\omega_A(\Phi) = \prod_{\overline{c} \in \Phi} (b^{|\overline{c}|} + c - 1), \quad |\overline{c}| \in \{0, 1, 2, \ldots, k\}.$$

Substituting $j_i = |\overline{c}|$, we obtain

$$\omega_A(\Phi) = \prod_{i=1}^{n-k} (b^{j_i} + c - 1).$$

If $c = c_1 + c_2$ for some numbers c_1 and c_2, then, by (4.1),

$$\begin{aligned} \omega_A(\Phi) &= \prod_{i=1}^{n-k} (c_1 + \omega^*(j_i)), \quad \omega^*(j_i) = b^{j_i} + c_2 - 1 \\ &= \sum_{r=0}^{n-k} c_1^{n-k-r} \sum_{\substack{q_1 \leq q_2 \leq \cdots \leq q_r, \\ q_i \in \{j_1, \ldots, j_{n-k}\}}} \prod_{i=1}^{r} \omega^*(q_i). \end{aligned}$$

Then

$$\begin{aligned} \sum_{\Phi \in T^A(k, n-k)} \omega_A(\Phi) &= \sum_{0 \leq j_1 \leq j_2 \leq \ldots \leq j_{n-k} \leq k} \sum_{r=0}^{n-k} c_1^{n-k-r} \sum_{\substack{q_1 \leq q_2 \leq \cdots \leq q_r, \\ q_i \in \{j_1, \ldots, j_{n-k}\}}} \prod_{i=1}^{r} \omega^*(q_i) \\ &= \sum_{r=0}^{n-k} \sum_{0 \leq j_1 \leq j_2 \leq \ldots \leq j_{n-k} \leq k} \sum_{\substack{q_1 \leq q_2 \leq \cdots \leq q_r, \\ q_i \in \{j_1, \ldots, j_{n-k}\}}} c_1^{n-k-r} \prod_{i=1}^{r} \omega^*(q_i) \end{aligned}$$

Now, we are going to count the number of tableaux with $n-k$ columns such that $n-k-r$ columns are of weight c_1 and r columns are of weight $\omega^*(q_i)$, $q_i \in \{0, 1, 2, \ldots, k\}$. Note that there are $\binom{n-k}{r}$ tableaux with r columns whose lengths are taken from the lengths of the columns of Φ. Since there is a one-to-one correspondence between weights $\omega(j_i)$ and A-tableaux, the number of A-tableaux Φ in $T^A(k,n-k)$ is equal to the number of possible multisets $\{j_1,j_2,\dots,j_{n-k}\}$ with j_i in $\{0,1,2,\dots,k\}$. That is,

$$|T^A(k,n-k)| = \binom{n-k+k}{n-k} = \binom{n}{n-k} = \binom{n}{k}.$$

Thus, for all $\Phi \in T^A(k,n-k)$, we can generate $\binom{n}{k}\binom{n-k}{r}$ tableaux with r columns whose weights are $\omega^*(j_i)$, $j_i \in \{0,1,2,\dots,k\}$. However, there are only

$$|T^A(k,r)| = \binom{r+k}{r}$$

distinct tableaux with r columns whose lengths are in $\{0,1,2,\dots,k\}$. Hence, every distinct tableau with $n-k$ columns, r of which are of weight other than c_1, appears

$$\frac{\binom{n}{k}\binom{n-k}{r}}{\binom{r+k}{r}} = \binom{n}{r+k}$$

times in the collection. Thus,

$$\sum_{\Phi\in T^A(k,n-k)} \omega_A(\Phi) = \sum_{r=0}^{n-k}\sum_{\phi\in\overline{B}_r}\binom{n}{r+k}c_1^{n-k-r}\prod_{\overline{c}\in\phi}\omega^*(|\overline{c}|)$$

where $\overline{B}_r$ denotes the set of all tableaux ϕ having r columns of weights $\omega^*(j_i) = b^{j_i}+c_2-1$. Reindexing the double sum, we get

$$\sum_{\Phi\in T^A(k,n-k)} \omega_A(\Phi) = \sum_{j=k}^{n}\binom{n}{j}c_1^{n-j}\sum_{\phi\in\overline{B}_{j-k}}\prod_{\overline{c}\in\phi}\omega^*(|\overline{c}|)$$

where $\overline{B}_{j-k}$ is the set of all tableaux with $j-k$ columns of weights $\omega^*(j_i) = b^{j_i}+c_2-1$ for each $i = 1,2,\dots,j-k$. Clearly, $\overline{B}_{j-k} = T^A(k,j-k)$. Therefore,

$$\sum_{\Phi\in T^A(k,n-k)} \omega_A(\Phi) = \sum_{j=k}^{n}\binom{n}{j}c_1^{n-j}\sum_{\phi\in T^A(k,j-k)}\omega_A(\phi).$$

Applying Corollary 2.2 completes the proof of the theorem. ∎

By taking $b=q$ and $c=1$ with $c_1=1$ and $c_2=0$, Theorem 4.1 gives

$$\begin{aligned}
S[n,k;1,q,1] &= \sum_{j=k}^{n}\binom{n}{j}S[j,k;1,q,0]\\
&= \sum_{j=k}^{n}\binom{n}{j}(q-1)^{j-k}(q-1)^{k-j}S[j,k;q^0,q^1,q^0-1]\\
&= \sum_{j=k}^{n}\binom{n}{j}(q-1)^{j-k}\sigma^1[j,k;0,1,0]_q.
\end{aligned}$$

From (1.5), we can have

$$S[n,k;1,q,1] = \sum_{j=k}^{n}\binom{n}{j}(q-1)^{j-k}S_q[j,k].$$

But using Corollary 2.2, we obtain

$$S[n,k;1,q,1]=\sum_{0\le j_1\le j_2\le\ldots\le j_{n-k}\le k}\prod_{i=1}^{n-k}q^{j_i}=\begin{bmatrix}n\\k\end{bmatrix}_q$$

which is the representation given in [13] for the q-binomial coefficients. Thus, we get

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\sum_{j=k}^{n}\binom{n}{j}(q-1)^{j-k}S_q[j,k]$$

Carlitz identity as special case of Theorem 4.1.

Moreover, if we let $b=q^\beta$ and $c=q^\gamma-1$ with $c_1=q^\gamma-q^{\gamma_2}$ and $c_2=q^{\gamma_2}-1$, Theorem 4.1 yields

$$S[n,k;q^0,q^\beta,q^\gamma-1]=\sum_{j=k}^{n}\binom{n}{j}\left(q^\gamma-q^{\gamma_2}\right)^{n-j}S[j,k;q^0,q^\beta,q^{\gamma_2}-1]. \tag{4.2}$$

Multiplying both sides of (4.2) by $(q-1)^{k-n}$, we obtain the following theorem.

Theorem 4.2. *For nonnegative integers n and k, and complex numbers β and γ, the q-analogue $\sigma[n,k]_q^{\beta,\gamma}$ satisfies the following identity*

$$\sigma[n,k]_q^{\beta,\gamma}=\sum_{j=k}^{n}\binom{n}{j}[\gamma_1]^{n-j}\sigma[j,k]_q^{\beta,\gamma_2}$$

where $[\gamma]=[\gamma_1]+[\gamma_2]$ for some numbers γ_1 and γ_2.

It can be seen from the above theorems how the combinatorics of 0-1 tableaux works in obtaining good and interesting results. A deeper discussion on this can further enrich the theory of the q-analogue of the generalized Stirling numbers. For instance, Leroux [18] proved that the q-Stirling numbers of the first and second kind, $c_q[n,k]$ and $S_q[n,k]$, are q-log concave for $n\ge 1$ and $2\le k\le n-1$ using their explicit formulas in symmetric function form of degree n and the deeper treatment of the concept of 0-1 tableau. Since we already have explicit formulas for $\sigma[n,k]_q^{\beta,\gamma}$ and $\widehat{\sigma}[n,k]_q^{\alpha,\gamma}$ in symmetric function form of degree n, we may possibly prove that $\sigma[n,k]_q^{\beta,\gamma}$ and $\widehat{\sigma}[n,k]_q^{\alpha,\gamma}$ are also q-log concave using the method of Leroux.

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