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Factorial Stirling matrix and related combinatorial sequences[☆]

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Abstract

In this paper, some relationships between the Stirling matrix, the Vandermonde matrix, the Benoulli numbers and the Eulerian numbers are studied from a matrix representation of $k!S(n,k)$ which will be called the factorial Stirling matrix, where $S(n,k)$ are the Stirling numbers of the second kind. As a consequence a number of interesting and useful identities are obtained.

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1. Introduction

In many contexts (see [1–4]), a number of interesting and useful identities involving binomial coefficients are obtained from a matrix representation of a particular counting sequence. Such a matrix representation provides a powerful computational tool for deriving identities and an explicit formula related to the sequence.

First, we begin our study with a well-known definition: for any real number x and an integer k, define

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$$\binom{x}{k} = \begin{cases} \frac{[x]_k}{k!} & \text{if } k \geqslant 1, \\ 1 & \text{if } k = 0, \\ 0 & \text{if } k < 0, \end{cases}$$

where $[x]_k = x(x-1)\cdots(x-k+1)$.

Then the *Stirling numbers of the second kind* $S(n,k)$ for integers n and k with $0 \leqslant k \leqslant n$ are defined implicitly by

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k.$$

We can rewrite the above equation for x^n as

$$x^n = \sum_{k=0}^{n} k!S(n,k)\binom{x}{k}.$$

This expression can be represented as a system of matrix equations for each $n = 0, 1, \ldots,$

$$\mathbf{v}(x) = ([1] \oplus \widetilde{S}_n)\mathbf{c}(x), \tag{1}$$

where

$$\mathbf{v}(x) = \begin{bmatrix} 1 & x & \cdots & x^n \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{c}(x) = \left[\binom{x}{0}\ \binom{x}{1}\ \cdots\ \binom{x}{n}\right]^{\mathrm{T}}$$

and $\widetilde{S}_n = [k!S(n,k)]$ is the $n \times n$ matrix whose (i,j)-entry is $j!S(i,j)$ if $i \geqslant j$ and otherwise 0.

For example, if $n = 4$, then

$$\begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 6 & 6 & 0 \\ 0 & 1 & 14 & 36 & 24 \end{bmatrix} \begin{bmatrix} \binom{x}{0} \\ \binom{x}{1} \\ \binom{x}{2} \\ \binom{x}{3} \\ \binom{x}{4} \end{bmatrix}.$$

Note that

$$\widetilde{S}_n = S_n \cdot \mathrm{diag}(1!, 2!, \ldots, n!),$$

where $S_n = [S(n,k)]$ is the $n \times n$ *Stirling matrix of the second kind* [3]. We will call $\widetilde{S}_n$ as a *factorial Stirling matrix*.

In many cases it is possible to obtain from a matrix representation of a particular counting sequence an identity and an explicit formula for the general term of the sequence.

In this paper, some relationships between the Stirling matrix, the Vandermonde matrix, the Benoulli numbers and the Eulerian numbers from the factorial Stirling matrix $\widetilde{S}_n$ are studied. As a consequence we obtain interesting combinatorial identities related to theses numbers.

2. Stirling matrix and Vandermonde matrix

The factorial Stirling matrix can be used to obtain a summation formula which contains one or more summations. We first obtain interesting determinants of certain matrices in the decomposition of the Vandermonde matrix.

Matrix equation (1) suggests the Vandermonde matrix which is defined by

$$V_{n+1}(x) := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x & x+1 & \cdots & x+n \\ x^2 & (x+1)^2 & \cdots & (x+n)^2 \\ \vdots & \vdots & & \vdots \\ x^n & (x+1)^n & \cdots & (x+n)^n \end{bmatrix}$$

can be factorized by

$$V_{n+1}(x) = ([1] \oplus \widetilde{S}_n) C_{n+1}(x), \tag{2}$$

where $C_{n+1}(x)$ is the $(n+1) \times (n+1)$ matrix whose (i, j)-entry is $\binom{x+j-1}{i-1}$.

Moreover, the following LU-factorization of $C_{n+1}(x)$ can be proved by repeated application of the Pascal formula:

$$C_{n+1}(x) = \Delta_{n+1}(x) P_{n+1}^{\mathrm{T}}, \tag{3}$$

where P_{n+1} is the $(n+1) \times (n+1)$ Pascal matrix (see [1]) whose (i, j)-entry is $\binom{i-1}{j-1}$ if $i \geqslant j$ and otherwise 0, and $\Delta_{n+1}(x)$ is the $(n+1) \times (n+1)$ lower triangular matrix whose (i, j)-entry is $\binom{x}{i-j}$ if $i \geqslant j$ and otherwise 0. Also applying the Pascal formula, it is easy to show that

$$C_n(x+1) = \Delta_n(1) C_n(x). \tag{4}$$

From (2) and (3), another factorization for the Vandermonde matrix $V_{n+1}(x)$ is obtained (see [3]).

Theorem 2.1. *For any real number x, we have*

$$V_{n+1}(x) = ([1] \oplus \widetilde{S}_n) \Delta_{n+1}(x) P_{n+1}^{\mathrm{T}}.$$

For example, if $x = 1$, then

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1^2 & 2^2 & 3^2 & 4^2 \\ 1^3 & 2^3 & 3^3 & 4^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 6 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For any $n \times n$ matrix A, let $A(i\,|\,j)$ be the $(n-1) \times (n-1)$ matrix obtained from A by deleting the ith row and jth column.

Corollary 2.2. *For any real number x, we have*:
(i) $\det C_{n+1}(x) = 1$,
(ii) $\det C_{n+1}(x)(1|n) = \binom{x+n-1}{n}$ $(x \geqslant 1)$.

Proof. (i) is an immediate consequence of (3). Noticing

$$V_{n+1}(x)(1|n) = \widetilde{S}_n C_{n+1}(x)(1|n) \tag{5}$$

from (2) and

$$\det(V_{n+1}(x)(1|n)) = \prod_{k=1}^{n-1} k![x+n-1]_n \quad \text{and} \quad \det(\widetilde{S}_n) = \prod_{k=1}^{n} k!,$$

we obtain (ii). □

For example,

$$\det C_4(x) = \det \begin{bmatrix} \binom{x}{0} & \binom{x+1}{0} & \binom{x+2}{0} & \binom{x+3}{0} \\ \binom{x}{1} & \binom{x+1}{1} & \binom{x+2}{1} & \binom{x+3}{1} \\ \binom{x}{2} & \binom{x+1}{2} & \binom{x+2}{2} & \binom{x+3}{2} \\ \binom{x}{3} & \binom{x+1}{3} & \binom{x+2}{3} & \binom{x+3}{3} \end{bmatrix} = 1$$

and

$$\det C_5(x)(1|5) = \det \begin{bmatrix} \binom{x}{1} & \binom{x+1}{1} & \binom{x+2}{1} & \binom{x+3}{1} \\ \binom{x}{2} & \binom{x+1}{2} & \binom{x+2}{2} & \binom{x+3}{2} \\ \binom{x}{3} & \binom{x+1}{3} & \binom{x+2}{3} & \binom{x+3}{3} \\ \binom{x}{4} & \binom{x+1}{4} & \binom{x+2}{4} & \binom{x+3}{4} \end{bmatrix} = \binom{x+3}{4}.$$

More interesting fact is that $\widetilde{S}_n$ can be used to derive a summation formula. For convenience, we define for each $k = 1, 2, \ldots, n$, the numbers $Z^1(n,k)$, $Z^2(n,k), \ldots$ by the recursive definition:

$$Z^1(n,k) = 1^k + 2^k + \cdots + n^k = \sum_{j=1}^{n} j^k,$$

$$Z^p(n,k) = \sum_{j=1}^{n} Z^{p-1}(j,k) \quad (p = 2, 3, \ldots),$$

and we denote for each $i = 1, 2, \ldots, n$ and for $p \geqslant 2$,

$$\mathbf{x}_i(p) = \left[\binom{p+i-1}{p}\binom{p+i-1}{p+1}\cdots\binom{p+i-1}{p+k-1}\right]^{\mathrm{T}}, \tag{6}$$

$$\mathbf{z}_i(p) = \left[Z^{p-1}(i,1)\ Z^{p-1}(i,2)\cdots Z^{p-1}(i,k)\right]^{\mathrm{T}}. \tag{7}$$

Now we are ready to prove the following:

Theorem 2.3. *For each $p = 1, 2, \ldots, n$, we have*

$$\widetilde{S}_k\left[\binom{n+p}{p+1}\cdots\binom{n+p}{p+k-1}\right]^{\mathrm{T}} = \left[Z^p(n,1)\cdots Z^p(n,k)\right]^{\mathrm{T}}. \tag{8}$$

Proof. For positive integers n, k with $n \geqslant k$, we prove by induction on $n + p$. Substituting $x = 1$ in (5) proves (8) for $p = 1$. Now let $p \geqslant 2$. Applying the identity

$$\binom{n+1}{k+1} = \sum_{l=0}^{n}\binom{l}{k},$$

and using (6) and (7), by the induction we have

$$\begin{aligned}\widetilde{S}_k\mathbf{x}_n(p+1) &= \widetilde{S}_k(\mathbf{x}_1(p) + \mathbf{x}_2(p) + \cdots + \mathbf{x}_n(p))\\ &= \mathbf{z}_1(p) + \mathbf{z}_2(p) + \cdots + \mathbf{z}_n(p)\\ &= \mathbf{z}_n(p+1),\end{aligned}$$

which completes the proof. □

Formula (8) can be used to obtain multiple sums of powers of integers. For example, if $p = 1$ then $Z^1(n,k)$ expresses the sum of powers of the first n positive integers, and if $p = 2$, then

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 1 & 2 & 0 & 0\\ 1 & 6 & 6 & 0\\ 1 & 14 & 36 & 24\end{bmatrix}\begin{bmatrix}\binom{n+2}{3}\\ \binom{n+2}{4}\\ \binom{n+2}{5}\\ \binom{n+2}{6}\end{bmatrix} = \begin{bmatrix}\frac{1}{6}n(n+1)(n+2)\\ \frac{1}{12}n(n+2)(n+1)^2\\ \frac{1}{60}n(n+1)(n+2)(3n^2+6n+1)\\ \frac{1}{60}n(n+2)(n+1)^2(2n^2+4n-1)\end{bmatrix}$$
$$= [Z^2(n,1)\ Z^2(n,2)\ Z^2(n,3)\ Z^2(n,4)]^{\mathrm{T}}$$

and $Z^2(n,k)$, $k = 1, 2, 3, 4$, expresses

$$1^k + (1^k + 2^k) + \cdots + (1^k + 2^k + \cdots + n^k) = \sum_{l=1}^{n}\left(\sum_{j=1}^{l} j^k\right).$$

More generally, we obtain the following corollary from Theorem 2.3 and the definition of $Z^p(n,k)$.

Corollary 2.4. *For each $p = 1, 2, \ldots, n$, we have*

$$\sum_{n_{p-1}=1}^{n_p} \sum_{n_{p-2}=1}^{n_{p-1}} \cdots \sum_{j=1}^{n_1} j^k = \sum_{i=1}^{k} i!S(k,i)\binom{n+p}{i+p},$$

where $n_p := n$.

3. Stirling matrix and Bernoulli numbers

The *Bernoulli numbers* B_n satisfy the recurrence relation

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \tag{9}$$

for all $n \geqslant 1$, with $B_0 = 1$, and the *Bernoulli polynomials* $B_n(x)$ are given by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \tag{10}$$

and satisfy

$$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n \quad (n \geqslant 0). \tag{11}$$

The Bernoulli polynomials are important in obtaining closed form expressions for the sums of powers of integers such as

$$Z^1(n,k) = \frac{1}{k+1}[B_{k+1}(n+1) - B_{k+1}(0)],$$

from (11) (also see [5, p. 199]), and it is known that the Bernoulli numbers $B_n = B_n(0)$ can be expressed in terms of the Stirling numbers such as (see [5, p. 147])

$$B_n = \sum_{k=0}^{n} (-1)^k S(n,k) \frac{k!}{k+1} \tag{12}$$

for all $n \geqslant 0$.

In this section, we obtain other interesting identities related to the Bernoulli numbers in the decomposition of the factorial Stirling matrix. First we prove a very useful lemma.

Lemma 3.1. *For any integer $n \geqslant 0$, we have*

$$(n+1)x^n = \sum_{k=0}^{n} \binom{n+1}{k} B_k(x). \tag{13}$$

Proof. Applying (10) and the identity

$$\binom{n+1}{k}\binom{n+1-k}{j} = \binom{n+1}{j+k}\binom{j+k}{k},$$

we have

$$\begin{aligned} B_{n+1}(x+1) - B_{n+1}(x) &= \sum_{k=0}^{n} B_k \sum_{j=0}^{n-k} \binom{n+1}{k}\binom{n+1-k}{j} x^j \\ &= \sum_{k=0}^{n} B_k \sum_{j=0}^{n-k} \binom{n+1}{j+k}\binom{j+k}{k} x^j. \end{aligned} \tag{14}$$

Expanding the last expression (14) gives

$$\begin{aligned} &\binom{n+1}{0} B_0 + \binom{n+1}{1}\left\{ B_0 \binom{1}{0} x + B_1 \binom{1}{1} \right\} \\ &\quad + \cdots + \binom{n+1}{n}\left\{ B_0 \binom{n}{0} x^n + B_1 \binom{n}{1} x^{n-1} + \cdots + B_n \binom{n}{n} \right\} \\ &= \sum_{k=0}^{n} \binom{n+1}{k} \sum_{j=0}^{k} \binom{k}{j} B_j x^{k-j} \\ &= \sum_{k=0}^{n} \binom{n+1}{k} B_k(x), \end{aligned}$$

which proves (13) from (11). □

Note that substituting $x = 0$ in (13) gives (9).

Defining the lower triangular matrix $\widehat{P}_n = [\binom{i}{j-1}]$ by

$$\widehat{P}_n = \begin{bmatrix} \binom{1}{0} & & & \\ \binom{2}{0} & \binom{2}{1} & & O \\ \vdots & \vdots & \ddots & \\ \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{n-1} \end{bmatrix},$$

gives the following matrix representation of (13):

$$[1 \; 2x \; \cdots \; (n+1)x^n]^{\mathrm{T}} = \widehat{P}_{n+1}[B_0(x) \; B_1(x) \; \cdots B_n(x)]^{\mathrm{T}}. \tag{15}$$

Thus from (15) we obtain the closed form for the Bernoulli polynomials.

For example, if $n = 4$, then

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 2x \\ 3x^2 \\ 4x^3 \\ 5x^4 \end{bmatrix} = \begin{bmatrix} 1 \\ x - \frac{1}{2} \\ x^2 - x + \frac{1}{6} \\ x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\ x^4 - 2x^3 + x^2 - \frac{1}{30} \end{bmatrix}.$$

Note that $\widehat{P}_n$ is the 'reverse' of the Pascal matrix $P_{n+1}(1|1)$, and the Bernoulli numbers appear in the first column of $\widehat{P}_n^{-1}$ (also see [4]).

Theorem 3.2. *The factorial Stirling matrix $\widetilde{S}_{n+1}$ can be factorized as*

$$\widetilde{S}_{n+1} = \widehat{P}_{n+1}([1] \oplus \widetilde{S}_n).$$

Proof. Applying the identity (see [2, p. 209])

$$S(n, k) = \sum_{l=k-1}^{n-1} \binom{n-1}{l} S(l, k-1),$$

we obtain

$$\begin{aligned}
[\widehat{P}_{n+1}([1] \oplus \widetilde{S}_n)]_{ij} &= \sum_{k=j}^{i} \binom{i}{i-k+1} (j-1)!S(k-1, j-1) \\
&= \sum_{k=j-1}^{i-1} \binom{i}{i-k} (j-1)!S(k, j-1) \\
&= (j-1)! \sum_{k=j-1}^{i-1} \binom{i}{k} S(k, j-1) \\
&= (j-1)! \left(\sum_{k=j-1}^{i} \binom{i}{k} S(k, j-1) - \binom{i}{i} S(i, j-1) \right) \\
&= (j-1)! \, (S(i+1, j) - S(i, j-1)) \\
&= j!S(i, j) = [\widetilde{S}_{n+1}]_{ij},
\end{aligned}$$

which completes the proof. □

Note that

$$\begin{aligned}[\widehat{P}_{n+1}^{-1}]_{ij} &= [([1] \oplus \widetilde{S}_n)\widetilde{S}_{n+1}^{-1}]_{ij} \\ &= \left[([1] \oplus S_n)\mathrm{diag}\left(1, \frac{1}{2}, \ldots, \frac{1}{n+1}\right) S_{n+1}^{-1}\right]_{ij} \\ &= \sum_{k=1}^{n+1} \frac{1}{k} S(i-1, k-1)s(k, j),\end{aligned}$$

where $s(n, k)$ is the Stirling number of the first kind (see [2, p. 213]), i.e., $[s(k, j)] = [S(k, j)]^{-1}$, which proves that the coefficients of the Bernoulli polynomial can be expressed by the Stirling numbers of both kinds.

Corollary 3.3. *Let $B_n(x) = \sum_{k=0}^{n} b_k x^k$ be the Bernoulli polynomial. Then for each $k = 0, 1 \ldots, n$, we have*

$$b_k = \sum_{l=1}^{n+1} \frac{k+1}{l} S(n, l-1)s(l, k+1).$$

In particular, since $s(l, 1) = (-1)^{l-1}(l-1)!$ for each $l = 1, 2, \ldots, n+1$, we see that

$$b_0 = \sum_{l=1}^{n+1} (-1)^{l-1} \frac{(l-1)!}{l} S(n, l-1) \quad (n = 0, 1, \ldots),$$

which is equal to the Bernoulli number B_n from (12).

Comparing the coefficients of x^{n-k} of $B_n(x)$ in both (10) and Corollary 3.3 gives the following corollary.

Corollary 3.4. *For any n with $k = 0, 1, 2, \ldots, n$, we have*

$$B_k = \frac{1}{\binom{n}{k}} \sum_{l=1}^{n+1} \frac{n-k+1}{l} S(n, l-1)s(l, n-k+1).$$

4. Stirling matrix and Eulerian numbers

For each $k = 0, 1, \ldots, n-1$, $(n \geqslant 1)$, the *Eulerian numbers* $E(n, k)$ are given by the sum

$$E(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k+1-j)^n,$$

which also satisfy the recurrence relation

$$E(n,k) = (k+1)E(n-1,k) + (n-k)E(n-1,k-1) \quad (n \geqslant 2),$$

where $E(n,0) = E(n,n-1) = 1$. These numbers are one of the most celebrated numbers associated with random permutations.

It is known [5, p. 149] that the Eulerian numbers $E(n,k)$ are closely related to the Stirling numbers of the second kind $S(n,k)$ via

$$S(n,m) = \frac{1}{m!}\sum_{k=0}^{n-1} E(n,k)\binom{k}{n-m} \quad (n \geqslant m,\ n \geqslant 1). \tag{16}$$

This suggests a means of finding relationships between the Stirling matrix and the *Eulerian matrix* obtained from the triangular array of the Eulerian numbers. In this section, we find the relationships between the factorial Stirling matrix and the Eulerian matrix, and we obtain some interesting identities in a decomposition of the Eulerian matrix.

The $n \times n$ *Eulerian matrix* $E_n = [E(n,k)]$ is the triangular matrix whose (i,j)-entry is $E(i, j-1)$ if $i \geqslant j$ and otherwise 0.

For example,

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 1 & 11 & 11 & 1 & 0 \\ 1 & 26 & 66 & 26 & 1 \end{bmatrix}.$$

The Eulerian matrix is closely related to the matrix $\widetilde{S}_n$. First, define $\widehat{S}_n$ to be the 'reverse' of $\widetilde{S}_n$ whose (i,j)-entry is $(i-j+1)!S(i,i-j+1)$ if $i \geqslant j$ and otherwise 0.

Theorem 4.1. *For the $n \times n$ Pascal matrix P_n, we have*

$$E_n P_n = \widehat{S}_n. \tag{17}$$

Proof. Applying (16), we have

$$\begin{aligned}(E_n P_n)_{ij} &= \sum_{k=1}^{n} (E_n)_{ik}(P_n)_{kj} \\ &= \sum_{k=1}^{n} E_n(i,k-1)\binom{k-1}{j-1} \\ &= \sum_{k=j}^{i} E_n(i,k-1)\binom{k-1}{j-1}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=j-1}^{i-1} E_n(i,k)\binom{k}{j-1} \\
&= \sum_{k=i-(i-j+1)}^{i-1} E_n(i,k)\binom{k}{i-(i-j+1)} \\
&= (i-j+1)!S(i,i-j+1) \\
&= (\widehat{S}_n)_{ij}. \qquad \square
\end{aligned}
$$

Thus we obtain the factorization of the Eulerian matrix:

$$E_n = \widehat{S}_n P_n^{-1}. \tag{18}$$

For example,

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 6 & 1 & 0 & 0 \\ 24 & 36 & 14 & 1 & 0 \\ 120 & 240 & 150 & 30 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ -1 & 3 & -3 & 1 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}.$$

In particular, from (18) we get an interesting combinatorial identity which gives other explicit formulas for the Eulerian numbers.

Corollary 4.2. *For each $k = 1, 2, \ldots, n$, we have*

$$E(n,k) = \sum_{j=1}^{n} (-1)^{n-k-j+1} j!S(n,j)\binom{n-j}{k-1}.$$

In [3], it was shown that the Vandermonde matrix $V_n(x)$ which is defined in Section 2 can be factorized as

$$P_n V_n(x-1) = V_n(x).$$

Since $P_n = E_n^{-1}\widehat{S}_n$, we see that

$$E_n V_n(x) = \widehat{S}_n V_n(x-1). \tag{19}$$

Thus we have the following result from the $(n,1)$-entries of both sides of (19).

Corollary 4.3.

$$\sum_{k=1}^{n} E(n,k)x^{k-1} = \sum_{k=1}^{n} k!S(n,k)(x-1)^{n-k}. \tag{20}$$

Substituting $x = 2$ in (20) gives

$$\sum_{k=1}^{n} E(n,k)2^{k-1} = \sum_{k=1}^{n} k!S(n,k),$$

which proves the *Fubini formula* [2, p. 228], and noticing the *Eulerian polynomial*

$$f_n(x) := \sum_{k=1}^{n} E(n,k)x^k$$

we obtain

$$f_n(x) = x\sum_{k=1}^{n} k!S(n,k)(x-1)^{n-k},$$

which proves the theorem of Frobenius [2, p. 244].

We end this section obtaining another factorization for the Eulerian matrix. It is easy to show that the following corollary holds from (2), (4), and (19).

Corollary 4.4

$$E_n = \widehat{S}_n([1]\oplus\widetilde{S}_{n-1})\Delta_n(1)^{-1}([1]\oplus\widetilde{S}_{n-1})^{-1}.$$

Thus from (18), we obtain the factorization for the Pascal matrix P_n.

$$P_n = ([1]\oplus\widetilde{S}_{n-1})\Delta_n(1)([1]\oplus\widetilde{S}_{n-1})^{-1}.$$

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