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# Stirling matrix via Pascal matrix

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**Abstract**

The Pascal-type matrices obtained from the Stirling numbers of the first kind $s(n, k)$ and of the second kind $S(n, k)$ are studied, respectively. It is shown that these matrices can be factorized by the Pascal matrices. Also the LDU-factorization of a Vandermonde matrix of the form $V_n(x, x+1, \ldots, x+n-1)$ for any real number $x$ is obtained. Furthermore, some well-known combinatorial identities are obtained from the matrix representation of the Stirling numbers, and these matrices are generalized in one or two variables. © 2001 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

For integers $n$ and $k$ with $n \geqslant k \geqslant 0$, the *Stirling numbers of the first kind* $s(n, k)$ and *of the second kind* $S(n, k)$ can be defined as the coefficients in the following expansion of a variable $x$ (see [3, pp. 271–279]):

$$[x]_n = \sum_{k=0}^{n} (-1)^{n-k} s(n, k) x^k$$

and

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PII: S 0 0 2 4 - 3 7 9 5 ( 0 1 ) 0 0 2 3 4 - 8

$$x^n = \sum_{k=0}^{n} S(n,k)[x]_k, \tag{1.1}$$

where

$$[x]_n = \begin{cases} x(x-1)\cdots(x-n+1) & \text{if } n \geqslant 1, \\ 1 & \text{if } n = 0. \end{cases} \tag{1.2}$$

It is known that for an $n,\ k \geqslant 0$, the $s(n,k),\ S(n,k)$ and $[n]_k$ satisfy the following Pascal-type recurrence relations:

$$\begin{aligned} s(n,k) &= s(n-1,k-1) + (n-1)s(n-1,k), \\ S(n,k) &= S(n-1,k-1) + kS(n-1,k), \\ [n]_k &= [n-1]_k + k[n-1]_{k-1}, \end{aligned} \tag{1.3}$$

where $s(n,0) = s(0,k) = S(n,0) = S(0,k) = [0]_k = 0$ and $s(0,0) = S(0,0) = 1$, and moreover the $S(n,k)$ satisfies the following formula known as 'vertical' recurrence relation:

$$S(n,k) = \sum_{l=k-1}^{n-1} \binom{n-1}{l} S(l,k-1). \tag{1.4}$$

As we did for the Pascal triangle, we can define the Pascal-type matrices from the Stirling numbers of the first kind and of the second kind, respectively. A matrix representation of the Pascal triangle has catalyzed several investigations (see [1,2,4,6,7]).

The $n \times n$ *Pascal matrix* [4] (also see [2]), $P_n$, is defined by

$$(P_n)_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

More generally, for a nonzero real variable *x*, the Pascal matrix was generalized in $P_n[x]$ and $Q_n[x]$, respectively which are defined in [6] (also see [1]), and these generalized Pascal matrices were also extended in $\Phi_n[x,y]$ (see [7]) for any two nonzero real variables *x* and *y* where

$$(\Phi_n[x,y])_{ij} = \begin{cases} x^{i-j} y^{i+j-2} \dbinom{i-1}{j-1} & \text{if } i \geqslant j, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

By the definition, we see that

$$\begin{aligned} &P_n[x] = \Phi_n[x,1], \quad Q_n[y] = \Phi_n[1,y], \\ &P_n = P_n[1] = Q_n[1] = \Phi_n[1,1]. \end{aligned} \tag{1.6}$$

Moreover, it is known that

$$P_n^{-1}[x] = P_n[-x] = \left[ (-1)^{i-j} \binom{i-1}{j-1} x^{i-j} \right], \tag{1.7}$$

and in particular, $P_n^{-1} = P_n^{-1}$ [1].

In [6] and [7], the factorizations of $P_n[x]$, $Q_n[x]$, and $\Phi_n[x, y]$ are obtained, respectively.

In Section 2, we study the Pascal-type matrices which will be called the Stirling matrices obtained from the Stirling numbers of the first kind $s(n, k)$ and second kind $S(n, k)$. As a consequence it is shown that such matrices can be factorized by the Pascal matrices. Also the LDU-factorization of a Vandermonde matrix of the form $V_n(x, x+1, \ldots, x+n-1)$ for any real number $x$ is obtained.

In Section 3, some well-known combinatorial identities are obtained from the matrix representation of the Stirling numbers.

Finally in Section 4, these matrices are generalized in one or two variables.

## 2. Stirling matrices of the second kind

For the Stirling numbers $s(i, j)$ and $S(i, j)$ of the first kind and of the second kind respectively, define $s_n$ and $S_n$ to be the $n \times n$ matrices by

$$(s_n)_{ij} = \begin{cases} s(i, j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$(S_n)_{ij} = \begin{cases} S(i, j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

We call the matrices $s_n$ and $S_n$ *Stirling matrix of the first kind* and *of the second kind*, respectively (see [5, p. 144]).

For example,

$$s_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 6 & 11 & 6 & 1 \end{bmatrix} \quad \text{and} \quad S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix}.$$

From now on, we will use the notation $\oplus$ for the direct sum of two matrices.

Using the definition of $S_n$, we can derive the following matrix representation from (1.1):

$$X_n = ([1] \oplus S_{n-1})F_n, \tag{2.1}$$

where $X_n = [1 x \ldots x^{n-1}]^{\mathrm{T}}$ and $F_n = [[x]_0 [x]_1 \ldots [x]_{n-1}]^{\mathrm{T}}$.

In this section, we mainly study Stirling matrix $S_n$ of the second kind since

$$S_n^{-1} = [(-1)^{i-j} s(i, j)] \;\text{ or }\; s_n^{-1} = [(-1)^{i-j} S(i, j)]. \tag{2.2}$$

First, we will discuss for a factorization of $S_n$.

For the $k \times k$ Pascal matrix $P_k$, we define the $n \times n$ matrix $\bar{P}_k$ by

$$\bar{P}_k = \begin{bmatrix} I_{n-k} & O \\ O & P_k \end{bmatrix}.$$

Thus, $\bar{P}_n = P_n$ and $\bar{P}_1$ is the identity matrix of order $n$.

**Lemma 2.1.** *For the $n \times n$ Pascal matrix $P_n$,*

$$S_n = P_n([1] \oplus S_{n-1}).$$

**Proof.** For each $i$ and $j$ with $i \geqslant j \geqslant 1$, since the $(i, j)$-entry of $[1] \oplus S_{n-1}$ is $S(i-1, j-1)$, from the definition of the matrix product and (1.4), we get

$$\begin{aligned}(P_n([1] \oplus S_{n-1}))_{ij} &= \sum_{l=j-1}^{i-1} p_{i\,l+1} S(l, j-1) \\ &= \sum_{l=j-1}^{i-1} \binom{i-1}{l} S(l, j-1) = S(i, j) = (S_n)_{ij}. \qquad \square\end{aligned}$$

For example,

$$S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}.$$

The following theorem is an immediate consequence of Lemma 2.1.

**Theorem 2.2.** *The Stirling matrix $S_n$ of the second kind can be factorized by the Pascal matrices $\bar{P}_k$'s:*

$$S_n = \bar{P}_n \bar{P}_{n-1} \cdots \bar{P}_2 \bar{P}_1.$$

For example,

$$S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

We now turn our attention to the special matrices which can be expressed by the Stirling matrices.

It is easy to see that Lemma 2.1 and (2.1) lead to

$$(x+1)^n = \sum_{k=0}^{n} S(n+1, k+1)[x]_k \tag{2.3}$$

for each $n = 0, 1, \ldots$ Thus (2.1) and (2.3) suggest how the Vandermonde matrix which is defined by the following way can be factorized.

Define $V_n(x)$ to be the $n \times n$ *Vandermonde matrix* by

$$V_n(x) := V_n(x, x+1, \ldots, x+n-1)$$

$$= \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x & x+1 & \cdots & x+n-1 \\ x^2 & (x+1)^2 & \cdots & (x+n-1)^2 \\ \vdots & \vdots & & \vdots \\ x^{n-1} & (x+1)^{n-1} & \cdots & (x+n-1)^{n-1} \end{bmatrix},$$

and use the definition of $[x]_n$ in (1.2) to define the $n \times n$ matrix $L_n$ by

$$(L_n)_{ij} = \begin{cases} [i-1]_{j-1} & \text{if } i \geqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

For example,

$$L_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 6 & 6 \end{bmatrix}.$$

By a simple computation we obtain

$$L_n = P_n D_n, \tag{2.4}$$

where $D_n = \operatorname{diag}(1, 1, 2!, \ldots, (n-1)!)$. Thus, we have

$$L_n^{-1} = D_n^{-1} P_n^{-1} = \left[ (-1)^{i-j} \frac{1}{(i-1)!} \binom{i-1}{j-1} \right].$$

Applying the binomial theorem, it is easy to show that for any real number $x$ and for the Pascal matrix $P_n$,

$$P_n V_n(x) = V_n(x+1).$$

Thus, we have

$$\det V_n(x) = \det V_n(x+1).$$

**Lemma 2.3.** *For the $n \times n$ Stirling matrix $S_n$ of the second kind,*

$$V_n(1) = S_n L_n^{\mathrm{T}}.$$

**Proof.** Applying (1.1) and (1.3) for each $i, j = 1, 2, \ldots, n$, we have

$$\begin{aligned} \left(S_n L_n^{\mathrm{T}}\right)_{ij} &= \sum_{k=1}^{i} S(i,k)[j-1]_{k-1} \\ &= \sum_{k=1}^{i} \{S(i-1,k-1) + kS(i-1,k)\}[j-1]_{k-1} \\ &= \sum_{k=1}^{i} \{S(i-1,k-1)[j-1]_{k-1} + S(i-1,k)k[j-1]_{k-1}\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{i} [S(i-1,k-1)[j-1]_{k-1} + S(i-1,k)\{[j]_k - [j-1]_k\}] \\
&= \sum_{k=1}^{i-1} S(i-1,k)[j]_k \\
&= j^{i-1} = (V_n(1))_{ij},
\end{aligned}
$$

since

$$
\sum_{k=1}^{i} S(i-1,k-1)[j-1]_{k-1} = \sum_{k=1}^{i} S(i-1,k)[j-1]_k,
$$

which completes the proof. $\square$

In the following theorem, we obtain the LDU factorization of $V_n(x)$ for any real number $x$.

**Theorem 2.4.** *For any real number $x$ and the generalized Pascal matrix $P_n[x]$ in* (1.6),

$$
V_n(x) = (P_n[x-1]S_n)D_n P_n^{\mathrm{T}}.
$$

**Proof.** From (2.4) and Lemma 2.3, for any real number $x$ we get

$$
\begin{aligned}
\left((P_n[x-1]S_n)L_n^{\mathrm{T}}\right)_{ij} &= (P_n[x-1]V_n(1))_{ij} \\
&= \sum_{k=0}^{i-1} \binom{i-1}{k} (x-1)^{i-1-k} j^k = (x+j-1)^{i-1} \\
&= (V_n(x))_{ij}. \qquad \square
\end{aligned}
$$

For example,

$$
\begin{aligned}
V_4(x) &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & x+1 & x+2 & x+3 \\ x^2 & (x+1)^2 & (x+2)^2 & (x+3)^2 \\ x^3 & (x+1)^3 & (x+2)^3 & (x+3)^3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x-1 & 1 & 0 & 0 \\ (x-1)^2 & 2(x-1) & 1 & 0 \\ (x-1)^3 & 3(x-1)^2 & 3(x-1) & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix}
\end{aligned}
$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x+1 & 1 & 0 \\ x^3 & 3x^2+3x+1 & 3x+3 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

**Corollary 2.5.** *For any real number $x$,*

$$\det V_n(x) = \prod_{k=0}^{n-1} k!$$

For the inverse of $V_n(x)$, from (1.7) and Theorem 2.4 we see that

$$V_n(x)^{-1} = P_n^{\mathrm{T}}[-1]D_n^{-1}S_n^{-1}P_n[1-x].$$

## 3. Some combinatorial identities

In this section, we obtain some well-known identities for a Stirling number from its matrix representation.

Applying Theorem 2.4 for $x = 1$, we obtain

$$S_n = V_n(1)\left(P_n^{-1}\right)^{\mathrm{T}} D_n^{-1}. \tag{3.1}$$

Computing the matrix product in (3.1) and comparing with the last row of $S_n$, we can obtain the following representation for $S(n,k)$, known as Stirling formula:

$$S(n,k) = \frac{1}{(k-1)!}\sum_{t=1}^{k}(-1)^{k-t}\binom{k-1}{t-1}t^{n-1} \quad (k = 1, 2, \ldots, n).$$

Again applying (1.7) and (2.2) to Lemma 2.1, since

$$s_n = ([1] \oplus s_{n-1}])P_n, \tag{3.2}$$

by a simple matrix product, it is easy to see that the Stirling number $s(n,k)$ of the first kind satisfies the following 'horizontal' recurrence relation which gives other explicit formula for the $s(n,k)$ (see [5, p. 215)]

$$s(n,k)=\sum_{l=k-1}^{n-1}\binom{l}{k-1}s(n-1,l).$$

Moreover, from Lemma 2.1 and (3.2), since

$$P_n=S_n([1]\oplus s_{n-1}) \quad\text{or}\quad P_n=([1]\oplus S_{n-1})s_n,$$

a binomial coefficient $\binom{n}{k}$ can be expressed by the Stirling numbers of the first kind and of the second kind as follows:

$$\binom{n}{k}=\sum_{t=k}^{n}(-1)^{t-k}S(n+1,t+1)s(t,k)$$

or

$$\binom{n}{k}=\sum_{t=k}^{n}(-1)^{n-t}S(n,t)s(t+1,k+1).$$

Finally, note that the *Bell number* $\omega(n)$ is defined by

$$\omega(n)=\sum_{k=1}^{n}S(n,k),\qquad n\geqslant 1.$$

By virtue of the matrix, the $i$th Bell number $\omega(i)$ is just the sum of the entries in the $i$th row of the Stirling matrix $S_n$ of the second kind. Thus, from Lemma 2.1 we get

$$P_n\begin{bmatrix}1 & \omega(1) & \cdots & \omega(n-1)\end{bmatrix}^{\mathrm{T}}=\begin{bmatrix}\omega(1) & \omega(2) & \cdots & \omega(n)\end{bmatrix}^{\mathrm{T}}.$$

More generally, if we note that for each $n=0,1,\ldots$

$$\Delta^m\omega(n)=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\omega(n+k)\quad(m=0,1,\ldots)$$

where $\omega(0):=1$ and $\Delta$ is the difference operator which is defined by

$$\Delta\omega(n)=\omega(n+1)-\omega(n)\quad\text{and}\quad\Delta^m\omega=\Delta(\Delta^{m-1}\omega)\ \ (m=2,3,\ldots),$$

by a simple matrix computation we get

$$\begin{aligned}&P_n\begin{bmatrix}\omega(0) & \omega(1) & \cdots & \omega(n-1)\\ \omega(1) & \omega(2) & \cdots & \omega(n)\\ \vdots & \vdots & \vdots & \vdots\\ \omega(n-1) & \omega(n) & \cdots & \omega(2n-2)\end{bmatrix}\\ &\quad=\begin{bmatrix}\omega(1) & \Delta\omega(1) & \cdots & \Delta^{n-1}\omega(1)\\ \omega(2) & \Delta\omega(2) & \cdots & \Delta^{n-1}\omega(2)\\ \vdots & \vdots & \vdots & \vdots\\ \omega(n) & \Delta\omega(n) & \cdots & \Delta^{n-1}\omega(n)\end{bmatrix}.\end{aligned}\tag{3.3}$$

From (3.3), for each $n$=1,2,... it is easy to establish the following identity:

$$\Delta^m \omega(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} \omega(m+k) \quad (m = 0, 1, \ldots, n-1), \tag{3.4}$$

where $\Delta^0\omega(n) := \omega(n)$.

In particular, from (3.4) we get the following well-known identities (see [5, pp. 210–211]):

$$\omega(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} \omega(k) \quad (n \geqslant 1)$$

and

$$\omega(n) = \Delta^n \omega(1).$$

## 4. Generalizations of the Stirling matrices

For any nonzero real number $x$, the $n \times n$ *generalized Stirling matrix of the first kind* $s_n[x]$ and *of the second kind* $S_n[x]$ are defined by

$$(s_n[x])_{ij} = \begin{cases} x^{i-j}s(i,j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(S_n[x])_{ij} = \begin{cases} x^{i-j}S(i,j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition, we see that $s_n[1] = s_n$ and $S_n[1] = S_n$.

Also, for the $k \times k$ generalized Pascal matrix $P_k[x]$ we define the $n \times n$ matrix $\bar{P}_k[x]$ by

$$\bar{P}_k[x] = \begin{bmatrix} I_{n-k} & O \\ O & P_k[x] \end{bmatrix}.$$

Since $s_n[1] = s_n$ and $S_n[1] = S_n$, it is easy to prove the following lemma.

**Lemma 4.1.** *Let $x$ be a nonzero real number. Then*
(a) $s_n^{-1}[x] = S_n[-x]$,
(b) $S_n^{-1}[x] = s_n[-x]$.

The following theorem follows from Lemmas 2.1 and 4.1.

**Theorem 4.2.** *Let $x$ be a nonzero real number. Then*
(a) $S_n[x] = P_n[x]([1] \oplus S_{n-1}[x])$,
(b) $S_n[x] = \bar{P}_n[x]\ \bar{P}_{n-1}[x] \ldots \bar{P}_2[x]\ \bar{P}_1[x]$,
(c) $S_n^{-1}[x] = \bar{P}_1[-x]\ \bar{P}_2[-x] \ldots \bar{P}_{n-1}[-x]\ \bar{P}_n[-x]$.

For example,

$$
\begin{aligned}
S_4[x] &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 3x & 1 & 0 \\ x^3 & 7x^2 & 6x & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x^2 & 2x & 1 & 0 \\ x^3 & 3x^2 & 3x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & x^2 & 3x & 1 \end{bmatrix}.
\end{aligned}
$$

Again, if we define the $n \times n$ matrices $t_n[x]$ and $T_n[x]$ by

$$(t_n[x])_{ij} = \begin{cases} x^{i+j-2}s(i,j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(T_n[x])_{ij} = \begin{cases} x^{i+j-2}S(i,j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise,} \end{cases}$$

it is easy to see that the following theorem holds by the similar arguments for $s_n[x]$ and $S_n[x]$.

**Theorem 4.3.** *Let $x$ be a nonzero real number. Then for the generalized Pascal matrices $P_n[x]$ and $Q_n[x]$ defined in* (1.6), *the following results hold*:

(a) $t_n^{-1}[x] = T_n\left[-\frac{1}{x}\right]$,

(b) $T_n^{-1}[x] = t_n\left[-\frac{1}{x}\right]$,

(c) $t_n[x] = ([1] \oplus s_{n-1}[x])Q_n[x]$,

(d) $T_n[x] = Q_n[x]\left([1] \oplus S_{n-1}\left[\frac{1}{x}\right]\right)$,

(e) $t_n[x] = \bar{P}_1[x]\bar{P}_2[x]\ldots\bar{P}_{n-1}[x]Q_n[x]$,

(f) $T_n[x] = Q_n[x]\bar{P}_{n-1}\left[\frac{1}{x}\right]\ldots\bar{P}_2\left[\frac{1}{x}\right]\bar{P}_1\left[\frac{1}{x}\right]$,

(g) $t_n^{-1}[x] = \bar{Q}_n\left[-\frac{1}{x}\right]\bar{P}_{n-1}[-x]\ldots\bar{P}_2[-x]\bar{P}_1[-x]$,

(h) $T_n^{-1}[x] = \bar{P}_1\left[-\frac{1}{x}\right]\bar{P}_2\left[-\frac{1}{x}\right]\ldots\bar{P}_{n-1}\left[-\frac{1}{x}\right]\bar{Q}_n\left[-\frac{1}{x}\right]$.

Furthermore, for any two nonzero real numbers $x$ and $y$ we define the $n \times n$ matrices $\Psi_n[x,y]$ and $\Omega_n[x,y]$ by

$$(\psi_n[x,y])_{ij} = \begin{cases} x^{i-j}y^{i+j-2}s(i,j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\omega_n[x,y])_{ij} = \begin{cases} x^{i-j}y^{i+j-2}S(i,j) & \text{if } i \geqslant j, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition, we see that

$$\begin{aligned} \Omega_n[x,1] &= S_n[x], \quad \Omega_n[1,y] = T_n[y], \\ \Psi_n[x,1] &= s_n[x], \quad \Psi_n[1,y] = t_n[y]. \end{aligned}$$

It is easy to see that the following theorems hold by the similar arguments for $s_n[x]$ and $S_n[x]$.

**Theorem 4.4.** *Let $x$ and $y$ be any nonzero real numbers. Then for the extended generalized Pascal matrix $\Phi_n[x,y]$ defined in* (1.5), *the following results hold*:

(a) $\Omega_n[-x,y] = \Omega_n[x,-y]$,

(b) $\Psi_n[-x,y] = \Psi_n[x,-y]$,

(c) $\Omega_n^{-1}[x,y] = \Psi_n\left[-x,\frac{1}{y}\right] = \Psi_n\left[x,-\frac{1}{y}\right]$,

(d) $\Psi_n^{-1}[x,y] = \Omega_n\left[-x,\frac{1}{y}\right] = \Omega_n\left[x,-\frac{1}{y}\right]$,

(e) $\Omega_n[x,y] = \Phi_n[x,y]\left([1]\oplus S_{n-1}\left[\frac{x}{y}\right]\right)$,

(f) $\Psi_n[x,y] = ([1]\oplus s_{n-1}[xy])\Phi_n[x,y]$,

(g) $\Omega_n[x,y] = \Phi_n[x,y]\bar{P}_{n-1}\left[\frac{x}{y}\right]\ldots\bar{P}_2\left[\frac{x}{y}\right]\bar{P}_1\left[\frac{x}{y}\right]$,

(h) $\Psi_n[x,y] = \bar{P}_1[xy]\bar{P}_2[xy]\ldots\bar{P}_{n-1}[xy]\Phi_n[x,y]$,

(i) $\Psi_n^{-1}[x,y] = \Phi_n\left[x,-\frac{1}{y}\right]\bar{P}_{n-1}[-xy]\ldots\bar{P}_2[-xy]\bar{P}_1[-xy]$,

(j) $\Omega_n^{-1}[x,y] = \bar{P}_1\left[-\frac{x}{y}\right]\bar{P}_2\left[-\frac{x}{y}\right]\ldots\bar{P}_{n-1}\left[-\frac{x}{y}\right]\Phi_n\left[x,-\frac{1}{y}\right]$.

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