Associated Lah numbers and r-Stirling numbers

Hacène Belbachir and Imad Eddine Bousbaa
USTHB, Faculty of Mathematics
RECITS Laboratory, DG-RSDT
BP 32, El Alia, 16111, Bab Ezzouar, Algiers, Algeria
hbelbachir@usthb.dz & ibousbaa@usthb.dz

May 13, 2014

Abstract

We introduce the associated Lah numbers. Some recurrence relations and convolution identities are established. An extension of the associated Stirling and Lah numbers to the r-Stirling and r-Lah numbers are also given. For all these sequences we give combinatorial interpretation, generating functions, recurrence relations, convolution identities. In the sequel, we develop a section on nested sums related to binomial coefficient.

AMS Classification: 11B73, 05A19.

Keywords: Generating functions; Associated Stirling and Lah numbers; Convolution; Combinatorial interpretation; Recurrence relation.

1 Introduction

The Stirling numbers of the first and second kind, denoted respectively $\left[{n \atop k}\right]$ and $\left\{{n \atop k}\right\}$, are defined by

$$x(x+1)\cdots(x+n-1)=\sum_{k=0}^{n}\left[{n \atop k}\right]x^{k}, \tag{1}$$

and

$$x^{n}=\sum_{k=0}^{n}\left\{{n \atop k}\right\}x(x-1)\cdots(x-k+1). \tag{2}$$

It is well known that $\left[{n \atop k}\right]$ is the number of permutations of the set $Z_n := \{1,2,\ldots,n\}$ with k cycles and that $\left\{{n \atop k}\right\}$ is the number of partitions of the set Z_n into k non empty subsets [17, Ch. 5], [23, Ch. 4].

The Lah numbers $\left\lfloor{n \atop k}\right\rfloor$ (Stirling numbers of the third kind), see [19, pp. 44], are defined as the sum of products of the Stirling number of the first kind and the Stirling numbers of the second kind

$$\left\lfloor{n \atop k}\right\rfloor=\sum_{j=k}^{n}\left[{n \atop j}\right]\left\{{j \atop k}\right\}, \tag{3}$$

and count the number of partitions of the set Z_n into k ordered lists. According to 1 and 2, they satisfy

$$x(x+1)\cdots(x+n-1)=\sum_{k=0}^{n}\left\lfloor{n \atop k}\right\rfloor x(x-1)\cdots(x-k+1),$$

see for instance [3, eq 8].

Broder [12] gives a generalization of the Stirling numbers of the first and second kind the so-called r-Stirling numbers of the first and second kind, denoted respectively $\left[{n \atop k}\right]_r$ and $\left\{{n \atop k}\right\}_r$, by adding restriction on the elements of Z_n : the $\left[{n \atop k}\right]_r$ is the number of permutations of the set Z_n with k cycles such that the r first elements are in distinct cycles and the $\left[{n \atop k}\right]_r$ is the number of partitions of the set Z_n into k subsets such that the r first elements are in distinct subsets. The r-Lah numbers $\left\lfloor{n \atop k}\right\rfloor_r$, see [3], count the number of partitions of the set Z_n into k ordered lists such that the r first elements are in distinct lists.

These three sequences satisfy respectively the following recurrence relations

$$\left[{n \atop k}\right]_r = \left[{n-1 \atop k-1}\right]_r + (n-1)\left[{n-1 \atop k}\right]_r, \tag{4}$$

$$\left\{{n \atop k}\right\}_r = \left\{{n-1 \atop k-1}\right\}_r + k\left\{{n-1 \atop k}\right\}_r, \tag{5}$$

$$\left\lfloor{n \atop k}\right\rfloor_r = \left\lfloor{n-1 \atop k-1}\right\rfloor_r + (n+k-1)\left\lfloor{n-1 \atop k}\right\rfloor_r. \tag{6}$$

with $\left[{n \atop k}\right]_r = \left\{{n \atop k}\right\}_r = \left\lfloor{n \atop k}\right\rfloor_r = \delta_{n,k}$ for $k = r$, where δ is the Kronecker delta, and $\left[{n \atop k}\right]_r = \left\{{n \atop k}\right\}_r = \left\lfloor{n \atop k}\right\rfloor_r = 0$ for $n < r$.

For $r = 1$ and $r = 0$, these numbers coincide with the classical Stirling numbers of both kinds and with the classical Lah numbers.

Comtet [17, pp. 222] define an other generalization of the Stirling numbers of both kinds by adding a restriction on the number of elements by cycle or subset and call them, for $s \geqslant 1$, the s-associated Stirling numbers of the first kind $\left[{n \atop k}\right]^{(s)}$ and of the second kind $\left\{{n \atop k}\right\}^{(s)}$. The $\left[{n \atop k}\right]^{(s)}$ is the number of permutations of the set Z_n with k cycles such that, each cycle has at least s elements. The $\left\{{n \atop k}\right\}^{(s)}$ is the number of partitions of the set Z_n into k subsets such that, each subset has at least s elements. They have, each one, an explicit formula, see for instance [20, Eq 4.2, Eq 4.9]:

$$\left[{n \atop k}\right]^{(s)} = \frac{n!}{k!} \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ i_j \geq s}} \frac{1}{i_1 i_2 \cdots i_k}, \tag{7}$$

$$\left\{{n \atop k}\right\}^{(s)} = \frac{n!}{k!} \sum_{\substack{i_1+i_2+\cdots+i_k=n \\ i_j \geq s}} \frac{1}{i_1! i_2! \cdots i_k!}. \tag{8}$$

The generating functions are respectively

$$\sum_{n \geq sk} \left[{n \atop k}\right]^{(s)} \frac{x^n}{n!} = \frac{1}{k!}\left(-\ln(1-x) - \sum_{i=1}^{s-1} \frac{x^i}{i}\right)^k, \tag{9}$$

$$\sum_{n \geq sk} \left\{{n \atop k}\right\}^{(s)} \frac{x^n}{n!} = \frac{1}{k!}\left(\exp(x) - \sum_{i=0}^{s-1} \frac{x^i}{i!}\right)^k. \tag{10}$$

For $s = 2$, these numbers are reduced to the specific associated Stirling numbers of both kinds, see for instance [23, pp. 73].

Note that, from (7) and (8), for $n = sk$, we get

$$\left[{sk \atop k}\right]^{(s)} = \frac{(sk)!}{k! s^k} \quad \text{and} \quad \left\{{sk \atop k}\right\}^{(s)} = \frac{(sk)!}{k! (s!)^k}. \tag{11}$$

Ahuja and Enneking [1] give a generalization of the Lah numbers called the associated Lah numbers using an analytic approach. In Sloane [24, A076126], we have a definition of the associated Lah numbers $\left\lfloor{n \atop k}\right\rfloor^{(2)}$ as the number of partitions of the set Z_n into k ordered lists such that each list has at least 2 elements. They satisfy the following explicit formula

$$\left\lfloor {n \atop k} \right\rfloor^{(2)} = \frac{n!}{k!}\binom{n-k-1}{k-1}, \tag{12}$$

and have the double generating function

$$\sum_{n\geqslant 2}\sum_{k=1}^{\lfloor n/2\rfloor} \left\lfloor {n \atop k} \right\rfloor^{(2)} y^k \frac{x^n}{n!} = \exp\left(y\frac{x^2}{1-x}\right) - 1, \tag{13}$$

they consider $k \geqslant 1$, which means there is at least one part.

Hsu and Shiue [22] defined a Stirling-type pair $\{S^1, S^2\}$ as a unified approach to the Stirling numbers, this approach generalize degenerate Stirling numbers [13], Weighted Stirling numbers [14, 15], r-Whitney numbers [9, 16] and many other ones. The authors and Belkhir in [4] and the authors in [7] give a combinatorial approach to special cases of the Stirling-type pair. Howard [21] extend the associated generalization to the Weighted Stirling numbers. Note that the Stirling-type pair does not generalize the associated Stirling numbers. Motivated by this, we introduce and develop the s-associated Lah numbers and the s-associated r-Stirling numbers.

In section 2, we define the s-associated Lah numbers $\left\lfloor {n \atop k} \right\rfloor^{(s)}$, $n \geqslant sk$, by a combinatorial approach analogous to Comtet's generalization. We derive an explicit formula, a triangular recurrence relation, a combinatorial identity and some generating functions. We study, in section 3, some nested sums related to binomial coefficients in order to develop, in section 4, a generalization of the Stirling numbers of the three kinds using the two restrictions (Broder's and Comtet's ones), we call them respectively the s-associated r-Stirling numbers of the first kind $\left[{n \atop k} \right]_r^{(s)}$, the s-associated r-Stirling numbers of the second kind $\left\{ {n \atop k} \right\}_r^{(s)}$ and the s-associated r-Lah numbers $\left\lfloor {n \atop k} \right\rfloor_r^{(s)}$. We give some recurrence relations and combinatorial identities in sections 5 and 6. Cross recurrences and convolution identities are established in sections 7 and 8. In section 9, we propose some generating functions of the s-associated r-Stirling numbers.

2 The s-associated Lah numbers

We start by introducing the s-associated Lah numbers.

Definition 1 *The s-associated Lah number, denoted by $\left\lfloor {n \atop k} \right\rfloor^{(s)}$, is the number of partitions of Z_n into k order lists such that each list contains at least s elements.*

Theorem 2 *The s-associated Lah numbers obey to the following 'triangular' recurrence relation, for $n \geqslant sk$,*

$$\left\lfloor {n \atop k} \right\rfloor^{(s)} = \binom{n-1}{s-1} s! \left\lfloor {n-s \atop k-1} \right\rfloor^{(s)} + (n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor^{(s)}, \tag{14}$$

with $\left\lfloor {n \atop 0} \right\rfloor^{(s)} = \delta_{n,0}$ for $k = 0$, where δ is the Kronecker delta, and $\left\lfloor {n \atop k} \right\rfloor^{(s)} = 0$ for $n < sk$

Proof. Let us consider the n^{th} elements, if it belongs to a list containing exactly s elements, so we have $\binom{n-1}{s-1}$ ways to choose the remaining $(s-1)$ elements and $s!$ ways to order them into the cited list, then distribute the $(n-s)$ remaining elements into the $(k-1)$ remaining lists such that each list have at least s elements and we have $\left\lfloor {n-s \atop k-1} \right\rfloor^{(s)}$ ways to do it. Thus, we get $\binom{n-1}{s-1} s! \left\lfloor {n-s \atop k-1} \right\rfloor^{(s)}$ possibilities. Else, we consider all the possibilities of ordering $(n-1)$ elements into k lists under the usual condition which can be done by $\left\lfloor {n-1 \atop k} \right\rfloor^{(s)}$ ways, then add the n^{th} elements next to an other and we have $n-1$ possibilities, or as head of each list and we have k possibilities, this gives $(n+k-1)\left\lfloor {n-1 \atop k} \right\rfloor^{(s)}$ possibilities. □

For $s=1$ and $s=2$, we get Lah numbers and associated Lah numbers respectively.

For $s=3$, we obtain the following table, for $n \leq 15$,

$n\backslash k$	1	2	3	4	5
3	6				
4	24				
5	120				
6	720	360			
7	5040	5040			
8	40320	60480			
9	362880	725760	60480		
10	3628800	9072000	1814400		
11	39916800	119750400	39916800		
12	479 001 600	1676 505 600	798 336 000	19 958 400	
13	6227 020 800	24 908 083 200	15 567 552 000	1037 836 800	
14	87 178 291 200	392 302 310 400	305 124 019 200	36 324 288 000	
15	1307 674 368 000	6538 371 840 000	6102 480 384 000	1089 728 640 000	10 897 286 400

The following result gives an explicit formula for the s-associated Lah numbers according to identities (7) and (8) for the s-associated Stirling numbers of both kinds.

Theorem 3 *Let s, k and n be nonnegative integers such that $n \geqslant sk$, we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{(s)} = \frac{n!}{k!}\binom{n-(s-1)k-1}{k-1}. \tag{15}$$

Proof 1. We order n elements on k ordered lists such that, each list contains at least s elements: first, we suppose that the lists are labeled $1, \ldots, k$ and for each list j we choose (i_j+s) $(0 \leqslant i_j \leqslant n-s)$ elements, we have $\binom{n}{i_1+s, i_2+s, \ldots, i_k+s}$ possibilities to constitute the k groups. The arrangement of the j^{th} subset gives $(i_j+s)!$ possibilities. It gets $\sum_{i_1+i_2+\cdots+i_k=n-sk}\binom{n}{i_1+s, i_2+s, \ldots, i_k+s}(i_1+s)!(i_2+s)!\cdots(i_k+s)! = n!\binom{n-(s-1)k-1}{k-1}$, we divide by $k!$ to unlabeled the lists. □

Proof 2. First we choose k elements to identify the k lists with $\binom{n}{k}$ possibilities, then we choose k groups of $s-1$ elements to retch the condition of having s elements by list and we have $\binom{n-k}{s-1}$ possibilities for the first list, and $\binom{n-k-(s-1)}{s-1}$ possibilities for the second one, and so on ... the last list have $\binom{n-k-(s-1)(k-1)}{s-1}$ possibilities. So we get $\binom{n-k}{s-1}\binom{n-k-(s-1)}{s-1}\cdots\binom{n-k-(s-1)(k-1)}{s-1} = \binom{n-k}{s-1, s-1, \ldots, s-1, n-sk}$ possibilities. We affect the remaining $n-sk$ elements to the lists and we have k ways for the first element, $k+1$ ways for the second one, and so on ... the last one have $k+n-sk-1 = n-(s-1)k-1$. So, we get $k(k+1)\cdots(n-(s-1)k-1) = \frac{(n-(s-1)k-1)!}{(k-1)!}$ possibilities. The result follows. □

Remark 4 *For $n = sk$, we get the following according to relations given by (11)*

$$\left\lfloor \begin{matrix} sk \\ k \end{matrix} \right\rfloor^{(s)} = \frac{(sk)!}{k!}. \tag{16}$$

Comparing to (14), an other recurrence relation, with rational coefficients, can be deduced form the explicit formula 15, as follows.

Theorem 5 *The s-associated Lah numbers satisfy the following recurrence relation*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor^{(s)} = \frac{n!}{(n-s)!k}\left\lfloor \begin{matrix} n-s \\ k-1 \end{matrix} \right\rfloor^{(s)} + n\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor^{(s)}. \tag{17}$$

Proof. Using Pascal's formula and relation (15), we get the result. □

Note that for $s = 1$, we get the relation given by the authors [8, eq 7] when $r = 0$. . The s-associated Lah numbers can be expressed as a Vandermonde type formula as follows.

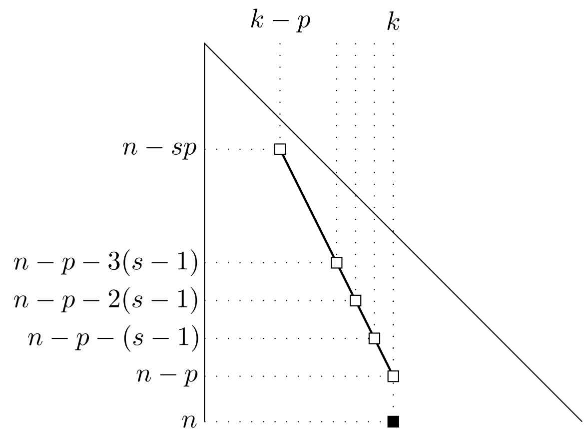

Figure 1: Value of $\left\lfloor {n \atop k} \right\rfloor^{(s)}$ (in black) as a inner product of a periodic sequence of elements of the same table (in white) with a sequence deriving from binomial coefficient.

Theorem 6 *The s-associated Lah numbers satisfy*

$$\left\lfloor {n \atop k} \right\rfloor^{(s)} = \frac{n!}{k!} \sum_{i=0}^{p} \frac{(k-i)!}{(n-p-(s-1)i)!} \binom{p}{i} \left\lfloor {n-(s-1)i-p \atop k-i} \right\rfloor^{(s)}. \tag{18}$$

Proof. Using the explicit formula (15) and the Vandermonde formula, we get the result. □

The special case $s = 1$ gives the identity given by the authors [8, eq 6] when $r = 1$.

The s-associated Lah numbers satisfy the following vertical recurrence relation.

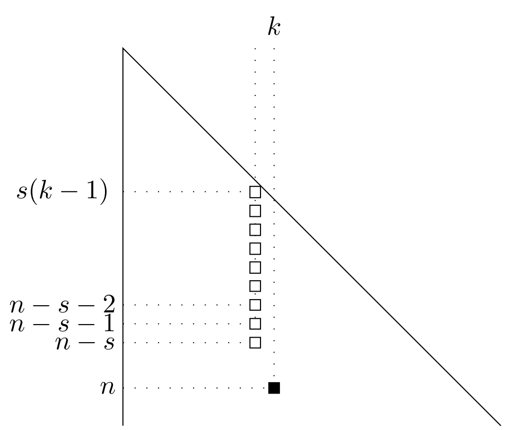

Figure 2: linear vertical recurrence relation

Theorem 7 *Let s, k and n be nonnegative integers such that $n \geqslant sk$, we have*

$$\left\lfloor {n \atop k} \right\rfloor^{(s)} = \sum_{i=s(k-1)}^{n-s} (n-i)! \binom{n-1}{i} \left\lfloor {i \atop k-1} \right\rfloor^{(s)}. \tag{19}$$

Proof. Let us consider the $(k-1)$ first lists, they contain i ($s(k-1) \leqslant i \leqslant n-s$) elements. So, we choose the i elements and we have $\binom{n-1}{i}$ ways to do it, and constitute the $k-1$ lists such that each list have at least s elements, which can be done by $\left\lfloor {i \atop k-1} \right\rfloor^{(s)}$ ways, then order the $(n-i)$ remaining elements in a list with $(n-i)!$ possibilities. We conclude by summing over i. □

The exponential generating function of the s-associated Lah numbers is given by the following. It is a complement list to (9) and (10).

Theorem 8 *Let n, k and s be integers, we have*

$$\sum_{n \geq sk} \left\lfloor {n \atop k} \right\rfloor^{(s)} \frac{x^n}{n!} = \frac{1}{k!} \left(\frac{x^s}{1-x} \right)^k. \tag{20}$$

Proof. Using the explicit formula (15), with the following identity for $x \in \mathbb{N}$, see for instance [18],

$$\sum_{n \geq 0} \binom{n+x}{x} t^n = \left(\frac{1}{1-t} \right)^{x+1},$$

we get the result. □

According to identity (13), the double generating function is given by

Theorem 9 *We have*

$$\sum_{n \geq sk} \sum_{k=0}^{\lfloor n/s \rfloor} \left\lfloor {n \atop k} \right\rfloor^{(s)} y^k \frac{x^n}{n!} = \exp\left\{ \frac{x^s}{1-x} y \right\}. \tag{21}$$

Proof. Interchanging the order of summation and using equation (20), we get the result. □

3 Nested sums related to binomial coefficients

In this section, we evaluate some symmetric functions. We start by the following result.

Lemma 10 *Let α and β be integers such that $\beta \geqslant \alpha$. We have*

$$\sum_{n \geq 0} \binom{n+\alpha}{\beta} z^n = \frac{z^{\beta-\alpha}}{(1-z)^{\beta+1}}. \tag{22}$$

Proof. $\sum\limits_{n \geq 0} \binom{n+\alpha}{\beta} z^n = \left(\sum\limits_{n \geq 0} \binom{n+\beta}{\beta} z^n \right) z^{\beta-\alpha} = \frac{z^{\beta-\alpha}}{(1-z)^{\beta+1}}.$ □

The following result seems to be nice as an independent one.

Theorem 11 *Let $\alpha_1, \ldots, \alpha_r, \alpha, \beta_1, \ldots, \beta_r, \beta, k_1, \ldots, k_r$ and k be integers such that $\alpha_1 + \cdots + \alpha_r = \alpha$, $\beta_1 + \cdots + \beta_r = \beta$ and $k_1 + \cdots + k_r = k$ with $\beta_i \geqslant \alpha_i$. The following identity holds*

$$\sum_{k_1 + \cdots + k_r = k} \binom{k_1+\alpha_1}{\beta_1} \cdots \binom{k_r+\alpha_r}{\beta_r} = \binom{k+\alpha+r-1}{\beta+r-1}. \tag{23}$$

Proof. By induction over r, we get the result. So It suffices to do the proof for $r=2$. Thus, we have to establish

$$\sum_{k_1+k_2=k} \binom{k_1+\alpha_1}{\beta_1} \binom{k_2+\alpha_2}{\beta_2} = \binom{k+\alpha+1}{\beta+1}. \tag{24}$$

We consider the following product $\sum\limits_{n\geq 0}\sum\limits_{k_1+k_1=k}\binom{k_1+\alpha_1}{\beta_1}\binom{k_2+\alpha_2}{\beta_2}t^n=\left(\sum\limits_{n\geq 0}\binom{k_1+\alpha_1}{\beta_1}t^n\right)\left(\sum\limits_{n\geq 0}\binom{k_2+\alpha_2}{\beta_2}t^n\right)$ using (22), we get $\frac{t^{\beta_1-\alpha_1}}{(1-t)^{\beta_1+1}}\frac{t^{\beta_2-\alpha_1}}{(1-t)^{\beta_2+1}}=\sum\limits_k\binom{k+\alpha+1}{\beta+1}t^k$. □

As a consequence, we evaluate the sum of all possible integer products having the same summation.

Corollary 12 *For $\alpha_i=0$ and $\beta_i=1$ we get*

$$\sum_{k_1+\cdots+k_r=n}k_1k_2\cdots k_r=\binom{n+r-1}{2r-1}. \tag{25}$$

The above identity can be interpreted as the number of ways to choose r leaders of r groups constituted from n persons: we choose one person of each group and we have $\binom{k_1}{1}\cdots\binom{k_r}{1}$ ways to do it. This is equivalent to choose r persons and $(r-1)$ separators from the n persons and the $r-1$ separators and we have $\binom{n+r-1}{2r-1}$ ways to do it.

Now, we are able to produce a general result. Also, it will be used to establish the next theorem.

Corollary 13 *Let r,p and k be integers such that $r\geqslant p$, we have*

$$\sum_{k_1+\cdots+k_p+\cdots+k_r=n}k_1k_2\cdots k_p=\binom{n+r-1}{r+p-1}. \tag{26}$$

Proof.

$$\sum_{k_1+\cdots+k_p+\cdots+k_r=k}k_1k_2\cdots k_p=\sum_{m=0}^{k}\left(\sum_{k_1+\cdots+k_p=m}k_1k_2\cdots k_p\right)\sum_{k_{p+1}+\cdots+k_r=k-m}1$$

using identity (25) and $\sum\limits_{i_1+i_2+\cdots+i_r=m}1=\binom{m+r-1}{r-1}$ we get

$$\sum_{k_1+\cdots+k_p+\cdots+k_r=k}k_1k_2\cdots k_p=\sum_{m=0}^{k}\binom{m+p-1}{2p-1}\binom{k-m+r-p-1}{r-p-1},$$

applying relation (24), we get the result. □

Now, we are able to evaluate the sum of all products of k terms, all translated by α, and having a fixed summation.

Theorem 14 *We have*

$$\sum_{i_1+\cdots+i_k=n}(i_1+\alpha)(i_2+\alpha)\cdots(i_k+\alpha)=\sum_{i=0}^{k}\binom{k}{j}\binom{n+k-1}{n-j}\alpha^{k-j}. \tag{27}$$

Proof. We have

$$\sum_{i_1+\cdots+i_k=n}(i_1+\alpha)\cdots(i_k+\alpha)=\sum_{j=0}^{k}\binom{k}{j}I_{k,k-j}\alpha^j,$$

where $I_{k,j}:=\sum\limits_{i_1+i_2+\cdots+i_k=n}i_1i_2\cdots i_j$ and from (26), we get the result. □

This nice result will be used to evaluate the explicit formula of the s-associated r-Stirling numbers which are introduced in the following section.

4 The s-associated r-Stirling numbers of the both kinds and the s-associated r-Lah numbers

Now, we introduce the s-associated r-Stirling numbers of the both kinds and the s-associated r-Lah numbers.

Definition 15 *The s-associated r-Stirling numbers of the first kind count the number of permutations of the set Z_n with k cycles such that the r first elements are in distinct cycles and each cycle contains at least s elements.*

The s-associated r-Stirling numbers of the second kind count the number of partitions of the set Z_n into k subsets such that the r first elements are in distinct subsets and each subset contains at least s elements.

The s-associated r-Lah numbers, called also the s-associated r-Stirling numbers of the third kind, count the number of partitions of the set Z_n into k ordered lists such that the r first elements are in distinct lists and each list contains at least s elements.

Here is given, for each kind, the table for $r = s = 2$.

$n\backslash k$	2	3	4	5	6	7
4	2					
5	12					
6	72	12				
7	480	160				
8	3600	1740	90			
9	30 240	18 648	2100			
10	282 240	207 648	35 840	840		
11	2903 040	2446 848	560 448	30 240		
12	32 659 200	30 702 240	8641 080	743 400	9450	
13	399 168 000	410 731 200	135 519 120	15 935 920	485 100	
14	5269 017 600	5852 753 280	2194 121 952	324 416 400	16 216 200	124 740
15	74 724 249 600	88 663 610 880	36 941 553 792	6522 721 920	455 975 520	8648 640
16	1133 317 785 600	1424 644 865 280	649 046 990 592	132 205 465 392	11 835 944 120	377 116 740

Table 1: Some values for the 2-associated 2-Stirling numbers of the first kind

$n\backslash k$	2	3	4	5	6	7	8
4	2						
5	6						
6	14	12					
7	30	80					
8	62	360	90				
9	126	1372	1050				
10	254	4788	7700	840			
11	510	15 864	45 612	15 120			
12	1022	50 880	239 190	163 800	9450		
13	2046	159 764	1161 270	1389 080	242 550		
14	4094	494 604	5353 392	10 182 480	3638 250	124 740	
15	8190	1516 528	23 800 920	67 822 040	41 771 730	4324 320	
16	16 382	4619 160	103 096 994	422 534 112	407 246 840	85 765 680	1891 890
17	32 766	14 004 876	438 124 050	2507 785 280	3555 852 300	1280 178 900	85 135 050

Table 2: Some values of the 2-associated 2-Stirling numbers of the second kind

$n\backslash k$	2	3	4	5	6	7
4	8					
5	72					
6	600	96				
7	5280	1920				
8	50 400	29 520	1440			
9	524 160	428 400	50 400			
10	5927 040	6249 600	1229 760	26 880		
11	72 576 000	93 985 920	26 490 240	1451 520		
12	958 003 200	1473 292 800	546 134 400	51 408 000	604 800	
13	13 571 712 000	24 189 580 800	11 176 704 000	1536 796 800	46 569 600	
14	205 491 686 400	416 731 392 000	231 357 772 800	42 471 475 200	2255 299 200	15 966 720
15	3312 775 065 600	7534 695 168 000	4894 438 348 800	1133 317 785 600	89 253 964 800	1660 538 880

Table 3: Some values for the 2-associated 2-Lah numbers

The s-associated r-Stirling numbers of the three kinds have the following explicit formulas.

Theorem 16 *For $n \geqslant sk$ and $k \geqslant r$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(s)} = \frac{(n-r)!}{(k-r)!} \sum_{m=0}^{n-sk} \binom{m+r-1}{r-1} \sum_{i_1+\cdots+i_{k-r}=n-sk-m} \frac{1}{(i_{r+1}+s)\cdots(i_k+s)}, \tag{28}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(s)} = \frac{(n-r)!}{(k-r)!} \sum_{i_1+\cdots+i_k=n-sk} \frac{1}{(i_1+s-1)!\cdots(i_r+s-1)!(i_{r+1}+s)!\cdots(i_k+s)!}, \tag{29}$$

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(s)} = \frac{(n-r)!}{(k-r)!} \sum_{j=0}^{r} \binom{r}{j} \binom{n-(s-1)\,k-1}{k+j-1} s^{r-j}. \tag{30}$$

Proof. We first proof the identity (29). To constitute a partition of Z_n into k parts such that each part has at least s elements and the r first elements are in distinct parts, we proceed as follows : we put the r first elements in r parts (one by part). Now we partition the $n-r$ remaining elements into k parts such that r parts have at least $s-1$ elements and $k-r$ parts have at least s elements, and we have $\frac{1}{(k-r)!}\binom{n-r}{i_1,i_2,\ldots,i_k}$ ways to do it, with $i_j \geqslant s-1$ for $j = 1,\ldots,r$ and $i_j \geqslant s$ for $j = r+1,\ldots,k$, which gives identity (29).

With the same specifications used to establish relation (29), to count the number of permutations of Z_n into k cycles it suffice to constitute the cycles by considering all the possible arrangement in the parts and we have $i_1!i_2!\cdots i_r!(i_{r+1}-1)!\cdots(i_k-1)!$ ways. So we can write:

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix}_r^{(s)} &= \frac{1}{(k-r)!} \sum_{i_1+\cdots+i_k=n-r} \binom{n-r}{i_1,\ldots,i_k} i_1!\cdots i_r!(i_{r+1}-1)!\cdots(i_k-1)!, \\
&= \frac{(n-r)!}{(k-r)!} \sum_{m=r(s-1)}^{n-r-s(k-r)} \sum_{\substack{i_{r+1}+\cdots+i_k=n-r-m \\ i_j\geqslant s}} \frac{1}{i_{r+1}\cdots i_k} \sum_{\substack{i_1+\cdots+i_r=m \\ i_j\geqslant s-1}} 1, \\
&= \frac{(n-r)!}{(k-r)!} \sum_{m=0}^{n-sk} \sum_{i_{r+1}+\cdots+i_k=n-sk-m} \frac{1}{(i_{r+1}+s)\cdots(i_k+s)} \sum_{i_1+\cdots+i_r=m} 1,
\end{aligned}$$

finally using $\sum_{i_1+\cdots+i_r=m} 1 = \binom{m+r-1}{r-1}$, we get identity (28).

The same approach works, to constitute partitions of Z_n into k ordered lists we have to consider the arrangement in the parts and we have $(i_1+1)!\,(i_2+1)!\cdots(i_r+1)!i_{r+1}!\cdots i_k!$ ways to do it. Thus we get

$$\begin{aligned}
\left\lfloor{n \atop k}\right\rfloor_r^{(s)} &= \frac{1}{(k-r)!}\sum_{i_1+\cdots+i_k=n-r}\binom{n-r}{i_1,\ldots,i_k}(i_1+1)!\cdots(i_r+1)!i_{r+1}!\cdots i_k! \qquad (31)\\
&= \frac{(n-r)!}{(k-r)!}\sum_{m=s(k-r)}^{n-sr}\ \sum_{\substack{i_1+\cdots+i_r=n-r-m\\ i_j\geqslant s-1}}(i_1+1)\cdots(i_r+1)\sum_{\substack{i_{r+1}+\cdots+i_k=m\\ i_j\geqslant s}}1\\
&= \frac{(n-r)!}{(k-r)!}\sum_{m=0}^{n-sk}\ \sum_{\substack{i_1+\cdots+i_r=n-sk-m\\ i_j\geqslant 0}}(i_1+s)\cdots(i_r+s)\sum_{\substack{i_{r+1}+\cdots+i_k=m\\ i_j\geqslant 0}}1,
\end{aligned}$$

using relation (27), we get

$$\left\lfloor{n \atop k}\right\rfloor_r^{(s)} = \frac{(n-r)!}{(k-r)!}\sum_{j=0}^{r}\binom{r}{j}\sum_{m=0}^{n-sk}\binom{m+k-r-1}{k-r-1}\binom{n-sk-m+r-1}{r-1+j}s^{r-j},$$

using relation (24), we get identity (30). □

From (31), we can write a second kind explicit formula according to (27) and generalizing relation (15).

$$\left\lfloor{n \atop k}\right\rfloor_r^{(s)} = \frac{(n-r)!}{(k-r)!}\sum_{i_1+i_2+\cdots+i_k=n-sk}(i_1+s)\,(i_2+s)\cdots(i_r+s)\,. \qquad (32)$$

The precedent theorem works for $k=r$. Furthermore, the identities are more explicit.

Remark 17 *For $k=r$, we get respectively*

$$\left[{n \atop r}\right]_r^{(s)} = (n-r)!\binom{n-r\,(s-1)-1}{r-1}, \qquad (33)$$

$$\left\{{n \atop r}\right\}_r^{(s)} = \sum_{\substack{i_1+i_2+\cdots+i_r=n-r\\ i_l\geqslant s-1}}\binom{n-r}{i_1,i_2,\ldots,i_r}, \qquad (34)$$

$$\left\lfloor{n \atop r}\right\rfloor_r^{(s)} = (n-r)!\sum_{i=0}^{r}\binom{r}{i}\binom{n-(s-1)r-1}{r+i-1}s^{r-i}. \qquad (35)$$

The following special values can be easily computed, extending relations given by relations (11) and (16)

$$\left[{sk \atop k}\right]_r^{(s)} = \frac{(n-r)!}{(k-r)!s^k}, \qquad (36)$$

$$\left\{{sk \atop k}\right\}_r^{(s)} = \frac{(n-r)!}{(k-r)!(s-1)!^r s!^{k-r}}, \qquad (37)$$

$$\left\lfloor{sk \atop k}\right\rfloor_r^{(s)} = \frac{(n-r)!}{(k-r)!s^r}. \qquad (38)$$

Here is given an other explicit formula of the s-associated r-Lah numbers. This one is more interesting than relation (32). It is evaluated using one summation

Theorem 18 *Let n,k,r and s be nonnegative integers such that $k\geqslant r$ and $n\geqslant sk$, we have*

$$\left\lfloor{n \atop k}\right\rfloor_r^{(s)} = \frac{(n-r)!}{(k-r)!}\sum_{j=0}^{r}\binom{r}{j}\binom{n+j-(s-1)k-1}{k+j-1}(s-1)^{r-j}. \qquad (39)$$

Proof. To constitute the k lists we use the r first elements which are supposed in different lists to identify the r first lists and we choose $k-r$ elements form the remaining elements, with $\binom{n-r}{k-r}$ possibilities, as head list of the $k-r$ remaining lists. Now to retch the condition that in each list we have at least s elements, we constitute k groups of $(s-1)$ elements form the $n-k$ remaining elements and we have $\binom{n-k}{s-1,\ldots,s-1,n-sk}$ ways to do it, and consider all the permutations of each group so we get $((s-1)!)^k$ possibilities. Now, for the r first elements we suppose that j of them are head lists so we choose them with $\binom{r}{j}$ ways and order the $r-j$ elements after an element of each group and we have $(s-1)^{r-j}$ possibilities. It remains to affect the $n-sk$ remaining elements, so the first one has $(k+j)$ possibilities (k : at the end of each lists or before the j supposed head lists), the second one have $(k+j+1)$ possibilities (one possibilities added by the previews element) and so one ... the last element have $(k+j)+(n-sk-1)=n+j-(s-1)k-1$ possibilities. This gives $\frac{(n+j-(s-1)k-1)!}{(k+j-1)!}=(k+j)\cdots(n+j-(s-1)k-1)$ possibilities. Summing over all possible values of j we get $\binom{n-r}{k-r}\binom{n-k}{s-1,\ldots,s-1,n-sk}\sum_{j=0}^{r}\binom{r}{j}\frac{(n+j-(s-1)k-1)!}{(k+j-1)!}(s-1)^{r-j}$ which, after simplification, gives the result. □

Note that the explicit formula of the s-associated Lah numbers (15) is obtained form (39) for $r=0$ and $r=1$. Also, for $s=1$, we get the explicit formula of the r-Lah numbers [3, Eq 3].

From (39) and (30) we can state the following, which is very nice in terms of identities related to binomial coefficients.

Proposition 19 *We have*

$$\sum_{j=0}^{r}\binom{r}{j}\binom{n+k+j-1}{n}(s-1)^{r-j}=\sum_{j=0}^{r}\binom{r}{j}\binom{n-(s-1)k-1}{k+j-1}s^{r-j}, \tag{40}$$

From (31) and (39) we get a second expression, dual to relation (27).

Proposition 20 *we have*

$$\sum_{i_1+i_2+\cdots+i_k=n}(i_1+s)(i_2+s)\cdots(i_r+s)=\sum_{j=0}^{r}\binom{r}{j}\binom{n+k+j-1}{n}(s-1)^{r-j}$$

5 Recurrence relations

The s-associated r-Stirling numbers satisfy recurrence relations as the regular s-associated Stirling numbers, using three terms of two triangles: the $(r-1)$-Stirling triangle and the r-Stirling triangle.

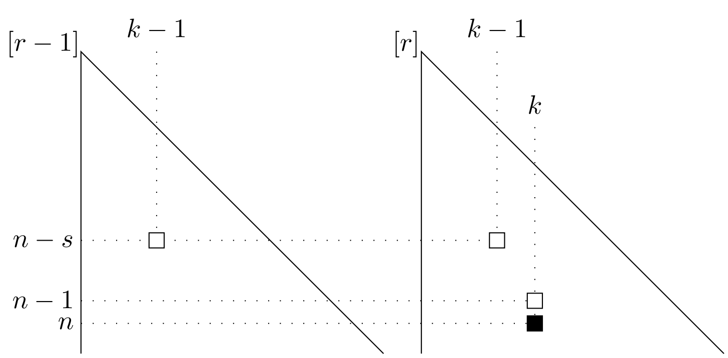

Figure 3: Triangular recurrence relation given the value of the black element as linear combination of the values of the three others, for the s-associate r-Stirling numbers of the three kinds

The recurrence relation of the s-associated r-Stirling numbers of the first kind is given as follows.

Theorem 21 *Let $r, k, s,$ and n be nonnegative integers such that $n \geqslant sk$ and $k \geqslant r$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(s)} = \binom{n-r-1}{s-1}(s-1)!\begin{bmatrix} n-s \\ k-1 \end{bmatrix}_r^{(s)} + r\binom{n-r-1}{s-2}(s-1)!\begin{bmatrix} n-s \\ k-1 \end{bmatrix}_{r-1}^{(s)} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix}_r^{(s)}. \quad (41)$$

Proof. Let us consider the n^{th} element, if it belongs to a cycle containing exactly s elements not from the r elements, we have $\binom{n-r-1}{s-1}$ ways to choose the $(s-1)$ remaining elements and $(s-1)!$ ways to constitute the cycle, then distribute the $(n-s)$ remaining elements on the $(k-1)$ remaining cycles such that each cycle has at least s element and the r first elements are in distinct cycles, so we have $\begin{bmatrix} n-s \\ k-1 \end{bmatrix}_r^{(s)}$ ways to do it. Thus we get $\binom{n-r-1}{s-1}(s-1)!\begin{bmatrix} n-s \\ k-1 \end{bmatrix}_r^{(s)}$ possibilities. Else, if one of the r first elements belongs to the cycle, we have r ways to choose one of the r first elements, $\binom{n-r-1}{s-2}$ ways to choose the remaining $(s-2)$ ones and $(s-1)!$ ways to constitute the cycle, then distribute the $(n-s)$ remaining elements on the $(k-1)$ remaining cycles such that, in each cycle, there is at least s elements and the $r-1$ first elements are in distinct cycles, so we have $\begin{bmatrix} n-s \\ k-1 \end{bmatrix}_{r-1}^{(s)}$ possibilities to do it. Thus we get $r\binom{n-r-1}{s-2}(s-1)!\begin{bmatrix} n-s \\ k-1 \end{bmatrix}_{r-1}^{(s)}$ possibilities. Else, we consider all the permutations of $(n-1)$ elements with k cycles under the usual conditions which can be done by $\begin{bmatrix} n-1 \\ k \end{bmatrix}_r^{(s)}$ ways, then add the n^{th} element to the k cycles and we have $(n-1)$ possibilities. □

For $s = 1$ we get relation (4), and for $r = 1$ using Pascal's formula we get the recurrence relation of the s-associated Stirling numbers of first kind [20, eq 4.8].

The s-associated r-Stirling numbers of the second kind satisfy the following recurrence relation.

Theorem 22 *Let $r, k, s,$ and n be nonnegative integers such that $n \geqslant sk$ and $k \geqslant r$, we have*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(s)} = \binom{n-r-1}{s-1}\begin{Bmatrix} n-s \\ k-1 \end{Bmatrix}_r^{(s)} + r\binom{n-r-1}{s-2}\begin{Bmatrix} n-s \\ k-1 \end{Bmatrix}_{r-1}^{(s)} + k\begin{Bmatrix} n-1 \\ k \end{Bmatrix}_r^{(s)}. \quad (42)$$

Proof. Let us consider the n^{th} elements, if it belongs to a part containing exactly s elements not from the r first ones, so we have $\binom{n-r-1}{s-1}$ ways to choose the remaining $(s-1)$ elements and $\begin{Bmatrix} n-s \\ k-1 \end{Bmatrix}_r^{(s)}$ ways to distribute the $(n-s)$ remaining elements on the $(k-1)$ remaining parts such that, the r first elements are in distinct parts, and each part, have at least s elements which gives $\binom{n-r-1}{s-1}\begin{Bmatrix} n-s \\ k-1 \end{Bmatrix}_r^{(s)}$ possibilities. Else, if one of the r first elements belongs to that part, we have r ways to choose it, and $\binom{n-r-1}{s-2}$ ways to choose the remaining $(s-2)$, then distribute the $(n-s)$ remaining elements on the $(k-1)$ remaining parts such that, the $r-1$ first elements are in distinct parts, and each part, have at least s elements which can be done by $\begin{Bmatrix} n-s \\ k-1 \end{Bmatrix}_{r-1}^{(s)}$ ways. So we have $r\binom{n-r-1}{s-2}\begin{Bmatrix} n-s \\ k-1 \end{Bmatrix}_{r-1}^{(s)}$ possibilities. Else, we consider all the partitions of $(n-1)$ elements on k blocks under the usual conditions which can be done by $\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(s)}$ ways, then add the n^{th} element to the k cycles with $(n-1)$ possibilities. □

For $s = 1$ we get relation (5), and for $r = 1$ using Pascal's formula we get the recurrence relation of the s-associated Stirling numbers of the second kind [20, eq 4.1].

The s-associated r-Lah numbers satisfy the following recurrence relation.

Theorem 23 *Let $r, k, s,$ and n be nonnegative integers such that $n \geqslant sk$ and $k \geqslant r$ we have*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r^{(s)} = \binom{n-r-1}{s-1}s!\left\lfloor \begin{matrix} n-s \\ k-1 \end{matrix} \right\rfloor_r^{(s)} + r\binom{n-r-1}{s-2}s!\left\lfloor \begin{matrix} n-s \\ k-1 \end{matrix} \right\rfloor_{r-1}^{(s)} + (n+k-1)\left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rfloor_r^{(s)}. \quad (43)$$

Proof. Let us consider the n^{th} element, if it belongs to a list containing exactly s elements not from the r first ones, we have $\binom{n-1}{s-1}$ ways to choose the remaining $(s-1)$ elements and $s!$ ways to constitute the list, then distribute the $(n-s)$ remaining elements into the $(k-1)$ remaining lists such that each list has at least s elements and the r first elements are in distinct lists with $\left\lfloor{n-s \atop k-1}\right\rfloor_r^{(s)}$ ways. Thus we get $\binom{n-1}{s-1}s!\left\lfloor{n-s \atop k-1}\right\rfloor^{(s)}$ possibilities. Else, if one of the r first elements belongs to the list, we have $\binom{r}{1} = r$ ways to choose one of the r first elements and $\binom{n-r-1}{s-2}$ ways to choose the remaining $(s-2)$ elements and $s!$ ways to constitute the list, then distribute the $(n-s)$ remaining elements into the $(k-1)$ remaining lists such that each list has at least s elements and the $r-1$ first elements are in distinct lists and we have $\left\lfloor{n-s \atop k-1}\right\rfloor_{r-1}^{(s)}$ ways to do it. Thus we get $r\binom{n-r-1}{s-2}s!\left\lfloor{n-s \atop k-1}\right\rfloor_{r-1}^{(s)}$ possibilities. Else, we consider all the partitions of $(n-1)$ elements into k lists under the usual conditions which can be done by $\left\lfloor{n-1 \atop k}\right\rfloor_r^{(s)}$ ways, then add the n^{th} element to the k lists and we have $(n-1)$ possibilities after each element or k possibilities as a head list, which gives $(n+k-1)\left\lfloor{n-1 \atop k}\right\rfloor_r^{(s)}$ possibilities. □

For $s = 1$ we get relation (6), and for $r = 1$ and using Pascal's formula we get the recurrence relation (14).

6 Combinatorial identities or convolution relations

In this section, we establish some combinatorial identities for the s-associated r-Stirling numbers using a combinatorial approach. we can also consider them as convolution relations.

The next identity is an expressions of s-associated r-Stirling numbers in terms of the s-associated r'-Stirling numbers with $r' \leqslant r$.

Theorem 24 *Let p, r, k and n be nonnegative integers such that $p \leqslant r \leqslant k$ and $n \geqslant sk$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(s)} = \sum_{i=(s-1)p}^{n-p-s(k-p)} \frac{(n-r)!}{(n-r-i)!} \binom{i-p\,(s-2)-1}{p-1} \begin{bmatrix} n-p-i \\ k-p \end{bmatrix}_{r-p}^{(s)}. \tag{44}$$

Proof. Let us consider the i $((s-1)p \leqslant i \leqslant n-p-s\,(k-p))$ elements which belongs to the p cycles containing the elements $1, \ldots, p$. We have $\binom{n-r}{i}$ possibilities to choose the i elements and $\left[{i+p \atop p}\right]_p^{(s)}$ ways to construct the corresponding cycles. The remaining $n-p-i$ elements must form the $k-p$ remaining cycles; this can be done in $\left[{n-p-i \atop k-p}\right]_{r-p}^{(s)}$ ways. Using equation (33) and summing for all i, we get the proof. □

For $p = r$, we obtain an expression of the s-associated r-Stirling numbers of the first kind in terms of the regular s-associated Stirling numbers of the first kind

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(s)} = \sum_{i=(s-1)r}^{n-r-s(k-r)} \frac{(n-r)!}{(n-r-i)!} \binom{i-r\,(s-2)-1}{r-1} \begin{bmatrix} n-r-i \\ k-r \end{bmatrix}^{(s)}. \tag{45}$$

For $s = 1$, we obtain the equation given by Broder [12, eq 26] and for $r = 1$, we get a vertical recurrence relation for the classical s-associated Stirling numbers of the first kind

$$\begin{bmatrix} n \\ k \end{bmatrix}^{(s)} = \sum_{i=s-1}^{n-s(k-1)-1} \frac{(n-1)!}{(n-i-1)!} \begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix}^{(s)}. \tag{46}$$

Theorem 25 *Let p, r, k and n be nonnegative integers such that $p \leqslant r \leqslant k$ and $n \geq sk$, we have*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(s)} = \sum_{i=p-r+s(k-p)}^{n-r-(s-1)p} \frac{(n-r)!}{((s-1)!)^p\,(n-p(s-1)-r)!} \binom{n-p(s-1)-r}{i} \begin{Bmatrix} i+r-p \\ k-p \end{Bmatrix}_{r-p}^{(s)} p^{n-p(s-1)-r-i}. \tag{47}$$

Proof. Let us consider p first elements ($p \leqslant r$), they constitute p parts with $p(s-1)$ elements so we choose those elements by $\binom{n-r}{s-1,\ldots,s-1,n-sp(s-1)-r} = \frac{(n-r)!}{((s-1)!)^p(n-p(s-1)-r)!}$ ways. Then we choose the i elements $((s-1)(r-p)+s(k-r) \leqslant i \leqslant n-r-(s-1)\,p)$ which belongs to the remaining $k-p$ parts and we have $\binom{n-p(s-1)-r}{i}$ ways to do it. Then, distribute them on $k-p$ parts such that the $r-p$ fixed elements are in distinct parts and each part have at least s elements, which can be done by $\left\{ {i+r-p \atop k-p} \right\}^{(s)}_{r-p}$ possibilities. It remains now to distribute the remaining $n-p(s-1)-r-i$ elements on the p first parts and we have $p^{n-p(s-1)-r-i}$ possibilities. We conclude by summing over all possible values of i. □

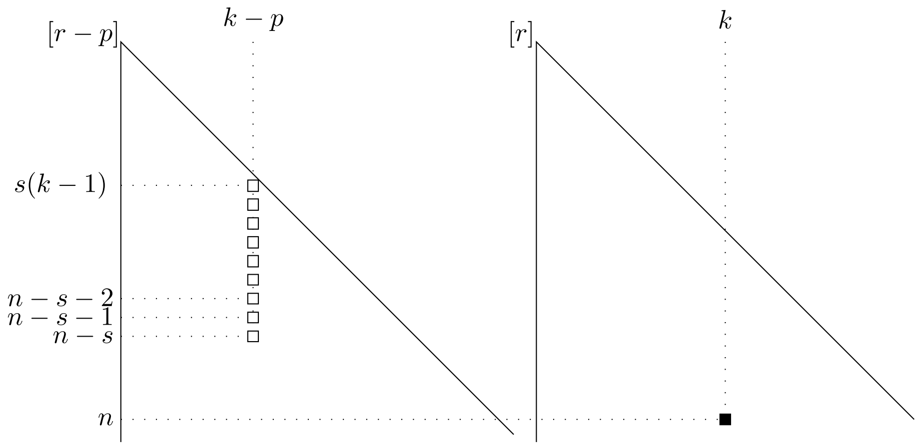

Figure 4: The value of an element in the s-associated r-Stirling table in terms of the consecutive vertical elements in the s-associated $r-p$-Stirling table as an inner product result

For $p = r$ we get an expression of the s-associated r-Stirling numbers of the second kind in terms of the regular s-associated Stirling numbers of the second kind

$$\left\{ {n \atop k} \right\}^{(s)}_{r} = \sum_{i=s(k-r)}^{n-sr} \frac{(n-r)!}{((s-1)!)^r\,(n-sr)!} \binom{n-sr}{i} \left\{ {i \atop k-r} \right\}^{(s)} r^{n-sr-i}, \tag{48}$$

also, for $s = 1$, we obtain the equation given by Broder [12, eq 31] and for $r = 1$, we get a vertical recurrence relation for the classical s-associated Stirling numbers of the second kind

$$\left\{ {n \atop k} \right\}^{(s)} = \sum_{i=s(k-1)}^{n-s} \binom{n-1}{s-1} \binom{n-s}{i} \left\{ {i \atop k-1} \right\}^{(s)}. \tag{49}$$

Theorem 26 *Let p, r, k and n be nonnegative integers such that $p \leqslant r \leqslant k$ and $n \geq sk$, we have*

$$\left\{ {n \atop k} \right\}^{(s)}_{r} = \sum_{i=0}^{p} \frac{(k-r+p-i)!}{(k-r)!} \binom{p}{i} \binom{n-r}{i(s-1)} \frac{(i(s-1))!}{((s-1)!)^i} \left\{ {n-p-i(s-1) \atop k-i} \right\}^{(s)}_{r-p}. \tag{50}$$

Proof. Let us consider the p first elements, and focus on the i ($0 \leq i \leq p$) parts containing exactly s elements, we have $\binom{p}{i}$ ways to choose the i elements from the p first ones, and $\binom{n-r}{i(s-1)}$ ways to choose the $i(s-1)$ remaining elements to have s elements by part, and $\left\{ {(i(s-1)) \atop i} \right\}^{(s-1)} = \frac{(i(s-1))!}{i!((s-1)!)^i}$ (from 11) ways to partition the $i(s-1)$ elements on i groups such that each group have at least $(s-1)$ elements, then affect each group to the i elements and we have $i!$. Then, we partition the $n-p-i(s-1)$ remaining elements into $(k-i)$ parts such that each group has at least s elements and the remaining $r-p$ elements are in distinct subsets, and we have $\left\{ {n-r-i(s-1) \atop k-i} \right\}^{(s)}_{r-p}$ ways to do it. Now, it reminds $(p-i)$ elements not yet affected.

Thus we have $(k-r+p-i)$ choice for the first one, $(k-r+p-i-1)$ choice for the second one and so on until the last one have $(k-r+1)$ which gives $(k-r+p-i)(k-r+p-i-1)\cdots(k-r+1)=\frac{(k-r+p-i)!}{(k-r)!}$ possibilities. We conclude by summing. $\square$

For $s=1$ we get the relation given by the authors [5, eq 5].

An expression of the s-associated r-Stirling numbers of the second kind in terms of the s-associated Stirling numbers can be deduced from equation (50), for $p=r$, as follows

$$\left\{ {n \atop k} \right\}_r^{(s)} = \sum_{i=0}^{r} \frac{(k-i)!}{(k-r)!} \binom{r}{i} \binom{n-r}{i(s-1)} \frac{(i(s-1))!}{((s-1)!)^i} \left\{ {n-r-i(s-1) \atop k-i} \right\}^{(s)}. \tag{51}$$

Also, for $r=1$, we obtain the recurrence relation of the s-associated Stirling numbers [20, eq 4.1].

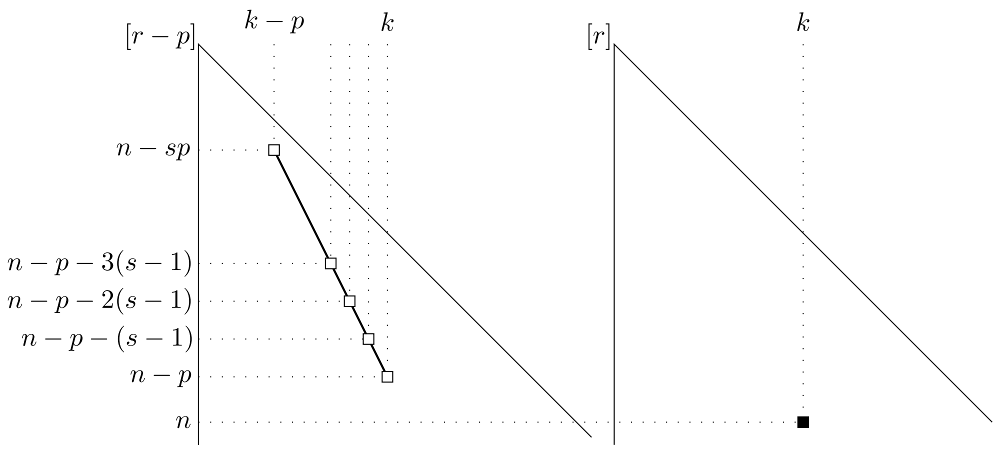

Figure 5: Value of s-associated r-Stirling element (in black) as a inner product of a periodic sequence of elements of the s-associated $r-p$-Stirling table (in white) with a sequence deriving from binomial coefficient.

Theorem 27 *Let p, r, k and n be nonnegative integers such that $p \leqslant r \leqslant k$ and $n \geq sk$, we have*

$$\left\lfloor {n \atop k} \right\rfloor_r^{(s)} = \sum_{i=0}^{p} \sum_{j=i}^{n-sk} \frac{(n-r)!}{(n-r-j+(s-1)p)!} \binom{p}{i} \binom{p+j-1}{j-i} \left\lfloor {n-sp-j \atop k-p} \right\rfloor_{r-p}^{(s)} s^{p-i}. \tag{52}$$

Proof. Let us consider the p first elements, they are in p distinct lists with i_j $(i_j \geqslant s-1;\ j=1..p)$ other elements, such that $i_1+i_2+\cdots i_p = j$ $((s-1)p \leqslant j \leqslant n-p-s(k-p))$. Then there are $\binom{n-r}{i_1,i_2,\ldots,i_p,n-r-j} = \binom{n-r}{i_1}\binom{n-r-i_1}{i_2}\cdots\binom{n-r-i_1-i_2-\ldots-i_{p-1}}{i_p}$ ways to choose the $i_1, i_2, \ldots, i_p$ elements and $(i_1+1)!(i_2+1)!\cdots(i_p+1)!$ ways to constitute the p lists. Now, it remains to distribute the $n-p-j$ remaining elements into $k-p$ lists such that each list have at least s elements and the $r-p$ elements are in distinct lists, which gives $\left\lfloor {n-p-j \atop k-p} \right\rfloor_{r-p}^{(s)}$ possibilities. we sum over all value of j we get

$$\left\lfloor {n \atop k} \right\rfloor_r^{(s)} = \sum_{j=(s-1)p}^{n-p-s(k-p)} \sum_{\substack{i_1+i_2+\cdots+i_p=j \\ i_l \geqslant s-1}} (i_1+1)!(i_2+1)!\cdots(i_p+1)! \binom{n-r}{i_1,i_2,\ldots,i_p,n-r-j} \left\lfloor {n-p-j \atop k-p} \right\rfloor_{r-p}^{(s)},$$

the inner summations can be evaluated using (27). This gives the result. $\square$

For $p=r$, we get an expression of the s-associated r-Lah numbers in terms of the s-associated Lah numbers

$$\left\lfloor {n \atop k} \right\rfloor_r^{(s)} = \sum_{i=0}^{r} \sum_{j=i}^{n-sk} \binom{r}{i} \binom{r+j-1}{j-i} \frac{(n-r)!}{(n-j+(s-2)r)!} s^{r-i} \left\lfloor {n-sr-j \atop k-r} \right\rfloor^{(s)}. \tag{53}$$

Also, For $r = 1$, we get relation (19), and for $s = 1$ we get the identity [3, eq 7].

7 Cross recurrence relations

From equations (44) and (52), for $p = 1$, we get some vertical cross recurrence relations.

Corollary 28 *We have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_r^{(s)} = \sum_{i=s-1}^{n-s(k-1)-1} \frac{(n-r)!}{(n-r-i)!} \begin{bmatrix} n-i-1 \\ k-1 \end{bmatrix}_{r-1}^{(s)}, \quad (54)$$

$$\begin{vmatrix} n \\ k \end{vmatrix}_r^{(s)} = \sum_{i=s-1}^{n-s(k-1)-1} (i+1)\frac{(n-r)!}{(n-r-i)!} \begin{vmatrix} n-i-1 \\ k-1 \end{vmatrix}_{r-1}^{(s)}. \quad (55)$$

For $r = 1$, we get relation (19) and for $s = 1$ we get the identity given by the authors in [8].

Theorem 29 *Let r, k, n be nonnegative integers such that $n \geq sk$, we have*

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_r^{(s)} = \binom{n-r}{s-1} \begin{Bmatrix} n-s \\ k-1 \end{Bmatrix}_{r-1}^{(s)} + (k-r+1) \begin{Bmatrix} n-1 \\ k \end{Bmatrix}_{r-1}^{(s)}. \quad (56)$$

Proof. Let us consider the r^{th} elements. If it belongs to a group containing exactly s elements, we have $\binom{n-r}{s-1}$ ways to choose the remaining $(s-1)$ elements and $\left\{ {n-s \atop k-1} \right\}_{r-1}^{(s)}$ ways to partition the remaining $(n-s)$ elements on $(k-1)$ parts such that the $(r-1)$ first elements are in distinct parts, and each parts, have at least s elements. Thus we get $\binom{n-r}{s-1}\left\{ {n-s \atop k-1} \right\}_{r-1}^{(s)}$ possibilities. Else, we have $\left\{ {n-1 \atop k} \right\}_{r-1}^{(s)}$ possibilities to partition the remaining $(n-1)$ elements into k parts such that the $(r-1)$ first elements are in distinct parts, and each parts, have at least s elements, then add the r^{th} elements to on of the $(k-(r-1))$ parts and we have $(k-r+1)$ possibilities. It gives $(k-r+1)\left\{ {n-1 \atop k} \right\}_{r-1}^{(s)}$. □

For $s = 1$ we get the cross recurrence [5, eq 3] and for $r = 1$ we get the recurrence relation of the s-associated Stirling numbers of the second kind [20, eq 4.1].

8 Convolution identities (revisited)

The s-associated r-Stirling numbers of the three kinds can be expressed as a convolution using the binomial coefficients.

Theorem 30 *Let r, k and n be nonnegative integers such that $n \geq sk$ with $k_1 + \cdots + k_p = k$ and $r_1 + \cdots + r_p = r$, we have*

$$\binom{k}{k_1, \ldots, k_p} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r^{(s)} = \sum_{\substack{l_1+\cdots+l_p=n \\ l_i \geqslant sk_i+(s-1)r_i}} \binom{n}{l_1, \ldots, l_p} \begin{bmatrix} l_1+r_1 \\ k_1+r_1 \end{bmatrix}_{r_1}^{(s)} \cdots \begin{bmatrix} l_p+r_p \\ k_p+r_p \end{bmatrix}_{r_p}^{(s)}. \quad (57)$$

Proof. We consider permutations of Z_{n+r} with $k+r$ cycles such that the r first elements are in distinct cycles and each cycle has at least s elements and we have $\left[{n+r \atop k+r} \right]_r^{(s)}$ possibilities. We color the cycles with p colors such that each r_i cycles containing the r_i elements with k_i other cycles have the same color, thus we choose the k_i cycles and we have $\binom{k}{k_1,\ldots,k_p}$ possibilities this is to choose the l_i elements that have the same color of the r_i first and we have $\binom{n}{l_1,\ldots,l_p}$ possibilities, then consider all the permutations of the l_i+r_i elements with k_i+r_i cycles such that the r_i elements are in distinct cycles and each cycle has at least s element and we have $\left[{l_i+r_i \atop k_i+r_i}\right]_{ri}^{(s)}$ ways to do it. Summing over all possible values of l_i gives the result. □

Theorem 31 *Let r,k and n be nonnegative integers such that $n\geq sk$ with $k_1+\cdots+k_p=k$ and $r_1+\cdots+r_p=r$, The s-associated r-Stirling numbers of the second kind satisfy*

$$\binom{k}{k_1,\ldots,k_p}\left\{{n+r \atop k+r}\right\}_r^{(s)}=\sum_{\substack{l_1+\cdots+l_p=n\\ l_i\geqslant sk_i+(s-1)r_i}}\binom{n}{l_1,\ldots,l_p}\left\{{l_1+r_1 \atop k_1+r_1}\right\}_{r_1}^{(s)}\cdots\left\{{l_p+r_p \atop k_p+r_p}\right\}_{r_p}^{(s)}. \tag{58}$$

Proof. We use an adapted analogous bijective proof as for the identity (57). □

Relations (57) and (58) extend those given by the others [5, Eq 8, Eq 12] to the s-associated situation.

Theorem 32 *Let r,k and n be nonnegative integers such that $n\geq sk$ with $k_1+\cdots+k_p=k$ and $r_1+\cdots+r_p=r$, The s-associated r-Lah numbers satisfy*

$$\binom{k}{k_1,\ldots,k_p}\left\lfloor{n+r \atop k+r}\right\rfloor_r^{(s)}=\sum_{\substack{l_1+\cdots+l_p=n\\ l_i\geqslant sk_i+(s-1)r_i}}\binom{n}{l_1,\ldots,l_p}\left\lfloor{l_1+r_1 \atop k_1+r_1}\right\rfloor_{r_1}^{(s)}\cdots\left\lfloor{l_p+r_p \atop k_p+r_p}\right\rfloor_{r_p}^{(s)}. \tag{59}$$

Proof. We use an adapted analogous bijective proof as for the identity (57). □

For $s=1$, we get

$$\binom{k}{k_1,\ldots,k_p}\left\lfloor{n+r \atop k+r}\right\rfloor_r=\sum_{\substack{l_1+\cdots+l_p=n\\ l_i\geqslant sk_i+(s-1)r_i}}\binom{n}{l_1,\ldots,l_p}\left\lfloor{l_1+r_1 \atop k_1+r_1}\right\rfloor_{r_1}\cdots\left\lfloor{l_p+r_p \atop k_p+r_p}\right\rfloor_{r_p}. \tag{60}$$

9 Generating functions

The s-associated r-Stirling numbers of the first kind have the following exponential generating function.

Theorem 33 *We have*

$$\sum_{n\geq sk+(s-1)r}\left[{n+r \atop k+r}\right]_r^{(s)}\frac{x^n}{n!}=\frac{(-1)^k}{k!}\left(\ln(1-x)+\sum_{i=1}^{s-1}\frac{x^i}{i}\right)^k\left(\frac{x^{s-1}}{1-x}\right)^r. \tag{61}$$

Proof. Using the identity (45), we get

$$\sum_{n\geq sk+(s-1)r}\left[{n+r \atop k+r}\right]_r^{(s)}\frac{x^n}{n!}=\sum_i\binom{i-r(s-2)-1}{r-1}x^i\sum_{n\geq sk+(s-1)r}\left[{n-i \atop k}\right]^{(s)}\frac{x^{n-i}}{(n-i)!},$$

from (9), we obtain

$$\sum_{n\geq sk+(s-1)r}\left[{n+r \atop k+r}\right]_r^{(s)}\frac{x^n}{n!}=\frac{1}{k!}\left(\ln\left(\frac{1}{1-x}\right)-\sum_{i=1}^{s-1}\frac{x^i}{i}\right)^k\sum_i\binom{i-r(s-2)-1}{r-1}x^i,$$

using relation (22) we get the result. □

The above theorem implies the double generating function.

Theorem 34 *The s-associated r-Stirling numbers of the first kind satisfy*

$$\sum_{n,k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r^{(s)} y^k \frac{x^n}{n!} = \exp\left(y \ln\left(\frac{1}{1-x}\right) - y \sum_{i=1}^{s-1} \frac{x^i}{i} \right) \left(\frac{x^{s-1}}{1-x}\right)^r. \tag{62}$$

Proof. Interchanging the order of summation and using equation (61) we get the result. □

The s-associated r-Stirling numbers of the second kind have the following exponential generating function

Theorem 35 *We have*

$$\sum_{n \geq sk+(s-1)r} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r^{(s)} \frac{x^n}{n!} = \frac{1}{k!} \left(\exp(x) - \sum_{i=0}^{s-1} \frac{x^i}{i!} \right)^k \left(\exp(x) - \sum_{i=0}^{s-2} \frac{x^i}{i!} \right)^r. \tag{63}$$

Proof. Using the identity (51), we get

$$\sum_{n \geq sk+(s-1)r} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r^{(s)} \frac{x^n}{n!} = \frac{(k+r-i)!}{k!} \sum_{i=0}^{r} \binom{r}{i} \left(\frac{x^{s-1}}{((s-1)!)}\right)^i \sum_{n \geq sk+(s-1)r} \begin{Bmatrix} n-i(s-1) \\ k+r-i \end{Bmatrix}^{(s)} \frac{x^{n-i(s-1)}}{(n-i(s-1))!},$$

the second summation can be evaluated using (10) and gives

$$\sum_{n \geq sk+(s-1)r} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r^{(s)} \frac{x^n}{n!} = \frac{1}{k!} \left(\exp(x) - \sum_{i=0}^{s-1} \frac{x^i}{i!} \right)^k \sum_{i=0}^{r} \binom{r}{i} \left(\frac{x^{s-1}}{((s-1)!)}\right)^i \left(\exp(x) - \sum_{i=0}^{s-1} \frac{x^i}{i!} \right)^{r-i},$$

using the binomial theorem we get the result. □

The double generating function for s-associated r-Stirling numbers of the second kind is

Theorem 36

$$\sum_{n,k} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r^{(s)} y^k \frac{x^n}{n!} = \exp\left(y \exp(x) - y \sum_{i=0}^{s-1} \frac{x^i}{i!} \right) \left(\exp(x) - \sum_{i=0}^{s-1} \frac{x^i}{i!} \right)^r. \tag{64}$$

The s-associated r-Lah numbers have the following exponential generating function

Theorem 37 *We have*

$$\sum_{n \geq sk+(s-1)r} \left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r^{(s)} \frac{x^n}{n!} = \frac{1}{k!} \left(\frac{x^s}{(1-x)} \right)^k \left(\frac{x^{s-1}}{(1-x)^2} (s-(s-1)x) \right)^r. \tag{65}$$

Proof. Using the explicit formula (39) in the left hand side we get

$$\sum_{n \geq sk+(s-1)r} \left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r^{(s)} \frac{x^n}{n!} = \frac{1}{k!} \sum_{j=0}^{r} \binom{r}{j} (s-1)^{r-j} \sum_{n \geq sk+(s-1)r} \binom{n+r+j-(s-1)(k+r)-1}{k+r+j-1} x^n,$$

the second summation in the right side, due to relation (22), gives

$$\begin{aligned} \sum_{n \geq sk+(s-1)r} \left\lfloor \begin{matrix} n+r \\ k+r \end{matrix} \right\rfloor_r^{(s)} \frac{x^n}{n!} &= \frac{1}{k!} \sum_{j=0}^{r} \binom{r}{j} (s-1)^{r-j} \frac{x^{(s-1)(k+r)+k}}{(1-x)^{k+r+j}} \\ &= \frac{1}{k!} \frac{x^{(s-1)r+sk}}{(1-x)^{k+2r}} \sum_{j=0}^{r} \binom{r}{j} ((s-1)(1-x))^{r-j} \end{aligned}$$

using the binomial theorem we get the result. □

The double generating function of the s-associated r-Lah numbers is given by

Theorem 38

$$\sum_{n\geq sk+(s-1)r}\sum_{k\geq 0}\left\lfloor {n+r \atop k+r} \right\rfloor_r^{(s)}\frac{x^n}{n!}y^k=\left[\exp\left\{y\frac{x^s}{1-x}\right\}\right]\left[\frac{x^{s-1}}{(1-x)^2}(s-(s-1)x)\right]^r. \tag{66}$$

Proof. Interchanging the order of summation and using equation (65) we get the result. □

10 Conclusion and perspectives

Roughly speaking, there are many recurrence and congruence relations known about the r-Stirling numbers and the associated Stirling numbers which can be generalized to $\left[{n \atop k}\right]_r^{(s)}$, $\left\{{n \atop k}\right\}_r^{(s)}$ and $\left\lfloor{n \atop k}\right\rfloor_r^{(s)}$. We have treated a few of these. In this section, we propose some problems :

- Howard [21] gave, as perspectives, an extension of the weighted associated Stirling numbers to the Weighted s-associated Stirling numbers without specifying the expressions. In this perspective, as continuity to our work, we propose the Weighted s-associated Stirling numbers of the first and the second kind, denoted $\left[{n \atop k}\right]_\lambda^{(s)}$ and $\left\{{n \atop k}\right\}_\lambda^{(s)}$ respectively, by the following

$$\sum_{n\geq sk}\left[{n \atop k}\right]_\lambda^{(s)}\frac{x^n}{n!} = \frac{(-1)^k}{k!}\left(\frac{1}{(1-x)^\lambda}-\sum_{i=1}^{s-1}\frac{(-\lambda)_i\,x^i}{i}\right)\left(\ln(1-x)+\sum_{i=1}^{s-1}\frac{x^i}{i}\right)^k, \tag{67}$$

$$\sum_{n\geq s}\left\{{n \atop k}\right\}_\lambda^{(s)}\frac{x^n}{n!} = \frac{1}{k!}\left(\exp(\lambda x)-\sum_{i=0}^{s-1}\frac{(\lambda x)^i}{i!}\right)\left(\exp(x)-\sum_{i=0}^{s-1}\frac{x^i}{i!}\right)^k. \tag{68}$$

  Note that for $s=2$, we get weighted associated Stirling numbers. It seems possible to derive analog relations of the weighted associated Stirling numbers, and establish other identities.

- By the same reasoning, it is interesting to extend these generalization to the r-Stirling numbers. We define the weighted s-associated r-Stirling numbers of the first and the second kind respectively

$$\sum_{n}\left[{n+r \atop k+r}\right]_{r,\lambda}^{(s)}\frac{x^n}{n!} = \frac{(-1)^k}{k!}\left(\frac{1}{(1-x)^\lambda}-\sum_{i=1}^{s-1}\frac{(-\lambda)_i\,x^i}{i}\right)\left(\ln(1-x)+\sum_{i=1}^{s-1}\frac{x^i}{i}\right)^k\left(\frac{x^{s-1}}{1-x}\right)^r, \tag{69}$$

$$\sum_{n}\left\{{n+r \atop k+r}\right\}_{r,\lambda}^{(s)}\frac{x^n}{n!} = \frac{1}{k!}\left(\exp(\lambda x)-\sum_{i=0}^{s-1}\frac{(\lambda x)^i}{i!}\right)\left(\exp(x)-\sum_{i=0}^{s-1}\frac{x^i}{i!}\right)^k\left(\exp(x)-\sum_{i=0}^{s-2}\frac{x^i}{i!}\right)^r \tag{70}$$

  It will be nice to investigate the combinatorial meaning and drive all the combinatorial identities. Also, for $s=1$, we get the definition of the weighted r-Stirling numbers as follows

$$\sum_{n\geq sk+(s-1)r}\left[{n+r \atop k+r}\right]_{r,\lambda}\frac{x^n}{n!} = \frac{(-1)^k}{k!}\frac{1}{(1-x)^{\lambda+r}}(\ln(1-x))^k, \tag{71}$$

$$\sum_{n\geq sk+(s-1)r}\left\{{n+r \atop k+r}\right\}_{r,\lambda}\frac{x^n}{n!} = \frac{1}{k!}(\exp(\lambda x)-1)(\exp(x)-1)^k\exp(rx). \tag{72}$$

- It will be nice to investigate the different generalization (weighted, degenerated) of the Lah numbers and r-Lah numbers.

- The authors and Belkhir [4] define the $\left\lfloor{n \atop k}\right\rfloor^{\alpha,\beta}$ as the weight of a partition of n elements into k lists such that the element inserted as head list has weight β except the first inserted one which has weight 1 and the element inserted after an other one has weight α. This interpretation allow an extension to the s-associated aspect by adding the known restriction (at least s elements by list).

- An other perspective of this work is to consider the Whitney numbers (see [9, 10, 11, 2]) and r-Whitney numbers (see [16]) and to introduce the s-associated situation by two approaches: the first one via the generating function and the second one using the combinatorial interpretation (see [6]).

11 Tables of the s-associated r-Stirling numbers of the three kinds

$n\backslash k$	3	4	5	6	7	8
6	6					
7	72					
8	720	60				
9	7200	1320				
10	75 600	21 420	630			
11	846 720	320 544	21 840			
12	10 160 640	4753 728	519 120	7560		
13	130 636 800	72 005 760	10 795 680	378 000		
14	1796 256 000	1129 788 000	213 804 360	12 335 400	103 950	
15	26 345 088 000	18 486 230 400	4191 881 760	339 255 840	7068 600	
16	410 983 372 800	316 406 787 840	83 018 048 256	8627 739 120	302 702 400	1621 620
17	6799 906 713 600	5670 985 582 080	1679 434 428 672	212 106 454 560	10 621 490 880	143 783 640

Some values of the 2-associated 3-Stirling numbers of the first kind

$n\backslash k$	2	3	4	5	6
6	24				
7	240				
8	2160				
9	20 160	1680			
10	201 600	36 960			
11	2177 280	616 896			
12	25 401 600	9616 320	201 600		
13	319 334 400	145 774 080	7761 600		
14	4311 014 400	2329 015 680	206 569 440		
15	62 270 208 000	39 165 984 000	4817 292 480	38 438 400	
16	958 961 203 200	672 898 786 560	106 815 893 184	2287 084 800	
17	15 692 092 416 000	12 080 986 444 800	2337 623 608 320	88 691 803 200	
18	271 996 268 544 000	226 839 423 283 200	51 485 284 730 880	2886 166 483 200	10 762 752 000
19	4979 623 993 344 000	4453 872 650 035 200	1153 763 447 316 480	86 362 805 168 640	914 833 920 000

Some values of the 3-associated 2-Stirling numbers of the first kind

$n\backslash k$	3	4	5	6
9	720			
10	15 120			
11	241 920			
12	3628 800	120 960		
13	54 432 000	4536 000		
14	838 252 800	117 754 560		
15	13 412 044 800	2682 408 960	26 611 200	
16	224 172 748 800	57 916 892 160	1556 755 200	
17	3923 023 104 000	1239 100 934 400	59 390 210 880	
18	71 922 090 240 000	26 544 536 282 880	1902 484 584 000	8072 064 000
19	1380 904 132 608 000	592 364 034 662 400	56 075 567 708 160	678 053 376 000
20	27 743 619 391 488 000	13 356 216 902 246 400	1589 118 272 501 760	35 651 077 862 400

Some values of the 3-associated 3-Stirling numbers of the first kind

$n\backslash k$	3	4	5	6	7	8
6	6					
7	36					
8	150	60				
9	540	660				
10	1806	4620	630			
11	5796	26 376	10 920			
12	18 150	134 316	114 660	7560		
13	55 980	637 020	947 520	189 000		
14	171 006	2882 220	6798 330	2772 000	103 950	
15	519 156	12 623 952	44 482 680	31 221 960	3534 300	
16	1569 750	54 031 692	273 060 216	299 459 160	68 918 850	1621 620
17	4733 820	227 425 380	1600 815 216	2578 495 920	1013 632 620	71 891 820
18	14 250 606	945 535 500	9069 810 750	20 561 420 880	12 509 597 100	1797 295 500
19	42 850 116	3895 163 928	50 074 806 600	154 904 109 360	136 912 175 400	33 423 390 000

Some values of the 2-associated 3-Stirling numbers of the second kind

$n\backslash k$	2	3	4	5	6	7
6	6					
7	20					
8	50					
9	112	210				
10	238	1540				
11	492	7476				
12	1002	30 240	12 600			
13	2024	110 550	161 700			
14	4070	379 764	1286 670			
15	8164	1252 680	8168 160	1201 200		
16	16 354	4020 016	45 411 366	23 823 800		
17	32 736	12 656 826	231 591 360	281 331 050		
18	65 502	39 315 588	1112 731 620	2574 371 800	168 168 000	
19	131 036	120 953 436	5122 253 136	20 176 035 880	4764 760 000	
20	262 106	369 535 392	22 845 529 356	142 501 719 360	78 189 711 600	
21	524 248	1123 340 382	99 494 683 548	934 588 410 756	973 654 882 200	32 590 958 400

Some values of the 3-associated 2-Stirling numbers of the second kind

$n\backslash k$	3	4	5	6	7
9	90				
10	630				
11	2940				
12	11 508	7560			
13	40 950	94 500			
14	137 610	734 580			
15	445 896	4569 180	831 600		
16	1410 552	24 959 220	16 216 200		
17	4390 386	125 381 256	188 558 370		
18	13 514 046	594 714 120	1701 649 950	126 126 000	
19	41 278 068	2707 865 160	13 172 479 320	3531 528 000	
20	125 405 532	11 965 834 608	92 024 532 600	57 320 062 800	
21	379 557 198	51 706 343 676	597 753 095 940	706 637 731 800	25 729 704 000
22	1145 747 538	219 672 404 652	3679 670 518 524	7344 721 664 280	977 728 752 000
23	3452 182 656	921 197 481 924	21 746 705 483 880	67 927 123 063 800	21 102 645 564 000
24	10 388 002 848	3824 306 218 236	124 527 413 730 720	577 211 131 256 760	340 398 980 922 000

Some values of the 3-associated 3-Stirling numbers of the second kind

$n\backslash k$	3	4	5	6
6	48			
7	864			
8	12 240	960		
9	166 320	31 680		
10	2298 240	735 840	20 160	
11	33 022 080	15 200 640	1048 320	
12	497 871 360	302 279 040	35 925 120	483 840
13	7903 526 400	5994 777 600	1043 280 000	36 288 000
14	132 204 441 600	120 708 403 200	28 101 427 200	1716 422 400
15	2328 905 779 200	2491 766 323 200	732 872 448 000	66 501 388 800
16	43 153 254 144 000	53 016 855 091 200	18 942 597 273 600	2325 792 268 800
17	839 788 479 129 600	1166 096 823 091 200	491 947 097 241 600	77 022 020 275 200

Some values for the 2-associated 3-Lah numbers

$n\backslash k$	2	3	4	5
6	216			
7	2880			
8	33 120			
9	383 040	45 360		
10	4636 800	1330 560		
11	59 512 320	28 667 520		
12	812 851 200	562 464 000	16 329 600	
13	11 815 372 800	10 777 536 000	838 252 800	
14	182 499 609 600	207 886 694 400	28 979 596 800	
15	2988 969 984 000	4097 379 686 400	859 328 870 400	9340 531 200
16	51 783 904 972 800	83 168 089 804 800	23 799 673 497 600	741 015 475 200
17	946 756 242 432 000	1745 745 281 280 000	640 760 440 320 000	37 486 665 216 000

Some values for the 3-associated 2-Lah numbers

$n\backslash k$	3	4	5	6
9	19 440			
10	544 320			
11	11 249 280			
12	212 647 680	9797 760		
13	3940 876 800	489 888 000		
14	73 766 246 400	16 525 555 200		
15	1414 970 726 400	479 001 600 000	6466 521 600	
16	28 021 593 600 000	12 989 565 388 800	504 388 684 800	
17	575 115 187 046 400	342 959 397 580 800	25 107 347 865 600	
18	12 255 524 176 896 000	9007 261 046 784 000	1030 447 401 984 000	5884 534 656 000
19	271 347 662 057 472 000	238 268 731 244 544 000	38 309 628 284 928 000	659 067 881 472 000
20	6242 314 363 084 800 000	6397 038 394 306 560 000	1350 900 851 908 608 000	45 350 147 082 240 000

Some values for the 3-associated 3-Lah numbers

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