JOURNAL OF APPROXIMATION THEORY **86**, 278–285 (1996)
ARTICLE NO. 0069

On Sobolev Orthogonality for the Generalized Laguerre Polynomials*

Teresa E. Pérez and Miguel A. Piñar†

Departamento de Matemática Aplicada, Universidad de Granada, Granada, Spain

Communicated by Walter Van Assche

Received December 27, 1994; accepted in revised form September 4, 1995

The orthogonality of the generalized Laguerre polynomials, $\{L_n^{(\alpha)}(x)\}_{n\geqslant 0}$, is a well known fact when the parameter α is a real number but not a negative integer. In fact, for $-1<\alpha$, they are orthogonal on the interval $[0, +\infty)$ with respect to the weight function $\rho(x)=x^{\alpha}e^{-x}$, and for $\alpha<-1$, but not an integer, they are orthogonal with respect to a non-positive definite linear functional. In this work we will show that, for every value of the real parameter α, the generalized Laguerre polynomials are orthogonal with respect to a non-diagonal Sobolev inner product, that is, an inner product involving derivatives.

1. INTRODUCTION

Classical Laguerre polynomials are the main subject of a very extensive literature. If we denote by $\{L_n^{(\alpha)}(x)\}_{n\geqslant 0}$ the sequence of monic Laguerre polynomials, their crucial property is the following orthogonality condition

$$(L_n^{(\alpha)}, L_m^{(\alpha)})_L^{(\alpha)} = \int_0^{+\infty} L_n^{(\alpha)}(x)\, L_m^{(\alpha)}(x)\, x^{\alpha} e^{-x}\, dx = k_n\, \delta_{n,m}, \tag{1.1}$$

where $m, n\in\mathbb{N}$, $k_n\neq 0$ and the parameter α satisfies the condition $-1<\alpha$ in order to assure the convergence of the integrals.

For the monic classical Laguerre polynomials, an explicit representation is well known (see Szegő [5, p. 102]):

$$L_n^{(\alpha)}(x) = (-1)^n\, n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n+\alpha}{n-j} x^j, \qquad n\geqslant 0, \tag{1.2}$$

where $\binom{a}{b}$ is the generalized binomial coefficient

$$\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\,\Gamma(a-b+1)}.$$

* This research was partially supported by Junta de Andalucía, Grupo de Investigación 1107.

† E-mail: mpinar@goliat.ugr.es.

0021-9045/96 $18.00

Now, we can consider (1.2) for an arbitrary value of the parameter $\alpha \in \mathbb{R}$. For a fixed value of α, expression (1.2) provides a family of polynomials, called the *generalized Laguerre polynomials*. Obviously, they constitute a basis for the linear space of real polynomials, since $\deg(L_n^{(\alpha)}) = n$ for all $n \geqslant 0$.

For any value of α we can obtain from expression (1.2) a three-term recurrence relation which is satisfied by the generalized Laguerre polynomials

$$\begin{gathered} L_{-1}^{(\alpha)}(x) = 0, \qquad L_0^{(\alpha)} = 1, \\ xL_n^{(\alpha)}(x) = L_{n+1}^{(\alpha)}(x) + \beta_n^{(\alpha)} L_n^{(\alpha)}(x) + \gamma_n^{(\alpha)} L_{n-1}^{(\alpha)}(x), \end{gathered} \tag{1.3}$$

where

$$\beta_n^{(\alpha)} = 2n + \alpha + 1, \qquad \gamma_n^{(\alpha)} = n(n+\alpha).$$

Therefore, for $\alpha \notin \{-1, -2, ...\}$ we have $\gamma_n^{(\alpha)} \neq 0$ for all $n \geqslant 0$, and from Favard's theorem (see Chihara [1, p. 21]), we conclude that the family of polynomials $\{L_n^{(\alpha)}(x)\}_{n \geqslant 0}$ is a monic orthogonal polynomial sequence associated with a quasi-definite linear functional. For $-1 < \alpha$, the linear functional is positive definite. However, for $\alpha \in \{-1, -2, ...\}$, no orthogonality results can be deduced from Favard's theorem since $\gamma_n^{(\alpha)}$ vanishes for some value of n.

In a recent paper [3], after several pages of hard computations, Kwon and Littlejohn establish the orthogonality of the generalized Laguerre polynomials $\{L_n^{(-k)}(x)\}_{n \geqslant 0}$, $k = 1, 2, ...$ with respect to an inner product involving some Dirac masses and derivatives.

In this work we will show that, for every value of the parameter α, the generalized Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n \geqslant 0}$ are orthogonal with respect to a non-diagonal Sobolev inner product, that is, an inner product $(f(x), g(x))_S^{(k)}$ defined by means of the following expression

$$(f(x), g(x))_S^{(k)} = \int_{\mathbb{R}} F(x)\, A G(x)^T,$$

where $k \geqslant 0$, $F(x)$, $G(x)$ are two vectors defined by

$$F(x) = (f(x), f'(x), ..., f^{(k)}(x)),$$

$$G(x) = (g(x), g'(x), ..., g^{(k)}(x)),$$

and A is a symmetric positive definite matrix whose elements are signed Borel measures.

In our case, we shall consider the following non-diagonal Sobolev inner product

$$(f(x), g(x))_S^{(k, \alpha)} = \int_0^{+\infty} F(x)\, M(k)\, G(x)^T x^\alpha e^{-x}\, dx, \tag{1.4}$$

where $M(k)$ is a matrix which (i, j) entry is defined by

$$m_{i,j}(k) = \sum_{p=0}^{\min\{i,j\}} (-1)^{i+j} \binom{k-p}{i-p} \binom{k-p}{j-p}, \qquad 0 \leqslant i, j \leqslant k,$$

and $\alpha > -1$, in order to ensure the convergence of the integrals.

When $k = 0$, the Sobolev inner product (1.4) coincides with the classical Laguerre inner product, because $M(0) = 1$, and therefore

$$(f(x), g(x))_S^{(0,\alpha)} = \int_0^{+\infty} f(x)\, g(x)\, x^{\alpha} e^{-x}\, dx = (f(x), g(x))_L^{(\alpha)}.$$

The main result in this paper is given by the following theorem

THEOREM. *Let $\{L_n^{(\alpha)}(x)\}_{n \geqslant 0}$, be the generalized monic Laguerre polynomial sequence, with α an arbitrary real number. Then $\{L_n^{(\alpha)}(x)\}_{n \geqslant 0}$ is the monic orthogonal polynomial sequence, (in short, the MOPS) with respect to the non-diagonal Sobolev inner product $(\cdot,\cdot)_S^{(k,\alpha+k)}$, where $k = \max\{0, [-\alpha]\}$, and $[\alpha]$ denotes the greatest integer less than or equal to α.*

As a consequence of this theorem, we can get a global understanding of the generalized Laguerre polynomials as a MOPS with respect to a non-diagonal Sobolev inner product like (1.4).

In Section 2, we will define the monic generalized Laguerre polynomials, for $\alpha \in \mathbb{R}$, by using their explicit expression, and we give some of their properties, which will be essential in this paper.

Section 3 is devoted to the definition of the non-diagonal Sobolev inner product (1.4). Finally, in Section 4 we will prove the theorem that shows the orthogonality of the generalized Laguerre polynomials, and we will obtain the results of Kwon and Littlejohn [3] as a particular case.

2. GENERALIZED LAGUERRE POLYNOMIALS

For $\alpha \in \mathbb{R}$, the nth *degree monic generalized Laguerre polynomial* is defined by means of the following expression

$$L_n^{(\alpha)}(x) = (-1)^n\, n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} \binom{n+\alpha}{n-j} x^j, \qquad n \geqslant 0. \tag{2.1}$$

Note that the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ satisfies

$$\deg(L_n^{(\alpha)}) = n \qquad n \geqslant 0.$$

In this way, for a given $\alpha \in \mathbb{R}$, the family of the generalized Laguerre polynomials is a basis for the linear space of the real polynomials, which we will denote by $\mathbb{P}$.

In the following lemma, we give some of the properties of the generalized Laguerre polynomials, which will be essential in the rest of the paper. The first three properties are well known and can be deduced from the explicit expression for these polynomials. Property (iv) can be shown by using induction on k, and applying properties (ii) and (iii).

LEMMA 2.1. *Given $\alpha \in \mathbb{R}$, let $k \geqslant 0$, $n \geqslant 1$ be integers. Then*

(i) (*Recurrence relation*)

$$\begin{gathered} L_{-1}^{(\alpha)}(x) = 0, \qquad L_0^{(\alpha)}(x) = 1, \\ xL_n^{(\alpha)}(x) = L_{n+1}^{(\alpha)} + \beta_n^{(\alpha)} L_n^{(\alpha)}(x) + \gamma_n^{(\alpha)} L_{n-1}^{(\alpha)}(x), \end{gathered} \tag{2.2}$$

where $\beta_n^{(\alpha)} = 2n + \alpha + 1$, $\gamma_n^{(\alpha)} = n(n+\alpha)$,

(ii) $$(L_n^{(\alpha)})'(x) = nL_{n-1}^{(\alpha+1)}(x), \tag{2.3}$$

(iii) $$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) + nL_{n-1}^{(\alpha+1)}(x), \tag{2.4}$$

(iv) $$L_n^{(\alpha)}(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (L_n^{(\alpha-k)})^{(i)}(x). \tag{2.5}$$

From Favard's theorem (Chihara [1, p. 21]) and property (i), we deduce that the generalized Laguerre polynomials are a MOPS with respect to a regular linear functional if

$$\gamma_n^{(\alpha)} \neq 0, \qquad n \geqslant 1,$$

that is, if $\alpha \notin \{-1, -2, ...\}$. Moreover, for $\alpha > -1$, the generalized Laguerre polynomials are the classical Laguerre polynomials which are associated with a positive definite linear functional.

3. THE NON-DIAGONAL SOBOLEV INNER PRODUCT

Let $k \geqslant 0$ be an integer. Let us define a matrix with dimension $k+1$ by

$$M(k) = (m_{i,j}(k))_{i,j=0}^{k},$$

where each element in the matrix is given by

$$m_{i,j}(k) = \sum_{p=0}^{\min\{i,j\}} (-1)^{i+j} \binom{k-p}{i-p} \binom{k-p}{j-p}, \qquad 0 \leqslant i,\ j \leqslant k.$$

LEMMA 3.1. *$M(k)$ is a positive definite matrix, with determinant equal to* 1.

Proof. Obviously $M(k)$ is a symmetric matrix and its positive definite character follows from the Cholesky factorization for $M(k)$ [4, p. 174], that is, for a lower triangular matrix $L(k)$ one has

$$M(k) = L(k)\, L(k)^T.$$

In fact, if we write

$$L(k) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\binom{k}{1} & 1 & 0 & \cdots & 0 \\ \binom{k}{2} & -\binom{k-1}{1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^k \binom{k}{k} & (-1)^{k-1}\binom{k-1}{k-1} & (-1)^{k-2}\binom{k-2}{k-2} & \cdots & 1 \end{pmatrix}$$

we can easily deduce

$$M(k) = L(k)\, L(k)^T.$$

Finally, $\det(M(k)) = \det(L(k)\, L(k)^T) = 1$. ∎

From the definition of the matrix $M(k)$, we can get a recursive scheme to compute it.

LEMMA 3.2. *Let $k \geqslant 1$. Then*

$$M(k) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & M(k-1) & \\ 0 & & \end{pmatrix}$$
$$+ \begin{pmatrix} \binom{k}{0} \\ -\binom{k}{1} \\ \vdots \\ (-1)^k \binom{k}{k} \end{pmatrix} \left(\binom{k}{0}, -\binom{k}{1}, ..., (-1)^k \binom{k}{k} \right).$$

Now, we can define the non-diagonal Sobolev inner product.

For an integer $k \geqslant 0$ and $\alpha > -1$, we define the symmetric bilinear form $(\ ,\ )_S^{(k,\alpha)}$ by means of the expression

$$(f, g)_S^{(k,\alpha)} = \int_0^{+\infty} F(x)\, M(k)\, G(x)^T\, x^\alpha e^{-x}\, dx, \tag{3.1}$$

where $F(x)$ and $G(x)$ are two vectors defined by

$$F(x) = (f(x), f'(x), ..., f^{(k)}(x)),$$
$$G(x) = (g(x), g'(x), ..., g^{(k)}(x)).$$

Note that, since $\alpha > -1$, all the integrals in this expression are finite, and as a consequence of the positive definite character of the symmetric matrix $M(k)$, we conclude that (3.1) is an inner product.

If we denote

$$(f, g)_L^{(\alpha)} = \int_0^{+\infty} f(x)\, g(x)\, x^\alpha e^{-x}\, dx, \tag{3.2}$$

the classical Laguerre inner product for $\alpha > -1$, then

$$(f, g)_S^{(0,\alpha)} = (f, g)_L^{(\alpha)},$$

since $M(0) = 1$ by definition.

From Lemma 3.2, we deduce that the non-diagonal Sobolev inner product defined in (3.1) can be written in a *recursive way*:

PROPOSITION 3.3. *Let $k \geqslant 1$ be an integer. Then*

$$(f, g)_S^{(k,\alpha)} = (f', g')_S^{(k-1,\alpha)} + \sum_{i,j=0}^{k} (-1)^{i+j} \binom{k}{i}\binom{k}{j} (f^{(i)}, g^{(j)})_S^{(0,\alpha)}. \tag{3.3}$$

4. ORTHOGONALITY OF THE GENERALIZED LAGUERRE POLYNOMIALS

In this section, we will show that the generalized Laguerre polynomials are orthogonal with respect to a non-diagonal Sobolev inner product like (3.1).

THEOREM 4.1. *Let $\alpha \in \mathbb{R}$. Then the monic generalized Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n \geqslant 0}$ are a MOPS with respect to the non-diagonal Sobolev inner product $(\cdot, \cdot)_S^{(k,\alpha+k)}$, where $k = \max\{0, [-\alpha]\}$.*

Proof. Obviously, for $\alpha > -1$ we have $k = 0$. Therefore, the monic generalized Laguerre polynomials are a classical MOPS, and from the properties of the Sobolev inner product, we have

$$(\cdot, \cdot)_S^{(0, \alpha+0)} = (\cdot, \cdot)_L^{(\alpha)}.$$

Given $\alpha \leqslant -1$, from the definition, we have $k = [-\alpha]$, and therefore all the integrals are finite. We will show the result by using induction on k.

For $k = 1$, we will multiply two generalized Laguerre polynomials with different degrees $L_n^{(\alpha)}(x)$ and $L_m^{(\alpha)}(x)$, where $n \neq m$, but $n, m \geqslant 1$. By using Proposition 3.3, and the properties of the generalized Laguerre polynomials (2.3) and (2.5), we get

$$\begin{aligned}
&(L_n^{(\alpha)}, L_m^{(\alpha)})_S^{(1, \alpha+1)} \\
&\quad = ((L_n^{(\alpha)})', (L_m^{(\alpha)})')_S^{(0, \alpha+1)} \\
&\qquad + \sum_{i, j=0}^{1} (-1)^{i+j} \binom{1}{i}\binom{1}{j} ((L_n^{(\alpha)})^{(i)}, (L_m^{(\alpha)})^{(j)})_S^{(0, \alpha+1)} \\
&\quad = nm(L_{n-1}^{(\alpha+1)}, L_{m-1}^{(\alpha+1)})_S^{(0, \alpha+1)} \\
&\qquad + \left(\sum_{i=0}^{1} (-1)^i \binom{1}{i} (L_n^{(\alpha)})^{(i)}, \sum_{j=0}^{1} (-1)^j \binom{1}{j} (L_m^{(\alpha)})^{(j)} \right)_S^{(0, \alpha+1)} \\
&\quad = (L_n^{(\alpha+1)}, L_m^{(\alpha+1)})_S^{(0, \alpha+1)} = 0.
\end{aligned}$$

The cases when $n = 0$ or $m = 0$ are trivial.

Now, we assume that the property is true for $k-1$ and we will show it for k. We will multiply two generalized Laguerre polynomials with different degrees $L_n^{(\alpha)}(x)$ and $L_m^{(\alpha)}(x)$, where $n \neq m$, but $n, m \geqslant 1$ (the cases $n = 0$ or $m = 0$ are trivial). Using again the Proposition 3.3, the properties (2.3) and (2.5), and the induction hypothesis, we get

$$\begin{aligned}
&(L_n^{(\alpha)}, L_m^{(\alpha)})_S^{(k, \alpha+k)} \\
&\quad = ((L_n^{(\alpha)})', (L_m^{(\alpha)})')_S^{(k-1, \alpha+k)} \\
&\qquad + \sum_{i, j=0}^{k} (-1)^{i+j} \binom{k}{i}\binom{k}{j} ((L_n^{(\alpha)})^{(i)}, (L_m^{(\alpha)})^{(j)})_S^{(0, \alpha+k)} \\
&\quad = nm(L_{n-1}^{(\alpha+1)}, L_{m-1}^{(\alpha+1)})_S^{(k-1, \alpha+1+k-1)} \\
&\qquad + \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} (L_n^{(\alpha)})^{(i)}, \sum_{j=0}^{k} (-1)^j \binom{k}{j} (L_m^{(\alpha)})^{(j)} \right)_S^{(0, \alpha+k)} \\
&\quad = (L_n^{(\alpha+k)}, L_m^{(\alpha+k)})_S^{(0, \alpha+k)} = 0. \quad \blacksquare
\end{aligned}$$

Remark. In the case when $\alpha \in \{-1, -2, ...\}$, we have $k = [-\alpha] = -\alpha$, and the non-diagonal Sobolev inner product can be written in the following way

$$(f, g)_S^{(k,0)} = \int_0^{+\infty} F(x)\, M(k)\, G(x)^T e^{-x}\, dx.$$

In this case, taking integration by parts we can write

$$(f, g)_S^{(k,0)} = \frac{1}{2} \sum_{i=0}^{k-1} \sum_{j=0}^{i} m_{i,j}(k)[f^{(i)}(0)\, g^{(j)}(0) + f^{(j)}(0)\, g^{(i)}(0)] + \int_0^{+\infty} f^{(k)}(x)\, g^{(k)}(x)\, e^{-x}\, dx. \tag{4.1}$$

This inner product is the same as the one introduced by Kwon and Littlejohn [3]. These authors, after several pages of combinatorial computations, show that the generalized Laguerre polynomials $\{L_n^{(-k)}(x)\}_{n \geqslant 0}$ are orthogonal with respect to the inner product (4.1).

REFERENCES

1. T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon and Breach, New York, 1978.
2. W. N. Everitt, K. H. Kwon, and L. L. Littlejohn, Spectral theory of the Laguerre polynomials when $-\alpha \in \mathbb{N}$, preprint.
3. K. H. Kwon and L. L. Littlejohn, The orthogonality of the Laguerre polynomials $\{L_n^{(-k)}(x)\}$ for positive integers k, *Ann. Numer. Anal.* **2** (1995), 1–4, 289–303.
4. J. Stoer and R. Burlirsch, "Introduction to Numerical Analysis," Springer-Verlag, New York, 1980.
5. G. Szegő, "Orthogonal Polynomials," Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1975.