The Generalized Stirling Numbers, Sheffer-type Polynomials and Expansion Theorems

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The concept of a generalized Stirling number pair can be characterized by a pair of inverse relations. A list of related work on the topic can be found from Jordan [39], Riordan [37, 58], Gould [61], Gould-Hopper [62], Rota [64], Comtet [74], Carlitz [80], Joni-Rota-Sagan [81], Charalambides [83, 84], Howard [84, 85], Cacoullos-Papageorgiou [84], Tsylova [85], Nandi-Dutta [87], Todorov [88], Hsu [93], Hsu-Yu [96], Hsu-Shiue [98, 99], Remmel-Wachs [04], and He-Hsu-Shiue [05, 06]. Here we present the generalized Stirling numbers and Sheffer-type polynomials generated by univariate power series and their applications in expansion problems. The higher dimensional setting will appear in He-Hsu-Shiue's paper (2006).

Definition 1 *Let* $\Gamma \equiv (\Gamma, +, \cdot)$ *be the commutative ring of formal power series over the real or complex field, in which the ordinary addition and Cauchy multiplication are defined. Then* *any two elements ϕ and ψ of Γ are said to be reciprocal (compositional inverse) of each other if and only if $\phi \circ \psi(t) = \psi \circ \phi(t) = t$ with $\phi(0) = \psi(0) = 0$.*

Definition 2 *Let $A(t)$, $g(t)$ and $f(t) \in \Gamma$ with $A(0) = 1$, $g(0) = 0$ and $g'(0) \neq 0$. Then the polynomials $p_n(x)$ ($n = 0, 1, 2, \cdots$) as defined by the generating function (GF)*

$$A(t)e^{xg(t)} = \sum_{n \geq 0} p_n(x)t^n$$

are called Sheffer-type polynomials with $p_0(x) = 1$. Accordingly, $p_n(D)$ with $D \equiv d/dt$ is called Sheffer-type differential operator of degree n associated with $A(t)$ and $g(t)$. In particular, $p_0(D) \equiv I$ is the identity operator.

Note that $\{p_k(x)\}$ is also called the sequence of Sheffer A-type zero, which has been treated throughly by Roman-Rota [78] and Rota [84] using umbral calculus (cf. also Boas-Buck [64]). For formula power series $f(t)$, the coefficient of t^k is usually denoted by $[t^k]f(t)$. Accordingly, we have the expression $p_k(x) = [t^k]A(t)e^{xg(t)}$. Also, we shall frequently use the denotation

$$p_k(D)f(0) = [p_k(D)f(t)]_{t=0}.$$

Throughout this talk all series expansions are formal, so that the symbolic calculus with operators *D* (formal differentiation) and *E* (shift) applies to all formal power series, where E is defined by $Ef(t) = f(t+1), E^x f(t) = f(t+x)$ ($x$is a real number), and satisfies the formal relations

$$E = e^D = \sum_{k\geq 0} \frac{1}{k!} D^k, E^x f(0) = e^{xD} f(0) = f(x).$$

Theorem 3 *(First Expansion theorem) Let* $A(t)$, $g(t)$ *and* $f(t)$ *be a formal power series over* $\mathbf{C}$, *with* $A(0) = 1, g(0) = 0$ *and* $g'(0) \neq 0$. *Then there holds an expansion formula of the form*

$$A(t)f(g(t)) = \sum_{k \geq 0} t^k p_k(D) f(0)$$

where $p_0(D)f(0) = f(0)$, *and* $p_k(D)$ *are Sheffer-type differential operators associated with* $A(t)$ *and* $g(t)$. *Moreover,* $p_k(D)(k = 0, 1, 2, \cdots)$ *satisfy the recurrence relations*

$$(k+1)p_{k+1}(D) = \sum_{j=0}^{k} (\alpha_j + \beta_j D) p_{k-j}(D)$$

with $p_0(D) = I$ *and* α_j, β_j *being given by*

$$\alpha_j = (j+1)[t^{j+1}] \text{ log } A(t), \; \beta_j = (j+1)[t^{j+1}]g(t).$$

Noted that Theorem 3 is equivalent to the computational rule:

$$\begin{aligned} & p_k(x) := [t^k]A(t)e^{xg(t)} \\ \Longrightarrow \quad & p_k(D)f(0) := [t^k]A(t)f(g(t)). \end{aligned}$$

Of course the number-sequence $\{p_k(D)f(0)\}_0^\infty$ has the $(GF) - A(t)f(g(t))$.

For the case $A(t) \equiv 1$, the expansion in the theorem is substantially equivalent to the Faa Di Bruno formula. Indeed, if $g(t) = \sum_{m\geq 1} a_m t^m/m!$, it follows that $e^{xg(t)}$ may be written in the form

$$\exp\,\{x \sum_{m\geq 1} a_m \frac{t^m}{m!}\} = 1 + \sum_{k\geq 1} \frac{t^k}{k!}\{\sum_{j=1}^{k} x^j B_{kj}(a_1, a_2, \cdots)\}$$

so that

$$p_k(x) = [t^k]e^{xg(t)} = \frac{1}{k!}\sum_{j=1}^{k} x^j B_{kj}(a_1, a_2, \cdots).$$

Consequently we have

$$[t^k]f(g(t)) = \frac{1}{k!}\sum_{j=1}^{k} B_{kj}(a_1, a_2, \cdots)D^j f(0).$$

This is precisely the Faa di Bruno formula

$$[(d/dt)^k f(g(t))]_{t=0} = \sum_{j=1}^{k} B_{kj}(g'(0), g''(0), \cdots)f^{(j)}(0).$$

Note that $B_{kj}(a_1, a_2, \cdots)$ is the so-called incomplete Bell polynomial whose explicit expression can easily be derived from the relation on $exp\{x\sum_{m\geq 1} a_m\frac{t^m}{m!}\}$ (cf. Comtet [74]), namely

$$B_{kj}(a_1, a_2, \cdots, a_{k-j+1}) = \sum_{(c)} \frac{k!}{c_1!c_2!\cdots}\left(\frac{a_1}{1!}\right)^{c_1}\left(\frac{a_2}{2!}\right)^{c_2}\cdots$$

where the summation extends over all integers $c_1, c_2, \cdots \geq 0$, such that $c_1 + 2c_2 + 3c_3 + \cdots = k, c_1 + c_2 + \cdots = j$.

More examples. Let $B_n(x), \widehat{C}_n^{(\alpha)}(x)$ and $T_n^{(p)}(x)$ be Bernouilli, Charlier and Touchard polynomials, respectively. Then, for any given formal power series $f(t)$ over **C** we have 3 weighted expansion formulas as follows

$$\frac{t}{e^t - 1} f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_n(D) f(0)$$

$$e^{-\alpha t} f(\log(1+t)) = \sum_{n=0}^{\infty} t^n \widehat{C}_n^{(\alpha)}(D) f(0) \quad (\alpha \neq 0)$$

$$(1-t)^p f(e^t - 1) = \sum_{n=0}^{\infty} t^n T_n^{(p)}(D) f(0). \quad (p > 0)$$

Definition 4 *Let A(t) and g(t) be given as in Theorem 1. Then we have a weighted Stirling-type pair* $\{\sigma(n,k), \sigma^*(n,k)\}$ *as defined by*

$$\frac{1}{k!}A(t)(g(t))^k = \sum_{n=k}^{\infty} \sigma(n,k)\frac{t^n}{n!}$$

$$\frac{1}{k!}A(g^*(t))^{-1}(g^*(t))^k = \sum_{n=k}^{\infty} \sigma^*(n,k)\frac{t^n}{n!},$$

where $g^* \equiv g^{\langle -1 \rangle}$ *is the compositional inverse of* g *with* $g^*(0) = 0$, $[t]g^*(t) \neq 0$, *and* $\sigma(0,0) = \sigma^*(0,0) = 1$.

Recall that classical Stirling numbers of the first and second kinds may be defined by the generating functions $A = 1$, $g(t) = (\ln(1 + t))^k/k!$, and $g^*(t) = (e^t - 1)^k/k!$.

Theorem 5 *The Sheffe-type operator* $p_n(D)$ *has an expression of the form*

$$p_n(D) = \frac{1}{n!} \sum_{k=0}^{n} \sigma(n,k) D^k,$$

where σ(n, k) (associated with A(t) and g(t)) may be written in the form

$$\sigma(n,k) = \sum_{r=k}^{k} \binom{n}{r} \alpha_{n-r} B_{rk}(a_1, a_2, \cdots)$$

provided that $A(t) = \sum_{m\geq 0} \alpha_m t^m/m!$ *and* $g(t) = \sum_{m\geq 1} a_m t^m/m!$ *with* $\alpha_0 = 1, a_1 \neq 0$.

Corollary 6 *The formula shown as in the expansion theorem may be rewritten in the form*

$$A(t)f(g(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\sum_{k=0}^{n} \sigma(n,k) f^{(k)}(0)),$$

where $\sigma(n,k)$'s are defined by Definition 4 and given by Theorem 5.

Corollary 7 *For the case $A(t) \equiv 1$, Theorem 5 gives $\sigma(n,k) = B_{nk}(a_1, a_2, \cdots)$.*

Corollary 8 *The generalized exponential polynomials related to the generalized Stirling numbers $\sigma(n,k)$ and $\sigma^*(n,k)$ are given, respectively by the following*

$$n! p_n(x) = \sum_{k=0}^{n} \sigma(n,k) x^k$$

and

$$n! p_n^*(x) = \sum_{k=0}^{n} \sigma^*(n,k) x^k,$$

where $p_n(x)$ *and* $p_n^*(x)$ *are Sheffer-type polynomials associated with* $\{A(t), g(t)\}$ *and* $\{A(g^*(t))^{-1}, g^*(t)\}$, *respectively*

Theorem 9 *Definition 4 implies the orghogonality relations*

$$\sum_{k \le n \le m} \sigma(m,n)\sigma^*(n,k) = \sum_{k \le n \le m} \sigma^*(m,n)\sigma(n,k) = \delta_{mk}$$

with δ_{mk} *denoting the Kronecker delta, and it follows that there hold the inverse relations*

$$f_n = \sum_{k=0}^{n} \sigma(n,k) g_k \Longleftrightarrow g_n = \sum_{k=0}^{n} \sigma^*(n,k) f_k.$$

Applying the reciprocal relations shown as in Theorem 9 to Corollary 8 we get

Corollary 10 *There hold the relations*

$$\sum_{k=0}^{n} \sigma^*(n,k)k!p_k(x) = x^n$$

and

$$\sum_{k=0}^{n} \sigma(n,k)k!p_k^*(x) = x^n.$$

These may be used as recurrence relations for $p_n(x)$ *and* $p_n^*(x)$ *respectively.*

Theorem 5 and Corollary 6 imply a higher derivative formula for $A(t)f(g(t))$ at $t = 0$, namely

$$\begin{aligned} &(\frac{d}{dt})_0^n (A(t)f(g(t))) \\ = &\sum_{k=0}^{n} \sigma(n,k)f^{(k)}(0) = n!p_n(D)f(0). \end{aligned}$$

Certainly, this will reduce to the Faa di Bruno formula when $A(t) \equiv 1$.

As an application of Theorem 9, we now turn to the problem for finding an expansion of a multivariate analytic function f in terms of a sequence of higher Sheffer-type polynomials $\{p_n\}$.

Theorem 11 *(Second Expansion Theorem) Let $f(z)$ be an analytic function defined on $\mathbb{C}$. Then we have the expansion of f in terms of a sequence of Sheffer-type polynomials $\{p_k\}$ as*

$$f(z) = \sum_{k \geq 0} \alpha_k p_k(z),$$

where

$$\alpha_k = \sum_{n \geq k} \frac{k!}{n!} \sigma^*(n,k) D^n f(0)$$

From the expression of α_k in Theorem 11, it is not hard to derive Boas-Buck formulas (7.3)-(7.4) of the coefficients of the series expansion of an entire function in terms of polynomial

$p_k(z)$ by using the expression of α_k, Definition 4, Cauchy's residue theorem, and careful discussion on the convergence.

We now give algorithms to derive the series expansion of $f(z)$ in terms of a Sheffer type polynomial set $\{p_n(x)\}_{n\in\mathbb{N}}$.

Algorithm 3.1

Step 1 For given Sheffer type polynomial $\{p_n(x)\}_{n\in\mathbb{N}}$, we determine its GF pair $(A(t), g(t))$ and the compositional inverse $g^*(t)$ of $g(t)$.

Step 2 Use Corollary 8 to evaluate set $\{\sigma^*(n,k)\}_{n\geq k}$ and substitute it into the corresponding expression in Theorem 11 to find α_k $(k \geq 0)$.

Algorithm 3.2

Step 1 For given Sheffer type polynomial $\{p_n(x)\}_{n\in\mathbb{N}}$, apply the first equation in Corollary 8 to obtain set $\{\sigma(n,k)\}_{n\geq k\geq 0}$.

Step 2 Use Theorem 9 to solve for set $\{\sigma^*(n,k)\}_{n\geq k}$ and substitute it into the corresponding expression in Theorem 11 to find α_k $(k\geq 0)$.

It is easy to see the equivalence of the two algorithms. However, the first algorithm is more readily applied than the second one.