The classical umbral calculus: Sheffer sequences

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Abstract[1]. Following the approach of Rota and Taylor [17], we present an innovative theory of Sheffer sequences in which the main properties are encoded by using umbrae. This syntax allows us noteworthy computational simplifications and conceptual clarifications in many results involving Sheffer sequences. To give an indication of the effectiveness of the theory, we describe applications to the well-known connection constants problem, to Lagrange inversion formula and to solving some recurrence relations.

1. Introduction

As well known, many polynomial sequences like Laguerre polynomials, first and second kind Meixner polynomials, Poisson-Charlier polynomials and Stirling polynomials are Sheffer sequences. Sheffer sequences can be considered the core of umbral calculus: a set of tricks extensively used by mathematicians at the beginning of the twentieth century.

Umbral calculus was formalized in the language of the linear operators by Gian-Carlo Rota in a series of papers (see [15], [16], and [14]) that have produced a plenty of applications (see [1]). In 1994 Rota and Taylor [17] came back to foundation of umbral calculus with the aim to restore, in light formal setting, the computational power of the original tools, heuristical applied by founders Blissard, Cayley and Sylvester. In this new setting, to which we refer as the classical umbral calculus, there are two basic devices. The first one is to represent a unital sequence of numbers by a symbol α, called an umbra, that is, to represent the sequence $1, a_1, a_2, \ldots$ by means of the sequence $1, \alpha, \alpha^2, \ldots$ of powers of α via an operator E, resembling the expectation operator of random variables. The second device is that distinct umbrae may represent the same sequence $1, a_1, a_2, \ldots$, as it happens also in probability theory for independent and identically distributed random variables. It is mainly thanks to these devices that the early umbral calculus has had a rigorous and simple formal look.

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At first glance, the classical umbral calculus seems just a notation for dealing with exponential generating functions. Nevertheless, this new syntax has given rise noteworthy computational simplifications and conceptual clarifications in different contexts. Applications are given by Zeilberger [23], where generating functions are computed for many difficult problems dealing with counting combinatorial objects. Applications to bilinear generating functions for polynomial sequences are given by Gessel [7]. Connections with wavelet theory have been investigated in [19] and [20]. In [4], the development of this symbolic computation has produced the theory of Bell umbrae, by which the functional composition of exponential power series has been interpreted in a effective way. On the basis of this result, the umbral calculus has been interpreted as a calculus of measures on Poisson algebras, generalizing compound Poisson processes [4]. A natural parallel with random variables has been further carried out in [5]. In [6], the theory of k-statistics and polykays has been completely rewritten, carrying out a unifying framework for these estimators, both in the univariate and multivariate cases. Moreover, very fast algorithms for computing these estimators have been carried out.

Apart from the preliminary paper of Taylor [22], Sheffer sequences have not been described in terms of umbrae. Here we complete the picture, giving many examples and several applications.

The paper is structured as follows. Section 2 is provided for readers unaware of the classical umbral calculus. We resume terminology, notation and some basic definitions. In Section 3, we introduce the notion of the adjoint of an umbra. This notion is the key to clarify the nature of the umbral presentation of binomial sequences. In Section 4, by introducing umbral polynomials, we stress a feature of the classical umbral calculus, that is the construction of new umbrae by suitable symbolic substitutions. Section 5 is devoted to Sheffer umbrae. We introduce the notion of Sheffer umbra from which we derive an umbral characterization of Sheffer sequences $\{s_n(x)\}$. Theorem 5.3 gives the umbral version of the well-known Sheffer identity with respect to the associated sequence. Theorem 5.4 gives the umbral version of a second characterization of Sheffer sequences, that is $\{s_n(x)\}$ is said to be a Sheffer sequence with respect to a delta operator Q, when $Qs_n(x) = ns_{n-1}(x)$ for all $n \geq 0$. In Section 6, the notion of Sheffer umbra is used to introducing two special umbrae: the one associated to an umbra, whose moments are binomial sequences, and the Appell umbra, whose moments are Appell polynomials. We easily state their main properties by umbral methods. In the last section, we discuss some topics to which umbral methods can be fruitfully applied. In particular we deal with the connection constants problem, that gives the coefficients in expressing a sequence of polynomials $\{s_n(x)\}$ in terms of a different sequence of polynomials $\{p_n(x)\}$ and viceversa. We give a very simple proof of the Lagrange inversion formula by showing that all polynomials of binomial type are represented by Abel polynomials. This last result was proved by Rota, Shen and Taylor in [18], but the authors did not make explicit the relations among the involved umbrae, and thus have not completely pointed out the powerfulness of the result. Moreover, the notion of a Sheffer umbra brings to the light the umbral connection between binomial sequences and Abel polynomials. This allows us to give a very handy umbral expression for the Stirling numbers of first and second type. Finally, we would like to stress how the connection between Sheffer umbrae and Lagrange inversion formula has smoothed the way to an umbral theory of free cumulants [3].

In closing, we provide some examples of exact solutions of linear recursions, which benefit of an umbral approach.

2. The classical umbral calculus

In the following, we recall terminology, notation and some basic definitions of the classical umbral calculus, as it has been introduced by Rota and Taylor in [17] and further developed in [4] and [5] . An umbral calculus consists of the following data:

- **a)** a set $A = \{\alpha, \beta, \ldots\}$, called the *alphabet*, whose elements are named *umbrae*;
- **b)** a commutative integral domain R whose quotient field is of characteristic zero;
- **c)** a linear functional E, called *evaluation*, defined on the polynomial ring $R[A]$ and taking values in R such that
    - **i)** $E[1] = 1$;
    - **ii)** $E[\alpha^i \beta^j \cdots \gamma^k] = E[\alpha^i]E[\beta^j] \cdots E[\gamma^k]$ for any set of distinct umbrae in A and for $i, j, \ldots, k$ nonnegative integers (*uncorrelation property*);
- **d)** an element $\epsilon \in A$, called *augmentation* [14], such that $E[\epsilon^n] = \delta_{0,n}$, for any nonnegative integer n, where

$$\delta_{i,j} = \left\{ \begin{array}{ll} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{array} \right. \qquad i, j \in N;$$

- **e)** an element $u \in A$, called *unity* umbra [4], such that $E[u^n] = 1$, for any nonnegative integer n.

A sequence $a_0 = 1, a_1, a_2, \ldots$ in R is umbrally represented by an umbra α when

$$E[\alpha^i] = a_i \quad, \quad \text{for } i = 0, 1, 2, \ldots .$$

The elements a_i are called *moments* of the umbra α.

Example 2.1. *Singleton umbra.*
The singleton umbra χ is the umbra such that

$$E[\chi^1] = 1 \quad, \quad E[\chi^n] = 0 \quad \text{for } n = 2, 3, \ldots .$$

The *factorial moments* of an umbra α are the elements

$$a_{(n)} = \left\{ \begin{array}{ll} 1, & n = 0, \\ E[(\alpha)_n], & n > 0, \end{array} \right.$$

where $(\alpha)_n = \alpha(\alpha - 1) \cdots (\alpha - n + 1)$ is the lower factorial.

Example 2.2. *Bell umbra.*
The Bell umbra β is the umbra such that

$$E[(\beta)_n] = 1 \qquad \text{for } n = 0, 1, 2, \ldots .$$

In [4] we prove that $E[\beta^n] = B_n$, where B_n is the n-th Bell number, i.e. the number of partitions of a finite nonempty set with n elements, or the n-th coefficient in the Taylor series expansion of the function $\exp(e^t - 1)$.

An umbral polynomial is a polynomial $p \in R[A]$. The support of p is the set of all umbrae occurring in p. If p and q are two umbral polynomials then

- *i)* p and q are *uncorrelated* if and only if their supports are disjoint;
- *ii)* p and q are *umbrally equivalent* if and only if $E[p] = E[q]$, in symbols $p \simeq q$.

2.1. **Similar umbrae and dot-product.** The notion of similarity among umbrae
comes in handy in order to manipulate sequences such

$$(1) \qquad \sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i} \quad , \quad \text{for } n = 0, 1, 2, \dots$$

as moments of umbrae. The sequence (1) cannot be represented by using only the
umbra α with moments $a_0 = 1, a_1, a_2, \dots$. Indeed, α being correlated to itself,
the product $a_i a_{n-i}$ cannot be written as $E[\alpha^i \alpha^{n-i}]$. So, as it happens for random
variables, we need two distinct umbrae having the same sequence of moments.
Therefore, if we choose an umbra α' uncorrelated with α but with the same sequence
of moments, we have

$$(2) \qquad \sum_{i=0}^{n} \binom{n}{i} a_i a_{n-i} = E\left[\sum_{i=0}^{n} \binom{n}{i} \alpha^i (\alpha')^{n-i} \right] = E[(\alpha + \alpha')^n] .$$

Then the sequence (1) represents the moments of the umbra $(\alpha + \alpha')$. In [17], Rota
and Taylor formalize this matter by defining an equivalence relation among umbrae.

Two umbrae α and γ are *similar* when α^n is umbrally equivalent to γ^n, for all
$n = 0, 1, 2, \dots$ in symbols

$$\alpha \equiv \gamma \Leftrightarrow \alpha^n \simeq \gamma^n \qquad n = 0, 1, 2, \dots .$$

Example 2.3. *Bernoulli umbra.*
The Bernoulli umbra ι (cf. [17]) satisfies the umbral equivalence $\iota + u \equiv -\iota$. Its
moments are the Bernoulli numbers B_n, such that

$$\sum_{k \geq 0} \binom{n}{k} B_k = B_n .$$

Thanks to the notion of similar umbrae, it is possible to extend the alphabet
A with the so-called *auxiliary* umbrae, resulting from operations among similar
umbrae. This leads to construct a saturated umbral calculus, in which auxiliary
umbrae are handled as elements of the alphabet. It can be shown that saturated
umbral calculi exist and that every umbral calculus can be embedded in a saturated
umbral calculus [17]. We shall denote by the symbol $n.\alpha$ the *dot-product* of n and
α, an auxiliary umbra (cf. [17]) similar to the sum $\alpha' + \alpha'' + \cdots + \alpha'''$ where
$\{\alpha', \alpha'', \dots, \alpha'''\}$ is a set of n distinct umbrae, each one similar to the umbra α. So
the sequence in (2) is umbrally represented by the umbra $2.\alpha$. We assume that $0.\alpha$
is an auxiliary umbra similar to the augmentation ϵ.

The next statements follow from the definition of the dot-product.

Proposition 2.1.

    i) $n.\alpha \equiv n.\beta$ for some integer $n \neq 0$ if and only if $\alpha \equiv \beta$;
    ii) if $c \in R$, then $n.(c\alpha) \equiv c(n.\alpha)$ for any nonnegative integer n;
    iii) $n.(m.\alpha) \equiv (nm).\alpha \equiv m.(n.\alpha)$ for any two nonnegative integers n, m;
    iv) $(n+m).\alpha \equiv n.\alpha + m.\alpha'$ for any two nonnegative integers n, m and any two
       distinct umbrae $\alpha \equiv \alpha'$;
    v) $(n.\alpha + n.\gamma) \equiv n.(\alpha + \gamma)$ for any nonnegative integer n and any two distinct
       umbrae α and γ.

Two umbrae α and γ are said to be *inverse* to each other when $\alpha + \gamma \equiv \varepsilon$. We denote the inverse of the umbra α by $-1.\alpha$. Note that they are uncorrelated. Recall that, in dealing with a saturated umbral calculus, the inverse of an umbra is not unique, but any two umbrae inverse to any given umbra are similar.

We shall denote by the symbol $\alpha^{\cdot\, n}$ the *dot-power* of α, an auxiliary umbra similar to the product $\alpha'\alpha'' \cdots \alpha'''$, where $\{\alpha', \alpha'', \ldots, \alpha'''\}$ is a set of n distinct umbrae, similar to the umbra α. We assume that $\alpha^{\cdot\, 0}$ is an umbra similar to the unity umbra u.

The next statements follow from the definition of the dot-power.

Proposition 2.2.

  i) If $c \in R$, then $(c\alpha)^{\cdot\, n} \equiv c^n \alpha^{\cdot\, n}$ for any nonnegative integer $n \neq 0$;

  ii) $(\alpha^{\cdot\, n})^{\cdot\, m} \equiv \alpha^{\cdot\, nm} \equiv (\alpha^{\cdot\, m})^{\cdot\, n}$ for any two nonnegative integers n, m;

  iii) $\alpha^{\cdot\, (n+m)} \equiv \alpha^{\cdot\, n}(\alpha')^{\cdot\, m}$ for any two nonnegative integers n, m and any two distinct umbrae $\alpha \equiv \alpha'$;

  iv) $(\alpha^{\cdot\, n})^k \equiv (\alpha^k)^{\cdot\, n}$ for any two nonnegative integers n, k.

By the statement *iv)*, the moments of $\alpha^{\cdot\, n}$ are:

$$(3) \qquad E[(\alpha^{\cdot\, n})^k] = E[(\alpha^k)^{\cdot\, n}] = a_k^n \quad , \quad k = 0, 1, 2, \ldots$$

for any nonnegative integer n. Hence the moments of the umbra $\alpha^{\cdot\, n}$ are the n-th power of the moments of the umbra α.

Moments of $n.\alpha$ can be expressed using the notions of integer partitions and dot-powers. Recall that a partition of an integer i is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$, where λ_j are weakly decreasing positive integers such that $\sum_{j=1}^{t} \lambda_j = i$. The integers λ_j are named *parts* of λ. The *lenght* of λ is the number of its parts and will be indicated by ν_λ. A different notation is $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$, where r_j is the number of parts of λ equal to j and $r_1 + r_2 + \cdots = \nu_\lambda$. We use the classical notation $\lambda \vdash i$ to denote "λ is a partition of i". By using an umbral version of the well-known multinomial expansion theorem, we have

$$(4) \qquad (n.\alpha)^i \simeq \sum_{\lambda \vdash i} (n)_{\nu_\lambda} d_\lambda \alpha_\lambda \,,$$

where the sum is over all partitions $\lambda = (1^{r_1}, 2^{r_2}, \ldots)$ of the integer i, $(n)_{\nu_\lambda} = 0$ for $\nu_\lambda > n$,

$$(5) \qquad d_\lambda = \frac{i!}{r_1! r_2! \cdots} \frac{1}{(1!)^{r_1} (2!)^{r_2} \cdots} \qquad \text{and} \qquad \alpha_\lambda = (\alpha_1)^{\cdot\, r_1} (\alpha_2)^{\cdot\, r_2} \cdots \,,$$

with $\alpha_1, \alpha_2, \ldots$ uncorrelated and similar umbrae.

2.2. The generating function of an umbra. The formal power series

$$(6) \qquad u + \sum_{n \geq 1} \alpha^n \frac{t^n}{n!}$$

is the *generating function* (g.f.) of the umbra α, and it is denoted by $e^{\alpha t}$. The notion of umbral equivalence and similarity can be extended coefficientwise to formal power series of $R[A][[t]]$

$$\alpha \equiv \beta \Leftrightarrow e^{\alpha t} \simeq e^{\beta t} \,,$$

(see [21] for a formal construction). Note that any exponential formal power series

$$(7) \qquad f(t) = 1 + \sum_{n \geq 1} a_n \, \frac{t^n}{n!}$$

can be umbrally represented by a formal power series (6) in $R[A][[t]]$. In fact, if the sequence $1, a_1, a_2, \dots$ is umbrally represented by α then

$$f(t) = E[e^{\alpha t}] \quad \text{i.e.} \quad f(t) \simeq e^{\alpha t} \, ,$$

assuming that we extend E by linearity. We denote the formal power series in (7) by $f(\alpha, t)$ and we will say that $f(\alpha, t)$ is umbrally represented by α. Henceforth, when no confusion occurs, we will just say that $f(\alpha, t)$ is the g.f. of α. For example the g.f. of the augmentation umbra ϵ is $f(\epsilon, t) = 1$, while the g.f. of the unity umbra u is $f(u, t) = e^t$. The g.f. of the singleton umbra χ is $f(\chi, t) = 1 + t$, the g.f. of the Bell umbra is $f(\beta, t) = \exp(e^t - 1)$ and the g.f. of the Bernoulli umbra is $f(\iota, t) = t/(e^t - 1)$.

The advantage of an umbral notation for g.f.'s is the representation of operations among g.f.'s through symbolic operations among umbrae. For example, the product of exponential g.f.'s is umbrally represented by a sum of the corresponding umbrae:

$$(8) \qquad f(\alpha, t) \, f(\gamma, t) \simeq e^{(\alpha + \gamma)t} \quad \text{with} \quad f(\alpha, t) \simeq e^{\alpha t} \ \text{and} \ f(\gamma, t) \simeq e^{\gamma t} \, .$$

Via (8), the g.f. of $n.\alpha$ is $f(\alpha, t)^n$. Note that

$$(9) \qquad e^{(n.\,\alpha)\,t} \simeq f(t)^n \simeq \left(e^{\alpha t} \right)^{\cdot n}$$

Via g.f., we have [4]

$$(10) \qquad E[(n.\alpha)^i] = \sum_{j=1}^{i} (n)_j \, B_{i,j}(a_1, a_2, \dots, a_{i-j+1}) \qquad i = 1, 2, \dots$$

where $B_{i,j}$ are the (partial) Bell exponential polynomials (cf. [12]) and a_i are the moments of the umbra α.

If α is an umbra with g.f. $f(\alpha, t)$, then

$$e^{(-1.\,\alpha)\,t} \simeq \frac{1}{f(\alpha, t)} \quad .$$

The sum of exponential g.f.'s is umbrally represented by a disjoint sum of umbrae. The *disjoint sum* (respectively *disjoint difference*) of α and γ is the umbra η (respectively δ) with moments

$$\eta^n \simeq \left\{ \begin{array}{ll} u \, , & n = 0 \\ \alpha^n + \gamma^n \, , & n > 0 \end{array} \right. \quad \left(\text{respectively} \quad \delta^n \simeq \left\{ \begin{array}{ll} u \, , & n = 0 \\ \alpha^n - \gamma^n \, , & n > 0 \end{array} \right. \right) ,$$

in symbols $\eta \equiv \alpha \dotplus \gamma$ (respectively $\delta \equiv \alpha \dotminus \gamma$). By the definition, we have

$$f(\alpha, t) \pm [f(\gamma, t) - 1] \simeq e^{(\alpha \pm \gamma)t} \, .$$

2.3. **Polynomial umbrae.** The introduction of the g.f. device leads to the definition of new auxiliary umbrae, improving the computational power of the umbral syntax. For this purpose, we could replace R by a suitable polynomial ring having coefficients in R and any desired number of indeterminates. Then, an umbra is said to be *scalar* if the moments are elements of R while it is said to be *polynomial* if the moments are polynomials. In this paper, we deal with $R[x, y]$. In particular,

we define the dot-product of x and α via g.f., i.e. $x.\alpha$ is the auxiliary umbra having g.f.

$$e^{(x.\alpha)} \simeq f(\alpha, t)^x .$$

Proposition 2.1 still holds, replacing n with x and m with y.

Example 2.4. *Bell polynomial umbra.*
The umbra $x.\beta$ is the Bell polynomial umbra. Its factorial moments are powers of x and its moments are the exponential polynomials (cf. [4])

$$(x.\beta)_n \simeq x^n \quad \text{and} \quad (x.\beta)^n \simeq \sum_{k=0}^{n} S(n, k)x^k .$$

Its g.f. is $f(x.\beta, t) = \exp[x(e^t - 1)]$.

2.4. **Special auxiliary umbrae.** A feature of the classical umbral calculus is the construction of new auxiliary umbrae by suitable symbolic substitutions. In $n.\alpha$ replace the integer n by an umbra γ. From (10), the new auxiliary umbra $\gamma.\alpha$ has moments

$$(11) \qquad E[(\gamma.\alpha)^i] = \sum_{j=1}^{i} g_{(j)} B_{i,j}(a_1, a_2, \ldots, a_{i-j+1}) \qquad i = 1, 2, \ldots$$

where $g_{(j)}$ are the factorial moments of the umbra γ. The auxiliary umbra $\gamma.\alpha$ is called *dot-product* of α and γ. The g.f. $f(\gamma.\alpha, t)$ is such that

$$e^{(\gamma.\alpha)t} \simeq [f(t)]^\gamma \simeq e^{\gamma \log f(t)} \simeq g[\log f(t)] .$$

Observe that $E[\gamma.\alpha] = g_1 a_1 = E[\gamma] E[\alpha]$. The following statements hold.

Proposition 2.3.
- **i)** $\eta.\alpha \equiv \eta.\gamma$ *for some umbra η if and only if $\alpha \equiv \gamma$;*
- **ii)** *if $c \in R$, then $\eta.(c\alpha) \equiv c(\eta.\alpha)$ for any two distinct umbrae α and η;*
- **iii)** *if $\gamma \equiv \gamma'$, then $(\alpha + \eta).\gamma \equiv \alpha.\gamma + \eta.\gamma'$;*
- **iv)** $\eta.(\gamma.\alpha) \equiv (\eta.\gamma).\alpha$.

For the proofs, see [4]. Observe that from property *ii)* it follows

$$(12) \qquad \alpha.x \equiv \alpha.(xu) \equiv x(\alpha.u) \equiv x\alpha .$$

In the following, we recall some useful dot-products of umbrae, whose properties have been investigated with full details in [4] and [5].

Example 2.5. *Exponential umbral polynomials.*
Suppose we replace x with a generic umbra α in the Bell polynomial umbra $x.\beta$. We get the auxiliary umbra $\alpha.\beta$, whose factorial moments are

$$(13) \qquad (\alpha.\beta)_n \simeq \alpha^n \qquad n = 0, 1, 2, \ldots .$$

The moments are given by the exponential umbral polynomials (cf. [4])

$$(14) \qquad (\alpha.\beta)^n \simeq \Phi_n(\alpha) \simeq \sum_{i=0}^{n} S(n, i)\alpha^i \qquad n = 0, 1, 2, \ldots .$$

The g.f. is $f(\alpha.\beta, t) = f(e^t - 1)$.

Example 2.6. α-*partition umbra.*

The α-partition umbra is the umbra $\beta.\alpha$, where β is the Bell umbra (see Example 2.2). Since the factorial moments of β are all equal to 1, equation (11) gives

$$(15) \qquad E[(\beta.\alpha)^i] = \sum_{j=1}^{i} B_{i,j}(a_1, a_2, \ldots, a_{i-j+1}) = Y_i(a_1, a_2, \ldots, a_i)$$

for $i = 1, 2, \ldots$, where Y_i are the complete exponential polynomials [12]. The umbra $x.\beta.\alpha$ is the polynomial α-partition umbra. Since the factorial moments of $x.\beta$ are powers of x, equation (11) gives

$$(16) \qquad E[(x.\beta.\alpha)^i] = \sum_{j=1}^{i} x^j \, B_{i,j}(a_1, a_2, \ldots, a_{i-j+1}) \, ,$$

so the g.f. is $f(x.\beta.\alpha, t) = \exp[x(f(t) - 1)]$.

The α-partition umbra $\beta.\alpha$ plays a crucial role in the umbral representation of the composition of exponential g.f.'s. Indeed, the *composition umbra* of α and γ is the umbra $\gamma.\beta.\alpha$. The g.f. is $f(\gamma.\beta.\alpha, t) = f[\gamma, f(\alpha, t) - 1]$. The moments are

$$(17) \qquad E[(\gamma.\beta.\alpha)^i] = \sum_{j=1}^{i} g_j \, B_{i,j}(a_1, a_2, \ldots, a_{i-j+1}) \, ,$$

where g_j and a_i are moments of the umbra γ and α respectively. From equivalences (4) and (13), we also have

$$(18) \qquad (\gamma.\beta.\alpha)^i \simeq \sum_{\lambda \vdash i} (\gamma.\beta)_{\nu_\lambda} d_\lambda \alpha_\lambda \simeq \sum_{\lambda \vdash i} \gamma^{\nu_\lambda} d_\lambda \alpha_\lambda \, ,$$

where d_λ and α_λ are given in (5).

We denote by $\alpha^{<-1>}$ the compositional inverse of α, i.e. the umbra having g.f. $f(\alpha^{<-1>}, t)$ such that $f[\alpha^{<-1>}, f(\alpha, t) - 1] = f[\alpha, f(\alpha^{<-1>}, t) - 1] = 1 + t$, i.e. $f(\alpha^{<-1>}, t) = f^{<-1>}(\alpha, t)$. So we have

$$\alpha.\beta.\alpha^{<-1>} \equiv \alpha^{<-1>}.\beta.\alpha \equiv \chi \, .$$

In particular for the unity umbra, we have

$$(19) \qquad \beta.u^{<-1>} \equiv u^{<-1>}.\beta \equiv \chi \, ,$$

by which the next fundamental equivalences follow

$$(20) \qquad \beta.\chi \equiv u \equiv \chi.\beta \, .$$

Since $\chi.\beta.\chi \equiv \chi.u \equiv \chi$, recalling i) of Proposition 2.3, the compositional inverse of the singleton umbra χ is the umbra χ itself.

Example 2.7. α-*cumulant umbra.*

The umbra $\chi.\alpha$ is the α-cumulant umbra, having g.f. $f(\chi.\alpha, t) = 1 + \log[f(t)]$. Then, the umbra χ is the cumulant umbra of u, the umbra u is the cumulant umbra of β, the umbra $u^{<-1>}$ is the cumulant umbra of χ. Properties of cumulant umbrae are investigated in details in [5]. A special role is held by the cumulant umbra of a polynomial Bell umbra. Indeed, as it has been proved in [5], the $(x.\beta)$-cumulant umbra has moments all equal to x:

$$(21) \qquad (\chi.x.\beta)^n \simeq x \, .$$

Example 2.8. α-*factorial umbra.*
The umbra $\alpha.\chi$ is the α-factorial umbra, since $(\alpha.\chi)^n \simeq (\alpha)_n$ for all nonnegative n. The g.f. is $f(\alpha.\chi, t) = f[\log(1 + t)]$. By using the α-factorial umbra, from equivalence (4) we also have

$$(\gamma.\alpha)^i \simeq \sum_{\lambda \vdash i} (\gamma)_{\nu_\lambda} d_\lambda \alpha_\lambda \simeq \sum_{\lambda \vdash i} (\gamma.\chi)^{\nu_\lambda} d_\lambda \alpha_\lambda \ ,$$

where d_λ and α_λ are given in (5).

3. Adjoint umbrae

Let γ be an umbra with $E[\gamma] = g_1 \neq 0$ so that the g.f. $f(\gamma, t)$ admits compositional inverse. In this section, we study some properties of a special partition umbra, i.e the $\gamma^{<-1>}$-partition umbra. As it will be clarified in the following, this is a key umbra in the theory of binomial polynomials.

Definition 3.1. The *adjoint umbra* of γ is the $\gamma^{<-1>}$-partition umbra:

$$\gamma^* = \beta.\gamma^{<-1>} \ .$$

The name parallels the adjoint of an umbral operator [14] since $\gamma.\alpha^*$ gives the umbral composition of γ and $\alpha^{<-1>}$.

Example 3.1. *Adjoint of the singleton umbra* χ.
The inverse of χ is the umbra χ itself. So we have

$$(22) \qquad\qquad \chi^* \equiv \beta.\chi^{<-1>} \equiv \beta.\chi \equiv u \ .$$

Example 3.2. *Adjoint of the unity umbra* u.
By virtue of equivalence (19), the adjoint of the unity umbra u is

$$(23) \qquad\qquad u^* \equiv \beta.u^{<-1>} \equiv \chi \ .$$

Example 3.3. *Adjoint of the Bell umbra* β.
We have $\beta^* \equiv u^{<-1>}$. Indeed $\beta.\beta.\beta^{<-1>} \equiv \chi$ and, taking the left-hand side dot-product by χ, we have

$$\chi.\beta.\beta.\beta^{<-1>} \equiv \chi.\chi \equiv u^{<-1>} \ .$$

The result follows recalling equivalence (20) and $\beta.\beta^{<-1>} = \beta^*$.

The adjoint umbra has g.f.

$$f(\gamma^*, t) = \exp[f^{<-1>}(\gamma, t) - 1] \ .$$

In particular the adjoint of the compositional inverse of an umbra is similar to its partition umbra, i.e.

$$(\gamma^{<-1>})^* \equiv \beta.\gamma \ .$$

From the previous equivalence, we have

$$(24) \qquad\qquad (\gamma^{<-1>})^*.\beta.\gamma^{<-1>} \equiv \beta.\gamma.\beta.\gamma^{<-1>} \equiv \beta.\chi \equiv u$$

and, replacing $\gamma^{<-1>}$ with γ, we have

$$(25) \qquad\qquad\qquad \gamma^*.\beta.\gamma \equiv u \ .$$

Example 3.4. *Adjoint of* $u^{<-1>}$.
The adjoint of the compositional inverse of the unity umbra is $(u^{<-1>})^* \equiv \beta.u \equiv \beta$.

Equivalences (24) and (25) may be rewritten in a more useful way. Indeed, we have

(26) $$(\gamma^{<-1>})^*.\gamma^* \equiv \gamma^*.(\gamma^{<-1>})^* \equiv u .$$

Note that the dot-product of an umbra α with the adjoint of γ is the composition umbra of α and $\gamma^{<-1>}$, i.e. $\alpha.\gamma^* \equiv \alpha.\beta.\gamma^{<-1>}$. In particular we have

(27) $$\gamma.\gamma^* \equiv \chi \Rightarrow \beta.\gamma.\gamma^* \equiv \beta.\chi \equiv u$$

and also

(28) $$\chi.\gamma^* \equiv \gamma^{<-1>} \quad \text{and} \quad \chi.(\gamma^{<-1>})^* \equiv \gamma .$$

Proposition 3.1. *The adjoint of the composition umbra of α and γ is the dot-product of the adjoints of γ and α, that is*

(29) $$(\alpha.\beta.\gamma)^* \equiv \gamma^*.\alpha^* .$$

  Proof.

$$(\alpha.\beta.\gamma)^* \equiv \beta.(\alpha.\beta.\gamma)^{<-1>} \equiv (\beta.\gamma^{<-1>}).(\beta.\alpha^{<-1>}) \equiv \gamma^*.\alpha^* .$$

$\square$

4. Umbral polynomials

Let $\{q_n(x)\}$ be a polynomial sequence of $R[x]$ such that $q_n(x)$ has degree n for any n :

$$q_n(x) = q_{n,\,n}x^n + q_{n,\,n-1}x^{n-1} + \cdots + q_{n,\,0} .$$

Moreover, let be α an umbra. The sequence $\{q_n(\alpha)\}$ consists of umbral polynomials with support α such that

$$E[q_n(\alpha)] = q_{n,\,n}a_n + q_{n,\,n-1}a_{n-1} + \cdots + q_{n,\,0}$$

for any nonnegative integer n. Now suppose $q_0(x) = 1$ and consider an auxiliary umbra η such that

$$E[\eta^n] = E[q_n(\alpha)] ,$$

for any nonnegative integer n. In order to underline that the moments of η depend on those of α, we add the subscript α to the umbra η so that we shall write

$$\eta_\alpha^n \simeq q_n(\alpha) \qquad \text{for } n = 0, 1, 2, \ldots .$$

If $\alpha \equiv x.u$, then $\eta_{x.u}$ is a polynomial umbra with moments $q_n(x)$, so we shall simply denote it by η_x.

Let us consider some simple consequences of the notations here introduced.

Proposition 4.1. *If η_x is a polynomial umbra and α and γ are umbrae both scalar either polynomial, then*

$$\eta_\alpha \equiv \eta_\gamma \Leftrightarrow \alpha \equiv \gamma .$$

  Proof. For any nonnegative integer n, there exist constants $c_{n,\,k}, k = 0, 1, \ldots, n$ such that $x^n = \sum_{k=0}^n c_{n,k}\, q_k(x)$. Since $\eta_\alpha \equiv \eta_\gamma$, then $q_k(\alpha) \simeq q_k(\gamma)$ for all nonnegative integers k and so for all nonnegative integers n we have

$$\alpha^n \simeq \sum_{k=0}^n c_{n,\,k}\, q_k(\alpha) \simeq \sum_{k=0}^n c_{n,\,k}\, q_k(\gamma) \simeq \gamma^n .$$

The other direction of the proof is straightforward.

$\square$

Proposition 4.2. *If η_x and ζ_x are polynomial umbrae and α is an umbra either scalar or polynomial, then*

$$\eta_\alpha \equiv \zeta_\alpha \Leftrightarrow \eta_x \equiv \zeta_x \; .$$

Proof. Suppose $\eta_\alpha \equiv \zeta_\alpha$. Let $\{q_n(x)\}$ be the moments of η_x and let $\{z_n(x)\}$ be the moments of ζ_x. For all nonnegative integers n, there exist constants $c_{n,k}, k = 0, 1, \ldots, n$ such that $q_n(x) = \sum_{k=0}^n c_{n,k} z_k(x)$. Being $q_n(\alpha) \simeq z_n(\alpha)$ for all nonnegative integers n, we have $c_{n,k} = \delta_{n,k}$ for $k = 0, 1, \ldots, n$ by which we have $q_n(x) = z_n(x)$. The other direction of the proof is straightforward.

$\square$

Proposition 4.3. *If η_x is a polynomial umbra, $\{\alpha_i\}_{i=1}^n$ are uncorrelated scalar umbrae and $\{w_i\}_{i=1}^n$ are some weights in R, then*

$$(30) \qquad \eta_{\dotplus_{i=1}^n \chi.w_i.\beta.\alpha_i} \equiv \dotplus_{i=1}^n \chi.w_i.\beta.\eta_{\alpha_i} \; .$$

Proof. Suppose $E[\eta_x^m] = q_m(x) = \sum_{k=0}^m q_{m,k} x^k$. Then for all nonnegative integers m, we have

$$q_m(\dotplus_{i=1}^n \chi.w_i.\beta.\alpha_i) \simeq \sum_{k=0}^m q_{m,k}(\dotplus_{i=1}^n \chi.w_i.\beta.\alpha_i)^k \simeq \sum_{k=0}^m q_{m,k} \left(\sum_{i=1}^n w_i \alpha_i^k \right) \simeq$$

$$\simeq \sum_{i=1}^n w_i \left(\sum_{k=0}^m q_{m,k} \alpha_i^k \right) \simeq \dotplus_{i=1}^n \chi.w_i.\beta.q_m(\alpha_i) \; ,$$

due to equivalence (21). The result follows by observing that

$$\dotplus_{i=1}^n \chi.w_i.\beta.q_m(\alpha_i) \simeq \left[\dotplus_{i=1}^n \chi.w_i.\beta.\eta_{\alpha_i} \right]^m \; .$$

$\square$

We achieve the proof of the next corollary choosing $w_i = 1$ for $i = 1, 2, \ldots, n$ in equivalence (30).

Corollary 4.1. *If η_x is a polynomial umbra and $\{\alpha_i\}_{i=1}^n$ are uncorrelated scalar umbrae, then*

$$\eta_{\dotplus_{i=1}^n \alpha_i} \equiv \dotplus_{i=1}^n \eta_{\alpha_i} \; .$$

5. SHEFFER SEQUENCES

In this section we give the definition of Sheffer umbra by which we recover fundamental properties of Sheffer sequences. In the following let us α and γ be scalar umbrae, with $g_1 = E[\gamma] \neq 0$.

Definition 5.1. A polynomial umbra σ_x is said to be a *Sheffer umbra* for (α, γ) if

$$(31) \qquad \sigma_x \equiv (-1.\alpha + x.u).\gamma^* \; ,$$

where γ^* is the adjoint umbra of γ.

In the following, we denote a Sheffer umbra by $\sigma_x^{(\alpha,\gamma)}$ in order to make explicit the dependence on α and γ.

We note that if α has g.f. $f(\alpha, t)$ and γ has g.f. $f(\gamma, t)$, the g.f of $(-1.\alpha + x.u).\gamma^*$ is the composition of $f(\gamma^{<-1>}, t) = f^{<-1>}(\gamma, t)$ and $f(-1.\alpha + x.u, t) = e^{xt}/f(\alpha, t)$, i.e.

$$(32) \qquad f(\sigma_x^{(\alpha,\gamma)}, t) = \frac{1}{f[\alpha, f^{<-1>}(\gamma, t) - 1]} \, e^{x \, [f^{<-1>}(\gamma,t)-1]} \; .$$

Theorem 5.1 (The expansion theorem). *If $\sigma_x^{(\alpha,\gamma)}$ is a Sheffer umbra for (α,γ), then*

$$\eta \equiv \alpha + \sigma_\eta^{(\alpha,\gamma)}.\beta.\gamma \tag{33}$$

for any umbra η.

Proof. Replacing x with η in equivalence (31), we obtain

$$\sigma_\eta^{(\alpha,\gamma)} \equiv (-1.\alpha + \eta).\gamma^* \ .$$

Take the right dot product with $\beta.\gamma$ of both sides, then

$$\sigma_\eta^{(\alpha,\gamma)}.\beta.\gamma \equiv (-1.\alpha + \eta).\gamma^*.\beta.\gamma \equiv (-1.\alpha + \eta) \ .$$

The result follows adding α to both sides of the previous equivalence.

$\square$

Theorem 5.2. *Let $\{s_n(x)\}$ be the moments of a Sheffer umbra $\sigma_x^{(\alpha,\gamma)}$. The polynomial sequence $\{s_n(x)\}$ is the unique polynomial sequence such that:*

$$s_n(\alpha + k.\gamma) \simeq (k.\chi)^n \qquad \forall\, n, k \geq 0 \ . \tag{34}$$

Proof. From equivalence (31), when x is replaced by $\alpha + k.\gamma$, we have

$$(-1.\alpha + \alpha + k.\gamma).\gamma^* \equiv k.\gamma.\gamma^* \equiv k.\chi \qquad \forall\, k \geq 0 \tag{35}$$

which gives (34). The uniqueness follows from Proposition 4.2.

$\square$

We explicitly note that any Sheffer umbra is uniquely determined by its moments evaluated at 0, since via equivalence (31) we have

$$\sigma_0^{(\alpha,\gamma)} \equiv -1.\alpha.\gamma^* \ .$$

In Theorem 5.3 we will prove that the moments of the umbra $\sigma_x^{(\alpha,\gamma)}$ satisfy the Sheffer identity.

Example 5.1. *Power polynomials.*
Choosing as umbra α the umbra ϵ and as umbra γ the umbra χ, from equivalence (31) we have

$$\sigma_x^{(\epsilon,\chi)} \equiv x.u \ ,$$

since $\chi^* \equiv u$. So the sequence of polynomials $\{x^n\}$ is a Sheffer sequence, being moments of the Sheffer umbra $\sigma_x^{(\epsilon,\chi)}$.

Example 5.2. *Poisson-Charlier polynomials.*
The Poisson-Charlier polynomials are

$$c_n(x;a) = \frac{1}{a^n} \sum_{k=0}^{n} \binom{n}{k} (-a)^{n-k}(x)_k \ ,$$

hence

$$c_n(x;a) \simeq \left\{ \frac{x.\chi - a}{a} \right\}^n \ ,$$

recalling that $(x.\chi)^k \simeq (x)_k$. Denoting by $\omega_{x,a}$ the polynomial umbra whose moments are $c_n(x;a)$, we have

$$\omega_{x,a} \equiv \frac{x.\chi - a}{a} \ .$$

The umbra $\omega_{x,a}$ is called the Poisson-Charlier polynomial umbra. We show that $\omega_{x,a}$ is a Sheffer umbra for $(a.\beta, \chi.a.\beta)$. Indeed

$$\omega_{x,a} \equiv \frac{x.\chi - a.\beta.\chi}{a} \equiv [-1.(a.\beta) + x.u]. \frac{\chi}{a} \ .$$

So the Poisson-Charlier polynomial umbra $\omega_{x,a}$ is a Sheffer umbra, being

$$\frac{\chi}{a} \equiv (\chi.a.\beta)^* \ .$$

Theorem 5.3 (The Sheffer identity). *A polynomial umbra σ_x is a Sheffer umbra if and only if there exists an umbra η, provided with a compositional inverse, such that*

$$(36) \qquad\qquad\qquad \sigma_{x+y} \equiv \sigma_x + y.\eta^* \ .$$

Proof. Let σ_x be a Sheffer umbra for (α, γ). By Definition 5.1, we have

$$\sigma_{x+y}^{(\alpha,\gamma)} \equiv (-1.\alpha + x.u + y.u).\gamma^* \equiv$$

$$\equiv (-1.\alpha + x.u).\gamma^* + y.\gamma^* \equiv \sigma_x^{(\alpha,\gamma)} + y.\gamma^* \ .$$

Viceversa, set $x = 0$ in equivalence (36). We have

$$(37) \qquad\qquad\qquad \sigma_y \equiv \sigma_0 + y.\eta^* \ .$$

On the other hand, known the moments of η, there exists an umbra α such that $-1.\alpha.\eta^* \equiv \sigma_0$. Then the polynomial umbra σ_y is a Sheffer umbra, being $\sigma_y \equiv (-1.\alpha + y.u).\eta^*$.

$\square$

Equivalence (36) gives the well-known Sheffer identity, because by using the binomial expansion we have

$$(38) \qquad\qquad s_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x) \, p_{n-k}(y)$$

where $s_n(x + y) = E[\sigma_{x+y}^n]$, $s_k(x) = E[\sigma_x^k]$ and $p_{n-k}(y) = E[(y.\eta^*)^{n-k}]$. In the next section, we will prove that the moments of umbrae such $y.\eta^*$ are binomial sequences.

Corollary 5.1. *If $\sigma_x^{(\alpha,\gamma)}$ is a Sheffer umbra for (α, γ), then*

$$\sigma_{\eta+\zeta}^{(\alpha,\gamma)} \equiv \sigma_\eta^{(\alpha,\gamma)} + \zeta.\gamma^* \ ,$$

where η and ζ are umbrae.

Theorem 5.4. *A polynomial umbra σ_x is a Sheffer umbra if and only if there exists an umbra η, provided with compositional inverse, such that*

$$(39) \qquad\qquad\qquad \sigma_{\eta+x.u} \equiv \chi + \sigma_x \ .$$

Proof. If σ_x is a Sheffer umbra for (α, γ) then

$$\sigma_{\gamma+x.u}^{(\alpha,\gamma)} \equiv (-1.\alpha + x.u + \gamma).\gamma^* \equiv (-1.\alpha + x.u).\gamma^* + \chi \ ,$$

recalling that $\gamma.\gamma^* \equiv \chi$. Then equivalence (39) follows from equivalence (31) choosing as umbra η the umbra γ. Viceversa, let σ_x be a polynomial umbra such that equivalence (39) holds for some umbra η, with $E[\eta] = g_1 \neq 0$. Set $x = 0$ in equivalence (39). We have

$$\sigma_\eta \equiv \chi + \sigma_0 \Rightarrow \sigma_\eta.\beta.\eta \equiv \chi.\beta.\eta + \sigma_0.\beta.\eta \equiv \eta + \sigma_0.\beta.\eta \ .$$

Known the moments of η, there exists an umbra α such that $-1.\alpha.\eta^* \equiv \sigma_0$ so that

$$\sigma_\eta.\beta.\eta \equiv \eta - 1.\alpha \ .$$

Due to equivalence (33), also the Sheffer umbra for (α, η) is such that $\sigma_\eta^{(\alpha,\eta)}.\beta.\eta \equiv \eta - 1.\alpha$, therefore $\sigma_\eta \equiv \sigma_\eta^{(\alpha,\eta)}$ and $\sigma_x \equiv \sigma_x^{(\alpha,\eta)}$ by Proposition 4.2.

$\square$

Corollary 5.2. *A polynomial umbra σ_x is a Sheffer umbra if and only if there exists an umbra η, provided with compositional inverse, such that*

$$\sigma_{\eta+x.u}^k \simeq \sigma_x^k + k\,\sigma_x^{k-1} \qquad for \ \ k = 1, 2, \dots \ .$$

Proof. Take the k-th moment of both sides in equivalence (39).

$\square$

Now suppose $s_n(x) = \sum_{k=0}^n s_{n,k}\, x^k$ be the moments of a Sheffer umbra for (α, γ) and $r_n(x)$ be the moments of a Sheffer umbra for (η, ζ). The umbra

$$[-1.\alpha + (-1.\eta + x.u).\zeta^*].\gamma^* \equiv [-1.(\alpha.\beta.\zeta + \eta) + x.u].\zeta^*.\gamma^*$$

has moments umbrally equivalent to

$$(40) \qquad \qquad \sum_{k=0}^n s_{n,k} r_k(x) \ ,$$

i.e. the Roman-Rota umbral composition $s_n(\mathbf{r}(x))$. So we have proved the following theorem.

Theorem 5.5 (Umbral composition and Sheffer umbrae)**.** *The polynomials given in (40) are moments of the Sheffer umbra for $(\alpha.\beta.\zeta + \eta, \gamma.\beta.\zeta)$.*

Two Sheffer sequences are said to be inverse of each other if and only if their Roman-Rota umbral composition (40) gives the sequence $\{x^n\}$.

Corollary 5.3 (Inverse of Sheffer sequences)**.** *The sequence of moments corresponding to the Sheffer umbra for $(-1.\alpha.\gamma^*, \gamma^{<-1>})$ are inverses of the sequence of moments corresponding to the Sheffer umbra for (α, γ).*

Proof. Set

$$\zeta \equiv \gamma^{<-1>} \quad and \quad \eta \equiv -1.\alpha.\gamma^* \ ,$$

in $(\alpha.\beta.\zeta + \eta, \gamma.\beta.\zeta)$. The corresponding Sheffer umbra has moments umbrally equivalent to x^n.

$\square$

6. Two special Sheffer umbrae

In this section, we study two special classes of Sheffer umbrae: the associated umbra and the Appell umbra. The associated umbrae are polynomial umbrae whose moments $\{p_n(x)\}$ satisfies the well-known binomial identity

$$(41) \qquad \qquad p_n(x + y) = \sum_{k=0}^n \binom{n}{k} p_k(x)\, p_{n-k}(y)$$

for all $n = 0, 1, 2, \dots$. Every sequence of binomial type is a Sheffer sequence but most Sheffer sequences are not of binomial type. The concept of binomial type has applications in combinatorics, probability, statistics, and a variety of other fields.

The Appell umbrae are polynomial umbrae whose moments $\{p_n(x)\}$ satisfies the identity

$$(42) \qquad \frac{d}{dx} p_n(x) = n p_{n-1}(x) \qquad n = 1, 2, \dots .$$

Among the most notable Appell sequences, besides the trivial example $\{x^n\}$, are the Hermite polynomials, the Bernoulli polynomials, and the Euler polynomials.

6.1. **Associated umbrae.** Let us consider a Sheffer umbra for the umbrae (ϵ, γ), where γ has compositional inverse and ϵ is the augmentation umbra.

Definition 6.1. A polynomial umbra σ_x is said to be the *associated umbra* of γ if

$$\sigma_x \equiv x.\gamma^* \, ,$$

where γ^* is the adjoint umbra of γ.

The g.f. of $x.\gamma^*$ is

$$f(x.\gamma^*, t) = e^{x[f^{<-1>}(\gamma, t) - 1]} \, ,$$

because in equation (32) we have $f(\alpha, t) = f(\epsilon, t) = 1$. The expansion theorem for associated umbrae (cf. Theorem 5.1) is

$$(43) \qquad \eta \equiv \eta.\gamma^*.\beta.\gamma \, .$$

Theorem 6.1. *An umbra $\sigma_x^{(\alpha,\gamma)}$ is a Sheffer umbra for (α, γ) if and only if $\sigma_{\alpha+x.u}^{(\alpha,\gamma)}$ is the umbra associated to γ.*

Proof. If $\sigma_x^{(\alpha,\gamma)}$ is a Sheffer umbra for (α, γ) then

$$\sigma_{\alpha+x.u}^{(\alpha,\gamma)} \equiv (-1.\alpha + \alpha.u + x.u).\gamma^* \, ,$$

by which

$$\sigma_{\alpha+x.u}^{(\alpha,\gamma)} \equiv x.\gamma^* \, .$$

From Definition 6.1, the umbra $\sigma_{\alpha+x.u}^{(\alpha,\gamma)}$ is the umbra associated to γ. Viceversa, let η_x be a polynomial umbra such that

$$\eta_{\alpha+x.u} \equiv x.\gamma^* \, .$$

Replacing x with $k.\gamma$, we have

$$\eta_{\alpha+k.\gamma} \equiv k.\gamma.\gamma^* \equiv k.\chi \, .$$

The result follows by equivalence (35). $\qquad\square$

We will say that a polynomial sequence $\{p_n(x)\}$ is *associated* to an umbra γ if and only if

$$p_n(x) \simeq (x.\gamma^*)^n \quad , \quad n = 0, 1, 2, \dots$$

or

$$(44) \qquad p_n(k.\gamma) \simeq (k.\chi)^n \qquad n, k = 0, 1, 2, \dots .$$

Theorem 6.2 (Umbral characterization of associated sequences)**.** *The sequence $\{p_n(x)\}$ is associated to the umbra γ if and only if:*

$$(45) \qquad p_n(\epsilon) \simeq \epsilon^n \qquad for \ n = 0, 1, 2, \dots$$

$$(46) \qquad p_n(\gamma + x.u) \simeq p_n(x) + n\, p_{n-1}(x) \qquad for \ n = 1, 2, \dots .$$

Proof. If the sequence $\{p_n(x)\}$ is associated to γ then

$$p_n(\epsilon) \simeq (\epsilon.\gamma^*)^n \simeq \epsilon^n$$

for all $n = 0, 1, 2, \ldots$. Equivalence (46) follows from Corollary 5.2 choosing as umbra α the umbra ϵ. Viceversa, if equivalences (45) and (46) hold, we prove by induction that the sequence $\{p_n(x)\}$ satisfies (44). Indeed, by equivalence (45) we have

$$p_n(0.\gamma) \simeq p_n(\epsilon) \simeq \begin{cases} 0 , & n = 1, 2, \ldots, j , \\ 1 , & n = 0 . \end{cases}$$

Suppose that equivalence (44) holds for $k = m$

(47) $$p_n(m.\gamma) \simeq (m.\chi)^n \qquad n = 0, 1, 2, \ldots .$$

By equivalence (46), we have

$$p_n[(m+1).\gamma)] \simeq p_n(\gamma + m.\gamma) \simeq p_n(m.\gamma) + n p_{n-1}(m.\gamma) \qquad n = 1, 2, \ldots .$$

Due to induction hypothesis (47), we have

$$p_n[(m+1).\gamma] \simeq (m.\chi)^n + n(m.\chi)^{n-1} \simeq (\chi + m.\chi)^n \simeq [(m+1).\chi]^n \qquad n = 1, 2, \ldots .$$

Since the sequence $\{p_n(x)\}$ verifies (44), it is associated to γ.

$\square$

Theorem 6.3 (The binomial identity). *The sequence $\{p_n(x)\}$ is associated to the umbra γ if and only if*

(48) $$p_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) \, p_{n-k}(y)$$

for all $n = 0, 1, 2, \ldots$.

Proof. If the sequence $\{p_n(x)\}$ is associated to the umbra γ, then identity (48) follows from the property

$$(x + y).\gamma^* \equiv x.\gamma^* + y.\gamma^* .$$

Viceversa, suppose the sequence $\{p_n(x)\}$ satisfies identity (48). Let η_x be a polynomial umbra such that

$$E[\eta_x^n] = p_n(x) .$$

By identity (48), we have

(49) $$\eta_{x+y} \equiv \eta_x + \eta_y'$$

with η_x similar to η_x' and uncorrelated. In particular, if we replace y with ϵ in equivalence (49), then $\eta_x \equiv \eta_x + \eta_\epsilon'$ and hence $\eta_\epsilon' \equiv \epsilon$. So the polynomials $\{p_n(x)\}$ are such that $p_n(\epsilon) \simeq \epsilon^n$, i.e. they satisfy equivalence (45). By induction on equivalence (49), we have

$$\underbrace{\eta_{x + \cdots + x}}_{k} \equiv \underbrace{\eta_x + \cdots + \eta_x'}_{k}$$

where the polynomial umbrae on the right-hand side are uncorrelated and similar to η_x. If the x's are replaced by uncorrelated umbrae similar to any umbra γ, provided of compositional inverse, then

$$\eta_{k.\gamma} \equiv k.\eta_\gamma .$$

Since $E[\gamma] \neq 0$, we can choose an umbra γ such that

$$\eta_\gamma \equiv \chi$$

thus

(50) $$\eta_{k.\gamma} \equiv k.\chi$$

and the result follows from equivalences (50) and (44).

$\square$

Example 6.1. The umbra $x.u$ is associated to the umbra χ. Indeed, the polynomial sequence $\{x^n\}$ is associated to the adjoint umbra $\chi^* \equiv \beta.\chi^{<-1>} \equiv u$. The g.f. is

$$f(x.u, t) = \sum_{k \geq 0} \frac{x^k}{k!} \, t^k$$

and the binomial identity becomes

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \, y^{n-k} \ .$$

Example 6.2. The umbra $x.u^* \equiv x.\chi$ is associated to the umbra u. The associated polynomial sequence is given by $\{(x.\chi)^n\} \simeq \{(x)_n\}$, see Example 2.8. The g.f. is

$$f(x.u^*, t) = (1 + t)^x$$

and the binomial identity becomes

$$(x + y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k \, (y)_{n-k} \ .$$

Example 6.3. The umbra $x.(u^{<-1>})^* \equiv x.\beta$ is associated to the umbra $u^{<-1>}$. The associated polynomial sequence is given by $\{(x.\beta)^n\} \simeq \{\Phi_n(x)\}$, where $\{\Phi_n(x)\}$ are the exponential polynomials (14).

Now, suppose $\{p_n(x)\}$ be the polynomial sequence associated to an umbra γ with g.f. $f(\gamma, t)$ and $\{q_n(x)\}$ be the polynomial sequence associated to an umbra ζ with g.f. $f(\zeta, t)$, i.e.

$$p_n(x) \simeq (x.\beta.\gamma^{<-1>})^n \quad \text{and} \quad q_n(x) \simeq (x.\beta.\zeta^{<-1>})^n$$

for $n = 0, 1, 2, \ldots$. On the other hand, due to Proposition 3.1, we have

$$(x.\beta.\gamma^{<-1>}).\beta.\zeta^{<-1>} \equiv x.(\beta.\gamma^{<-1>}).(\beta.\zeta^{<-1>}) \equiv x.\gamma^*.\zeta^* \equiv x.(\zeta.\beta.\gamma)^* \ .$$

Following Roman-Rota notation, the umbra $x.(\zeta.\beta.\gamma)^*$ has moments

(51) $$q_n(\mathbf{p}(x)) = \sum_{k=0}^{n} q_{n,k} p_k(x) \ .$$

This proves the next theorem.

Theorem 6.4 (Umbral composition of associated sequences). *The polynomial sequence* (51) *is associated to the compositional umbra of ζ and γ.*

Corollary 6.1 (Inverse of associated sequences). *The polynomial sequence associated to the umbra $\gamma^{<-1>}$ is inverse of the polynomial sequence associated to the umbra γ.*

Proof. Choosing as umbra γ the umbra $\zeta^{<-1>}$ in the previous theorem, we have

$$x.\zeta^*.\beta.\zeta \equiv x.u .$$

$\square$

Example 6.4. Choose as umbra ζ the umbra u. Then $x.u^*.\beta.u \equiv x.\chi.\beta \equiv x.u$ and so

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k ,$$

which is the well-known formula giving powers in terms of lower factorials.

Finally, via Proposition 3.1, it is easy to prove the following recurrence formula

$$(x.\gamma^*)^{n+1} \simeq x\,\gamma^{<-1>}\,[(x+\chi).\gamma^*]^n .$$

6.2. **Appell umbrae.** In this section, we consider a second kind of special Sheffer umbra, that is a Sheffer umbra for (α, χ).

Definition 6.2. A polynomial umbra σ_x is said to be the *Appell umbra* of α if

$$\sigma_x \equiv -1.\alpha + x.u .$$

By equivalence (32), being $f(\chi, t) = 1 + t$, the g.f. of $(-1.\alpha + x.u)$ is

$$f(-1.\alpha + x.u, t) = \frac{1}{f(\alpha, t)}\,e^{xt} .$$

We will say that a polynomial sequence $\{p_n(x)\}$ is an Appell sequence if and only if

$$p_n(x) \simeq (-1.\alpha + x.u)^n \quad , \quad n = 0, 1, 2, \ldots .$$

The expansion theorem for Appell umbrae (cf. Theorem 5.1) easily follows by observing that

$$\eta \equiv \alpha + (-1.\alpha + \eta) .$$

Theorem 6.5 (The Appell identity)**.** *The polynomial umbra σ_x is an Appell umbra for some umbra α if and only if*

$$\sigma_{x+y} \equiv \sigma_x + y .$$

Proof. The result follows immediately, choosing as umbra γ the singleton umbra χ in the Sheffer identity.

$\square$

Corollary 6.2. *A polynomial umbra σ_x is an Appell umbra for some umbra α if and only if*

(52) $$\sigma_{\chi+x.u}^n \simeq \sigma_x^n + n\sigma^{n-1}(x) .$$

Proof. The result follows from Corollary 5.2, choosing as umbra γ the singleton umbra χ.

$\square$

Roughly speaking, Corollary 6.2 says that, when in the Appell umbra we replace x by $\chi + x.u$, the umbra χ acts as a derivative operator. Indeed by the binomial expansion, we have

$$
(-1.\alpha + \chi + x.u)^n \simeq \sum_{k \geq 0} \binom{n}{k} (-1.\alpha)^{n-k} (\chi + x.u)^k \simeq
$$

$$
\simeq \sum_{k \geq 0} \binom{n}{k} (-1.\alpha)^{n-k} [(x.u)^k + k(x.u)^{k-1}] \simeq
$$

$$
\simeq (-1.\alpha + x.u)^n + \sum_{k \geq 0} \binom{n}{k} (-1.\alpha)^{n-k} D_x[(x.u)^k] \ .
$$

So equivalence (52) umbrally expresses equation (42).

Theorem 6.6 (The multiplication theorem)**.** *For any constant $c \in R$, we have*

$$
-1.\alpha + (c\,x).u \equiv c(-1.\alpha + x.u) + (c - 1).\alpha \ .
$$

Proof. We have

$$
c(-1.\alpha + x.u) + (c - 1).\alpha \equiv -c.\alpha + (c\,x).u + c.\alpha - 1.\alpha \equiv -1.\alpha + (c\,x).u \ .
$$

$\square$

Example 6.5. *Bernoulli polynomials.*
The Appell umbra for the inverse of the Bernoulli umbra is $\iota + x.u$. From the binomial expansion, its moments are the Bernoulli polynomials

$$
E[(\iota + x.u)^n] = \sum_{k \geq 0} \binom{n}{k} B_{n-k}\, x^k \ ,
$$

where B_n are the Bernoulli numbers.

7. Topics in umbral calculus

In this section, we apply the language of umbrae to some topics which benefit from this approach. In particular we discuss the well-known connection constants problem, the Lagrange inversion formula, and we solve some recurrence relations to give an indication of the effectiveness of the method.

7.1. **The connection constants problem.** The connection constants problem consists in determining the connection constants $c_{n,k}$ in the expression

$$
s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x) \ ,
$$

where $s_n(x)$ and $r_n(x)$ are sequences of polynomials. When $s_n(x)$ and $r_n(x)$ are Sheffer sequences, umbrae provide an easy solution to this problem. Indeed, suppose η_x be a polynomial umbra such that

$$
E[\eta_x^n] = q_n(x) = \sum_{k=0}^{n} c_{n,k} x^k \ .
$$

The theorem we are going to prove states that η_x is a Sheffer umbra whenever $s_n(x)$ and $r_n(x)$ are Sheffer sequences.

Theorem 7.1. *If η_x is a polynomial umbra such that*

$$(53) \qquad\qquad (-1.\alpha + x.u).\gamma^* \equiv \eta_{(-1.\delta + x.u).\zeta^*} \; ,$$

then η_x is a Sheffer umbra for $((\delta - 1.\alpha).\zeta^*, \gamma.\beta.\zeta^{<-1>})$.

Note that equivalence (53) is the way to transform the Sheffer umbra $(-1.\delta + x.u).\zeta^*$ in the Sheffer umbra $(-1.\alpha + x.u).\gamma^*$ by using the polynomial umbra η_x.

Proof. In equivalence (53), replace x with $\delta + x.\beta.\zeta$. The right-hand side of equivalence (53) becomes

$$\eta_{[-1.\delta + (\delta + x.\beta.\zeta).u].\zeta^*} \equiv \eta_{x.\beta.\zeta.\zeta^*} \equiv \eta_{x.u}$$

due to equivalence (27), whereas the left-hand side gives

$$(54) \qquad [-1.\alpha + (\delta + x.\beta.\zeta).u].\gamma^* \equiv (\delta - 1.\alpha).\gamma^* + x.\beta.\zeta.\gamma^* \; .$$

By Proposition 3.1, we have

$$x.\beta.\zeta.\gamma^* \equiv x.(\zeta^{<-1>})^*.\gamma^* \equiv x.(\gamma.\beta.\zeta^{<-1>})^*$$

and by equivalence (26)

$$(\delta - 1.\alpha).\gamma^* \equiv (\delta - 1.\alpha).u.\gamma^* \equiv (\delta - 1.\alpha).\zeta^*.(\zeta^{<-1>})^*.\gamma^* \; .$$

Replacing $(\delta - 1.\alpha).\gamma^*$ and $x.\beta.\zeta.\gamma^*$ in equivalence (54), we have

$$(55) \qquad [-1.\alpha + (\delta + x.\beta.\zeta).u].\gamma^* \equiv [(\delta - 1.\alpha).\zeta^* + x.u].(\gamma.\beta.\zeta^{<-1>})^*$$

and so equivalence (53) returns

$$(56) \qquad\qquad \eta_x \equiv [(\delta - 1.\alpha).\zeta^* + x.u].(\gamma.\beta.\zeta^{<-1>})^* \; .$$

The result follows from Definition 5.1.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

To get an explicit expression of the connection constants, we can use equivalence (18) to expand the n-th moment of η_x in (56)

$$\eta_x^n \simeq \sum_{\lambda \vdash n} [(\delta - 1.\alpha).\zeta^* + x.u]^{\nu_\lambda} \, d_\lambda \, (\zeta.\beta.\gamma^{<-1>})_\lambda \; .$$

The connection constants $c_{n,k}$ are the coefficient of x^k in the previous equivalence.

Example 7.1. Consider

$$c_n(x; b) = \sum_{k=0}^{n} c_{n,k} \, c_k(x; a)$$

where $c_n(x; a)$ and $c_n(x; b)$ are Poisson-Charlier polynomials. As shown in Example 5.2, $c_n(x; b)$ are the moments of a Sheffer umbra for $(b.\beta, \chi.b.\beta)$ and $c_n(x; a)$ are the moments of a Sheffer umbra for $(a.\beta, \chi.a.\beta)$. By equivalence (56), we have

$$\eta_x \equiv [(a.\beta - b.\beta).(\chi.a.\beta)^* + x.u].[(\chi.b.\beta).\beta.(\chi.a.\beta)^{<-1>}]^*$$

or, equivalently, via equivalence (55)

$$\eta_x \equiv (-b.\beta + a.\beta + x.\beta.\chi.a.\beta).(\chi.b.\beta)^* \; .$$

Observe that $x.\beta.\chi.a.\beta \equiv xa.\beta$, so

$$(57) \qquad (-b.\beta + a.\beta + x.\beta.\chi.a.\beta).(\chi.b.\beta)^* \equiv (a - b + xa).\beta.(\chi.b.\beta)^* \; .$$

Moreover $(\chi.b.\beta)^* \equiv (\chi.b.\beta.u)^* \equiv u^*.(b\chi)^* \equiv \chi.(b\chi)^*$, from which

$$\beta.(\chi.b.\beta)^* \equiv \beta.\chi.(b\chi)^* \equiv (b\chi)^* \; .$$

Since $\chi^* \equiv u$ then $(b\chi)^* \equiv u/b$ and

$$\beta.(\chi.b.\beta)^* \equiv \frac{u}{b} \ .$$

Replacing this last result in equivalence (57), we have

$$\eta_x \equiv \frac{a - b + xa}{b} \ .$$

The binomial expansion gives

$$c_{n,k} = \binom{n}{k} \left(\frac{a}{b}\right)^n \left(1 - \frac{b}{a}\right)^{n-k} \ .$$

7.2. Abel polynomials and Lagrange inversion formula. Abel polynomials play a leading role in the theory of associated sequences of polynomials. The main result of this section is the proof that any sequence of binomial type can be represented as Abel polynomials, heart of the paper [18]. The proof given in [18] was a hybrid, based both on the early Roman-Rota version of the umbral calculus and the last version, introduced by Rota-Taylor. Here, we give a very simple proof by introducing the notion of the derivative of an umbra.

Definition 7.1. The *derivative umbra* α_D of an umbra α is the umbra whose moments are:

$$(\alpha_D)^n \simeq \partial_\alpha \alpha^n \simeq n\alpha^{n-1} \qquad \text{for } n = 1, 2, \dots \ .$$

We have

$$f(\alpha_D, t) = 1 + t\, f(\alpha, t) \ ,$$

since

$$e^{\alpha_D t} \simeq u + \sum_{n \geq 1} n\alpha^{n-1} \frac{t^n}{n!} \simeq u + te^{\alpha t} \ .$$

In particular, we have

(58) $$\left(e^{\alpha_D t} - u\right)^{.k} \simeq t^k \left(e^{\alpha t}\right)^{.k} \simeq t^k e^{(k.\alpha)t} \ .$$

Note that $E[\alpha_D] = 1$.

Example 7.2. *Singleton umbra.*
The singleton umbra χ is the derivative umbra of the augmentation umbra ϵ, that is $\epsilon_D \equiv \chi$.

Example 7.3. *Bernoulli umbra.*
We have $u \equiv (-1.\iota)_D$. Indeed, we have

$$f(-1.\iota, t) = \frac{e^t - 1}{t}$$

so that

$$f[(-1.\iota)_D, t] = 1 + t\frac{e^t - 1}{t} = e^t = f(u, t) \ .$$

Example 7.4. *Bernoulli-factorial umbra.*
We have $u^{<-1>} \equiv (\iota.\chi)_D$. Indeed

$$f(\iota.\chi, t) = \frac{\log(1 + t)}{e^{\log(1+t)} - 1} = \frac{\log(1 + t)}{t}$$

so that

$$f[(\iota.\chi)_D, t] = 1 + t\frac{\log(1+t)}{t} = 1 + \log(1+t) = f(u^{<-1>}, t) \ .$$

Theorem 7.2 (Abel representation of binomial sequences). *If γ is an umbra provided with a compositional inverse, then*

$$(59) \qquad\qquad (x.\gamma_D^*)^n \simeq x(x - n.\gamma)^{n-1} \quad , \quad n = 1, 2, \ldots$$

for all $x \in R$.

In the following, we refer to polynomials $x(x - n.\gamma)^{n-1}$ as *umbral Abel polynomials*.

Proof. On the basis of Theorem 6.2, the result follows showing that umbral Abel polynomials are associated to the umbra γ, i.e. showing that such polynomials satisfy equivalences (45) and (46). Since $\epsilon(\epsilon - n.\gamma)^{n-1} \simeq \epsilon^n$, equivalences (45) are satisfied. Moreover, it is easy to check by simple calculations that

$$(x.u + \gamma_D)^n - x^n \simeq n(x.u + \gamma)^{n-1} \quad , \quad n = 1, 2, \ldots$$

and more in general

$$p(x.u + \gamma_D) - p(x) \simeq \frac{d}{dx} p(x.u + \gamma)$$

for any polynomial $p(x) \in R[A][x]$. In particular for $p_n(x) = x(x - n.\gamma)^{n-1}$, we have

$$p_n(x.u + \gamma_D) - p_n(x) \simeq \frac{d}{dx} p_n(x.u + \gamma) \ .$$

We state equivalences (46) by proving that

$$\frac{d}{dx} p_n(x.u + \gamma) \simeq np_{n-1}(x) \ .$$

To this aim, we have

$$\frac{d}{dx} p_n(x) \simeq n(x - n.\gamma)^{n-2}(x - 1.\gamma)$$

so

$$\frac{d}{dx} p_n(x.u + \gamma) \simeq nx(x - (n-1).\gamma)^{n-2} \simeq np_{n-1}(x) \ .$$

$\square$

According to Corollary 6.1, the inverses of umbral Abel polynomials with respect to the Roman-Rota umbral composition are the moments of $x.\beta.\gamma_D$. An umbral expression of the inverses of umbral Abel polynomials will be given in Corollary 7.2.

Since the g.f. of $x.\gamma_D^*$ is $\exp[x\,(f^{<-1>}(\gamma_D, t) - 1)]$, we have

$$\exp[x\,(f^{<-1>}(\gamma_D, t) - 1)] \simeq 1 + \sum_{k \geq 1} \frac{t^k}{k!} [x(x - k.\gamma)]^{k-1} \ ,$$

which is the g.f. of umbral Abel polynomials.

Theorem 7.2 includes the well-known Transfer Formula. In the following we state various results usually derived by Transfer Formula. We start with the Lagrange inversion formula.

Corollary 7.1. *For any umbra γ, we have*

(60) $$(\gamma_D^{<-1>})^n \simeq (-n.\gamma)^{n-1} \quad , \quad n = 1, 2, \dots .$$

Proof. Since $\chi.\beta \equiv u$ then $\gamma_D^{<-1>} \equiv \chi.\beta.\gamma_D^{<-1>}$. From equivalence (59), with x replaced by χ, we have

$$(\gamma_D^{<-1>})^n \simeq (\chi.\beta.\gamma_D^{<-1>})^n \simeq \chi(\chi - n.\gamma)^{n-1} \quad , \quad n = 1, 2, \dots .$$

Since $\chi^{k+1} \simeq 0$ for $k = 1, 2, \dots, n-1$, we have

$$\chi(\chi - n.\gamma)^{n-1} \simeq \sum_{k=0}^{n-1} \binom{n-1}{k} \chi^{k+1}(n.\gamma)^{n-1-k} \simeq (n.\gamma)^{n-1}$$

by which equivalence (60) follows.

$\square$

Remark 7.1. Theorem 7.2 is referred to normalized binomial polynomials, i.e. sequences $\{p_n(x)\}$ such that $p_1(x)$ is monic. This is why the Lagrange inversion formula (60) refers to umbrae having first moment equal to 1. More in general, if one would consider umbrae having first moment different from zero, one step more is necessary. By way of an example, we do this for the Lagrange inversion formula.

For any umbra γ such that $E[\gamma] = g_1 \neq 0$, there exists[2] an umbra α such that $\gamma/g_1 \equiv \alpha_D$. Indeed such an umbra α has moments

$$\alpha^{n-1} \simeq \frac{\gamma^n}{n\, g_1^n} \qquad n = 1, 2, \dots$$

and g.f. $f(\alpha, t) = [f(\gamma, t/g_1) - 1]/t$. In particular $g_1 \alpha \equiv \overline{\gamma}$, where $\overline{\gamma}$ is the umbra introduced in [4] having moments

$$E[\overline{\gamma}^n] = \frac{g_{n+1}}{g_1(n+1)} \qquad n = 0, 1, \dots ,$$

with $E[\gamma^n] = g_n, n = 1, 2, \dots$. Multiplying for g_1^{n-1} both sides of equivalence (60), written for the umbra α, we have

$$g_1^{n-1}(\alpha_D^{<-1>})^n \simeq g_1^{n-1}(-n.\alpha)^{n-1} \simeq [-n.(g_1\alpha)]^{n-1} \simeq (-n.\overline{\gamma})^{n-1}$$

for $n = 1, 2, \dots$. Being $\gamma.\beta.\gamma^{<-1>} \equiv \chi$, recalling equivalence (12) and $\beta.\chi \equiv u$, we have

$$\frac{\gamma}{g_1}\, g_1.\beta.\gamma^{<-1>} \Leftrightarrow \frac{\gamma}{g_1}.g_1.\beta.\gamma^{<-1>} \equiv \chi \Leftrightarrow \frac{\gamma}{g_1}.\beta.\chi.g_1.\beta.\gamma^{<-1>} \equiv \chi .$$

So, we have

$$\left(\frac{\gamma}{g_1}\right)^{<-1>} \equiv \chi.g_1.\beta.\gamma^{<-1>}$$

and from equivalence (18) we have

$$\left[\left(\frac{\gamma}{g_1}\right)^{<-1>}\right]^n \simeq (\chi.g_1.\beta.\gamma^{<-1>})^n \simeq g_1(\gamma^{<-1>})^n .$$

Finally, being

$$g_1^{n-1}(\alpha_D^{<-1>})^n \simeq g_1^{n-1}\left[\left(\frac{\gamma}{g_1}\right)^{<-1>}\right]^n \simeq g_1^n(\gamma^{<-1>})^n$$

---

[2]In this case, in the setting of the umbral calculus R must be a field.

we have

$$(61) \qquad \gamma^{\cdot n}(\gamma^{<-1>})^n \simeq (-n.\overline{\gamma})^{n-1} \ .$$

This last equivalence is the generalized Lagrange inversion formula.

The Lagrange inversion formula (60) allows us to express Stirling numbers of first kind in terms of the Bernoulli umbra. An analogous result was proved by Rota and Taylor in [17] via Nörlund sequences.

Proposition 7.1. *If ι is the Bernoulli umbra and $s(n,1) = (-1)^{n-1}(n-1)!$ for $n = 1, 2, \ldots$ are the Stirling numbers of first kind, then*

$$(62) \qquad s(n,1) \simeq (n.\iota)^{n-1} \quad , \quad n = 1, 2, \ldots \ .$$

Proof. From Example 7.3, we have $u^{<-1>} \equiv (-1.\iota)_D^{<-1>}$, where ι is the Bernoulli umbra. From equivalence (60), we have

$$(u^{<-1>})^n \simeq \left[(-1.\iota)_D^{<-1>)} \right]^n \simeq [-n.(-1.\iota)]^{n-1} \simeq (n.\iota)^{n-1} \quad , \quad n = 1, 2, \ldots \ .$$

Equivalence (62) follows recalling that $1 + \log(1+t)$ is the g.f. of $u^{<-1>}$ and that

$$1 + \log(1+t) = 1 + \sum_{n=1}^{\infty} s(n,1)\frac{t^n}{n!}$$

with $s(k,1)$ the Stirling numbers of first kind.

$\square$

One more application of Theorem 7.2 is the proof of the following proposition, giving a property of Abel polynomials, known as *Abel identity*.

Proposition 7.2 (Abel identity). *If $\gamma \in A$, then*

$$(63) \qquad (x+y)^n \simeq \sum_{k \geq 0} \binom{n}{k} y(y - k.\gamma)^{k-1}(x + k.\gamma)^{n-k} \ .$$

Proof. Recall that

$$e^{(y.\beta.\gamma_D)t} \simeq \sum_{k \geq 0} y^k \frac{(e^{\gamma_D t} - u)^{\cdot k}}{k!} \ .$$

Replace y by $y.\gamma_D^*$. Since $\gamma_D^*.\beta.\gamma_D \equiv u$ we have

$$(64) \qquad e^{yt} \simeq \sum_{k \geq 0} (y.\gamma_D^*)^k \frac{(e^{\gamma_D t} - u)^{\cdot k}}{k!} \simeq \sum_{k \geq 0} (y.\gamma_D^*)^k \frac{t^k e^{(k.\gamma)t}}{k!} \simeq$$

$$(65) \qquad \simeq \sum_{k \geq 0} y(y - k.\gamma)^{k-1} \frac{t^k e^{(k.\gamma)t}}{k!}$$

where the second equivalence in (64) follows from (58), and equivalence (65) follows from Theorem 7.2. Multiplying both sides by e^{xt}, we have

$$(66) \qquad e^{(x+y)t} \simeq \sum_{k \geq 0} y(y - k.\gamma)^{k-1}\frac{t^k e^{(x+k.\gamma)t}}{k!} \ .$$

Since $(x+y)^n = D_t^{(n)}[e^{(x+y)t}]_{t=0}$, where $D_t^{(n)}[\cdot]_{t=0}$ is the n-th derivative with respect to t evaluated at $t = 0$, the result follows taking the n-th derivative with

respect to t of the right-hand side of (66) and evaluating it at $t = 0$. Indeed, by using the binomial property of the derivative operator we have

$$D_t^{(n)}[t^k \, e^{(x+k \cdot \gamma)t}] \simeq \sum_{j=0}^{k} \binom{n}{j} D_t^{(j)}[t^k] \, D_t^{(n-j)}[e^{(x+k \cdot \gamma)t}]$$

and, setting $t = 0$, we have

$$D_t^{(n)}[t^k e^{(x+k \cdot \gamma)t}]_{t=0} \simeq (n)_k (x + k \cdot \gamma)^{n-k}$$

by which equivalence (63) follows.

<div style="text-align: right">□</div>

Setting $x = 0$ in equivalence (63), we obtain

(67) $$y^n \simeq \sum_{k \geq 0} \binom{n}{k} (k \cdot \gamma)^{n-k} y(y - k \cdot \gamma)^{k-1} \ .$$

This proves the following polynomial expansion theorem in terms of Abel polynomials.

Proposition 7.3. *If $p(x) \in R[x]$, then*

$$p(x) \simeq \sum_{k \geq 0} p^{(k)}(k \cdot \gamma) \frac{y(y - k \cdot \gamma)^{k-1}}{k!} \quad .$$

The following corollary gives the umbral expression of the Bell exponential polynomials in (16).

Corollary 7.2 (Umbral representation of Bell exponential polynomials)**.** *For all nonnegative n, we have*

(68) $$(x \cdot \beta \cdot \gamma_D)^n \simeq \sum_{k \geq 0} \binom{n}{k} (k \cdot \gamma)^{n-k} x^k \ .$$

Proof. From equivalence (67) and by using Theorem 7.2, we have

(69) $$y^n \simeq \sum_{k \geq 0} \binom{n}{k} (k \cdot \gamma)^{n-k} (y \cdot \gamma_D^*)^k \ .$$

Replace y with $x \cdot \beta \cdot \gamma_D$. We have

$$(x \cdot \beta \cdot \gamma_D)^n \simeq \sum_{k \geq 0} \binom{n}{k} (k \cdot \gamma)^{n-k} (x \cdot \beta \cdot \gamma_D \cdot \gamma_D^*)^k \simeq \sum_{k \geq 0} \binom{n}{k} (k \cdot \gamma)^{n-k} x^k \ ,$$

by which the result follows immediately recalling equivalence (26) for γ_D.

<div style="text-align: right">□</div>

The generalization of equivalence (68) to umbrae γ with first moment g_1 different from zero can be stated by using the same arguments given in Remark 7.1:

$$(x \cdot \beta \cdot \gamma)^n \simeq \gamma^{\cdot n} (x \cdot \beta \cdot \alpha_D)^n \simeq \sum_{k \geq 0} \binom{n}{k} \gamma^{\cdot n} (k \cdot \alpha)^{n-k} x^k \simeq$$

$$\simeq \sum_{k \geq 0} \binom{n}{k} \gamma^{\cdot k} [k \cdot (g_1 \alpha)]^{n-k} x^k \simeq \sum_{k \geq 0} \binom{n}{k} \gamma^{\cdot k} [k \cdot \overline{\gamma}]^{n-k} x^k \ .$$

Example 7.5. *Stirling numbers of second kind.*
In equivalence (68), choose $-1 \boldsymbol{.} \iota$ as umbra γ, then $x \boldsymbol{.} \beta \boldsymbol{.} (-1 \boldsymbol{.} \iota)_D \equiv x \boldsymbol{.} \beta$ (see Example 7.3). Comparing equivalence (68) with $(x \boldsymbol{.} \beta)^n \simeq \sum_{k \geq 0} S(n,k) x^k$, where $\{S(n,k)\}$ are Stirling numbers of second kind (see [4]), we have

$$S(n,k) \simeq \binom{n}{k} (-k \boldsymbol{.} \iota)^{n-k} \qquad k = 0, 1, \ldots, n \ .$$

This last equivalence was already proved by Rota and Taylor in [17] through a different approach.

The umbral version of Stirling numbers of first kind is given in the following proposition.

Proposition 7.4. *If $\{s(n,k)\}$ are Stirling numbers of first kind, then*

(70)
$$s(n,k) \simeq \binom{n}{k} (k \boldsymbol{.} \iota \boldsymbol{.} \chi)^{n-k} \qquad k = 0, 1, \ldots, n \ .$$

Proof. Recalling Example 7.4, we have

$$(x)_n \simeq (x \boldsymbol{.} \chi)^n \simeq (x \boldsymbol{.} \beta \boldsymbol{.} u^{<-1>})^n \simeq [x \boldsymbol{.} \beta \boldsymbol{.} (\iota \boldsymbol{.} \chi)_D]^n \ .$$

The result follows from equivalence (68), being

$$(x \boldsymbol{.} \chi)^n \simeq \sum_{k \geq 0} s(n,k) \, x^k \ .$$

$\square$

7.3. **Solving recursions.** In many special combinatorial problems, the hardest part of the solution may be the discovery of an effective recursion. Once a recursion has been established, Sheffer polynomials are often a simple and general tool for finding answers in closed form. Main contributions in this respect are due to Niederhausen [8, 9, 10]. Further contributions are given by Razpet [11] and Di Bucchianico and Soto y Koelemeijer [2].

Example 7.6. Suppose we are asked to solve the difference equation

(71)
$$s_n(x+1) = s_n(x) + s_{n-1}(x)$$

under the condition $\int_0^1 s_n(x) \, dx = 1$ for all nonnegative integers n. Equation (71) fits the Sheffer identity (38) if we set $y = 1$, choose the sequence $\{p_n(x)\}$ such that $p_0(x) = 1, p_1(1) = 1$ and $p_n(1) = 0$ for all $n \geq 2$ and consider the Sheffer sequence $\{n! s_n(x)\}$. The sequence $\{p_n(x)\}$ is associated to the umbra χ, so we are looking for solutions of (71) such that $n! s_n(x) \simeq (\sigma_x^{(\alpha, \gamma)})^n$ with $\gamma^* \equiv \chi$, i.e. $\gamma \equiv u$ (cf. Example 3.2). The condition $\int_0^1 s_n(x) \, dx = 1$ can be translated in umbral terms by looking for an umbra δ such that $E[s_n(\delta)] = 1$ for all nonnegative integers n. Such an umbra has g.f.

$$\int_0^1 e^{xt} \, dx = \frac{e^t - 1}{t}$$

so that $\delta \equiv -1 \boldsymbol{.} \iota$, with ι the Bernoulli umbra. Therefore, the umbra α satisfies the following identity

$$(-1 \boldsymbol{.} \alpha + -1 \boldsymbol{.} \iota) \boldsymbol{.} \chi \equiv u \ .$$

Due to statement i) of Proposition 2.3, being $\beta.\chi \equiv u$, we have $-1.\alpha + -1.\iota \equiv \beta$ and so $\alpha \equiv -1.(\iota + \beta)$. Solutions of (71) are moments of the Sheffer umbra $(\iota + \beta + x.u).\chi$ normalized by $n!$.

Example 7.7. Suppose we are asked to solve the difference equation

$$(72) \qquad s_n(x) = s_n(x-1) + s_{n-1}(x)$$

that satisfies the initial condition

$$(73) \qquad s_n(1-n) = \sum_{i=0}^{n-1} s_i(n-2\,i) \qquad \text{for all } n \geq 1, \;\; s_0(-1) = 1 .$$

If we rewrite (72) as $s_n(x-1) = s_n(x) - s_{n-1}(x)$, we note that this equation fits the Sheffer identity (38) if we set $y = -1$, choose the sequence $\{p_n(x)\}$ such that $p_0(x) = 1$, $p_1(-1) = -1$ and $p_n(-1) = 0$ for all $n \geq 2$, and consider the Sheffer sequence $\{n! s_n(x)\}$. In particular, from (36) we have $-1.\gamma^* \equiv -\chi$ so $\gamma* \equiv -1. - \chi$ which has g.f. $f(-1. - \chi, t) = (1-t)^{-1}$. Suppose to set $\overline{u} = -1. - \chi$. We have $E[\overline{u}^n] = n!$ for all $n \geq 1$ and $x.\overline{u} \equiv -x. - \chi$ with g.f.

$$f(x.\overline{u}, t) = (1-t)^{-x} = \sum_{n\geq 0} (x)^n \frac{t^n}{n!}$$

and $(x)^n = x(x+1)\cdots(x+n-1) = (x+n-1)_n$. In particular we have $\gamma \equiv (\chi.\overline{u})^{<-1>}$. Solutions of (72) are therefore of the type

$$s_n(x) \simeq \frac{[(-1.\alpha + x.u).\overline{u}]^n}{n!} \;\; .$$

Now we need to identify α. As in the previous example, the moments of such an umbra depend on the initial condition (73). Observe that, if $s_n(x)$ is a Sheffer sequence with associated polynomials $(x.\overline{u})^n/n!$, then $s_n(x-n)$ is a Sheffer sequence with associated polynomials

$$\frac{[(x-n).\overline{u}]^n}{n!} \simeq \binom{x-n+n-1}{n} \simeq \frac{[(x-1).\chi]^n}{n!} \;\; ,$$

since $[(x-1).\chi]^n \simeq (x-1)_n$. Therefore due to Theorem 6.1 we have

$$(74) \qquad s_n(x-n) \simeq \frac{[(-1.\alpha + (x-1).u).\chi]^n}{n!} \;\; .$$

So the values of $s_n(1-n)$ in the initial condition (73) give exactly the moments of $-1.\alpha.\chi$ normalized by $n!$. Therefore, by observing that

$$s_n(x) \simeq \frac{[(-1.\alpha + x + n - 1).\chi]^n}{n!} \simeq \sum_{k=0}^{n} \frac{(-1.\alpha.\chi)^k}{k!} \binom{x+n-1}{n-k}$$

we have

$$s_n(x) = \sum_{k=0}^{n} s_k(1-k) \binom{x+n-1}{n-k} \;\; .$$

From a computational point of view, this formula is very easy to implement by using the recursion of the initial condition. If one would evaluate the moments of

$-1.\alpha.\chi$ not by using the recursion of the initial condition, but with a closed form, some different considerations must be done. By using equivalence (74) we have

$$\sum_{i=0}^{n-1} s_i(n-2i) = \sum_{i=0}^{n-1} s_i(x-i)|_{x=n-i} \simeq \sum_{i=0}^{n-1} \frac{[-1.\alpha.\chi + (n-i-1).\chi]^i}{i!}$$

and from the initial condition we have

(75)
$$\frac{(-1.\alpha.\chi)^n}{n!} \simeq \sum_{j=0}^{n-1} \frac{(-1.\alpha.\chi)^j}{j!} \sum_{i=0}^{n-j-1} \frac{[(n-i-j-1).\chi]^i}{i!} \simeq$$

$$\simeq \frac{(-1.\alpha.\chi + \overline{\delta})^{n-1}}{(n-1)!}$$

where $\overline{\delta}$ is an umbra such that

(76)
$$\overline{\delta}^k \simeq k! \sum_{i=0}^{k} \frac{[(k-i).\chi]^i}{i!} \simeq k!\delta^k .$$

Observe that

$$\overline{\delta}^k \simeq k! \sum_{t=0}^{k} \frac{(t.\chi)^{k-t}}{(k-t)!} \simeq \sum_{t=0}^{k} \binom{k}{t}(t.\chi)^{k-t}\overline{u}^t \simeq (\overline{u}.\beta.\chi_D)^k$$

by using Corollary 7.2. The umbra δ is said to be the boolean cumulant umbra of χ_D (cf. [3]). In particular the umbra δ has moments equal to the Fibonacci numbers, since $\overline{\delta}$ has g.f.

$$f(\overline{\delta}, t) = \frac{1}{1 - t(1+t)} = \frac{1}{1 - t - t^2} .$$

In terms of g.f.'s, equivalence (75) gives

$$\sum_{n\geq 0} \frac{(-1.\alpha.\chi)^n}{n!} t^n \simeq 1 + t \sum_{n\geq 1} \frac{(-1.\alpha.\chi + \overline{\delta})^{n-1}}{(n-1)!} t^{n-1}$$

so that

$$\frac{1}{f(\alpha.\chi, t)} = 1 + t\frac{f(\overline{\delta}, t)}{f(\alpha.\chi, t)} \Leftrightarrow f(-1.\alpha.\chi, t) = \frac{1}{1 - tf(\overline{\delta}, t)}$$

and

$$-1.\alpha.\chi \equiv \overline{u}.\beta.\overline{\delta}_D .$$

Therefore the solution in closed form is

$$s_n(x) \simeq \frac{[\overline{u}.\beta.\overline{\delta}_D + (x+n-1).\chi]^n}{n!} .$$

Example 7.8. Suppose we are asked to solve the difference equation

(77)
$$F_n(m) = F_n(m-1) + F_{n-1}(m-2)$$

under the condition $F_n(0) = 1$ for all nonnegative integers n. Replace m with $x + n + 1$. Then equation (77) can be rewritten as

(78)
$$F_n(x+n+1) = F_n(x+n) + F_{n-1}(x+n-1) .$$

Equation (78) fits the Sheffer identity (38) if we set $y = 1$, choose the sequence $\{p_n(x)\}$ such that $p_0(x) = 1, p_1(1) = 1$ and $p_n(1) = 0$ for all $n \geq 2$ and consider

the Sheffer sequence $\{n!F_n(x)\}$. As in Example 7.6, we are looking for solutions of (78) such that
$$F_n(x+n) \simeq \frac{[(-1.\alpha + x.u).\chi]^n}{n!} \ .$$
Let us observe that equation (78), for $x = 0$, gives the well-known recurrence relation for Fibonacci numbers so that
$$F_n(n) \simeq \frac{(-1.\alpha.\chi)^n}{n!} \simeq \delta^n \ .$$
Therefore we have $-1.\alpha.\chi \equiv \bar{\delta}$, with $\bar{\delta}^n \simeq n!\delta^n$ as given in equivalence (76), and $\alpha \equiv -1.\bar{\delta}.\beta$. Then, solutions of (78) are such that
$$F_n(x+n) \simeq \frac{(\bar{\delta} + x.\chi)^n}{n!} \ .$$
In particular
$$F_n(x+n) \simeq \sum_{k=0}^{n} \frac{(x.\chi)^{n-k}}{(n-k)!} \, \delta^k \simeq \sum_{k=0}^{n} \frac{(x.\chi)^{n-k}}{(n-k)!} \, \sum_{j=0}^{k} \frac{(j.\chi)^{k-j}}{(k-j)!} \simeq$$
$$\simeq \sum_{k=0}^{n}\sum_{j=0}^{n} \frac{(x.\chi)^{n-k-j}}{(n-k-j)!} \ \frac{(k.\chi)^j}{j!} \simeq \sum_{k=0}^{n} \frac{[(x+k).\chi]^{n-k}}{(n-k)!} \simeq$$
$$\simeq \sum_{k=0}^{n} \binom{x+k}{n-k}$$
by which we can verify that the initial conditions $F_n(0) = F_n(-n+n) = 1$ hold.

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