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Continuous Sheffer families II ☆

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ABSTRACT

Continuous Sheffer families have been recently introduced by the authors. These are continuous versions of the Sheffer sequences arising in the umbral calculus. We show here that quite a number of classical special functions are examples of such families.

0. Introduction

A sequence of polynomials $(p_n(t))_{n=0}^{\infty}$ is said to be binomial if, for $s, t \in \mathbb{R}$, it satisfies the rule

$$p_n(s+t) = \sum_{k=0}^{n} \binom{n}{k} p_k(t) p_{n-k}(s),$$

or, equivalently,

$$\widetilde{p}_n(t+s) = \sum_{k=0}^{n} \widetilde{p}_k(t) \widetilde{p}_{n-k}(s),$$

where $\widetilde{p}_n(t) = \widetilde{p}_n(t)/n!$. This means that the family $(\widetilde{p}^t)_{t\in\mathbb{R}}$ of sequences $\widetilde{p}^t = (\widetilde{p}_n(t))_{n=0}^{\infty}$ is a group in the convolution algebra of all complex sequences. For simplicity we limit ourselves to positive parameter $t > 0$, so that we will deal with *semigroups* $(\widetilde{p}^t)_{t>0}$ in the sequel.

A Sheffer (polynomial) sequence for a given binomial sequence $(p_n(t))_n$ is another polynomial sequence $(\sigma_n(t))_n$ such that, for all $n \in \mathbb{N} \cup \{0\}$,

$$\sigma_n(s+t) = \sum_{k=0}^{n} \binom{n}{k} \sigma_k(s) p_{n-k}(t).$$

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Putting $\widetilde{\sigma}_n = \sigma_n/n!$, one has

$$\widetilde{\sigma}_n(s+t) = \sum_{k=0}^{n} \widetilde{\sigma}_k(s)\widetilde{p}_{n-k}(t),$$

that is to say, $\widetilde{\sigma}_{s+t} = \widetilde{\sigma}_s * \widetilde{p}^t$ $(s, t > 0)$, for the convolution in the algebra of complex sequences, where $\widetilde{\sigma}_s(n) := \widetilde{\sigma}_n(s)$.

Binomial sequences and Sheffer sequences are important study objects in the umbral calculus, and most classical sequences in the theory of orthogonal polynomials are examples of Sheffer sequences; see [5,11]. Sheffer families associated with the binomial sequence $(t^n)_n$ are called Appell sequences, see [5, p. 8].

In [4], a continuous setting for the preceding notions is proposed, by introducing a metrizable complete locally convex algebra $\mathcal{U}$ which contains the Mellin transforms of generating functions of Sheffer sequences; see [4, Introduction] for more details. This algebra is formed by holomorphic functions on the right-hand half-plane, and is endowed with a convolution. Binomial sequences correspond to semigroups in $\mathcal{U}$, and in particular the fundamental sequence $(t^n)_n$ of the umbral calculus corresponds to the semigroup $(\gamma^t)_{t>0}$ in $\mathcal{U}$ given by $\gamma^t : z \mapsto t^{-z}\Gamma(z)$, $t > 0$. Also, a Sheffer family is defined in [4] as a family $(F_t)_{t>0} \subseteq \mathcal{U}$ for which there exists a semigroup $(f^t)_{t>0} \subseteq \mathcal{U}$ such that $F_{s+t} = F_s * f^t$, $s, t > 0$. It is shown in [4] that the Hermite function and Lerch functions are Sheffer families with respect to the semigroup $(\gamma^t)_{t>0}$.

In the present paper we give more examples of Sheffer families associated with the semigroup $(\gamma^t)_{t>0}$. Such families are, or are related with, the special functions which correspond to sequences of orthogonal polynomials of types Charlier, Gegenbauer, Abel, Laguerre, Jacobi. This is done in Section 3. Previously to that section we discuss the derivation and translation semigroups in the algebra of continuous operators on $\mathcal{U}$; see Section 2. Such semigroups of operators are important because they play the role of infinitesimal generators of remarkable semigroups in $\mathcal{U}$. One is the Gamma semigroup $(\gamma^t)_{t>0}$, another one is the Gaussian semigroup defined by the Gaussian function, which is also analyzed in Section 2. We include Section 1 on preliminaries, where the definition and main properties of the algebra $\mathcal{U}$ are collected.

1. Preliminaries

Put $\mathbb{C}^+ := \{z \in \mathbb{C}:\ \Re z > 0\}$. Let $\mathrm{Hol}(\mathbb{C}^+)$ denote the usual topological algebra of holomorphic functions on $\mathbb{C}^+$ endowed with the compact convergence topology τ_c. We define $\mathcal{U}$ as the space of functions $F \in \mathrm{Hol}(\mathbb{C}^+)$ such that

$$\|F\|_{a,b} := \sup_{a \leqslant x \leqslant b} \int_{-\infty}^{\infty} \left|F(x+iy)\right| dy, \quad 0 < a \leqslant b,$$

endowed with the locally convex vector space topology generated by the system of norms $\{\|\cdot\|_{a,b}\}_{a \leqslant b}$. Every function $F \in \mathcal{U}$ can be represented as

$$F(z) = \frac{1}{2\pi i} \int_{\gamma_{a,b}} \frac{F(w)}{w-z}\, dw, \quad 0 < a < \Re z < b, \tag{1.1}$$

where, for $x > 0$, γ_x is the vertical line $x + i\mathbb{R}$ parameterized from $-\infty$ to ∞ and $\gamma_{a,b} = \{-\gamma_a\} \cup \gamma_b$. This integral formula allows us to prove that the space $\mathcal{U}$ is metrizable and complete, and that the inclusion mapping $\mathcal{U} \hookrightarrow \mathrm{Hol}(\mathbb{C}^+)$ is continuous with dense range. Besides this, for

$$\mathcal{U}_\tau := \left\{G \in \mathcal{U}:\ G = F(\cdot + z);\ F \in \mathcal{U}, z \in \mathbb{C}^+\right\},$$

we have that $\mathcal{U}_\tau$ is a dense vector subspace of $\mathcal{U}$, with the restrictions on $i\mathbb{R}$ of functions in $\mathcal{U}_\tau$ lying in $L^1(i\mathbb{R})$.

The space $\mathcal{U}$ is moreover a commutative locally convex algebra in the sense defined in [1], with multiplication given by the convolution

$$F * G(z) := \frac{1}{2\pi i} \int_{\Re w = c} F(z-w)G(w)\, dw,$$

for $F, G \in \mathcal{U}$, $z \in \mathbb{C}^+$ and $0 < c < \Re z$, where the integral does not depend on c. Actually,

$$\|F * G\|_{a,b} \leqslant \|F\|_{a-\alpha, b-\alpha} \|G\|_{\alpha,\alpha}$$

for every α, a, b such that $0 < \alpha < a < b$.

Recall that a *character* φ of the algebra $\mathcal{U}$ is a complex algebra homomorphism $\varphi : \mathcal{U} \longrightarrow \mathbb{C}$. It is shown in [4] that all non-zero continuous characters of $\mathcal{U}$ are of the form $\varphi = \varphi_\lambda$, $\lambda > 0$, where

$$\varphi_\lambda(F) := \frac{1}{2\pi i} \int_{\Re z = c} \lambda^{-z} F(z)\, dz, \quad F \in \mathcal{U},\ c > 0.$$

For $F \in \mathcal{U}$ and $\lambda > 0$, set $\widehat{F}(\lambda) := \varphi_\lambda(F)$ and $\mathcal{G}(F) := \widehat{F}$. Then $\widehat{F}$ is continuous on $\mathbb{R}^+ := (0, \infty)$ and

$$\lim_{\lambda \to 0^+} \lambda^c f(\lambda) = 0, \qquad \lim_{\lambda \to \infty} \lambda^c f(\lambda) = 0, \quad \text{for all } c > 0.$$

Moreover, the correspondence $F \mapsto \mathcal{G}(F)$ is injective. We call $\mathcal{G}(F)$ the *Gelfand transform* of F.

We refer prospective readers to [4] for the above definitions and properties.

2. Semigroups associated with the space $\mathcal{U}$

2.1. Semigroups of operators on $\mathcal{U}$

We first give examples of semigroups living in the (locally convex) algebra $\mathcal{B}(\mathcal{U})$ of continuous linear operators on $\mathcal{U}$. They are related with generators of semigroups in the algebra $\mathcal{U}$, as we will see in Section 2.2 below.

Fractional derivation semigroup. We build up this semigroup on the base of formula (1.1). In the sequel, we denote by $\log\zeta$ the analytic branch of the logarithm with argument $\arg\zeta$ taking values in the interval $[-\pi/2, 3\pi/2)$. We put $\zeta^s := e^{s\log\zeta}$ for every $\zeta \in \mathbb{C} \setminus i(-\infty, 0)$ and $s > 0$. Let a, b be such that $0 < a < b$. For $F \in \mathcal{U}$ and $\nu \geqslant 0$ define

$$D^\nu F(z) = \frac{\Gamma(\nu+1)}{2\pi i}\left(\int_{\Re w = b} \frac{F(w)}{(w-z)^{\nu+1}}\,dw - \int_{\Re w = a} \frac{e^{2\pi\nu i}F(w)}{(w-z)^{\nu+1}}\,dw\right) \tag{2.1}$$

whenever $a < \Re z < b$. The above choice of the logarithm implies that the function $w \mapsto F(w)(w-z)^{-(\nu+1)}$ is holomorphic in the strip $c < \Re w < C$ for $0 < c < C < \Re z$ or $0 < \Re z < c < C$, and then, applying [4, Proposition 1.2(i)] to $F \in \mathcal{U}$, we get that the definition of $D^\nu F(z)$ is independent of a and b for fixed $z \in \mathbb{C}^+$. It is a simple matter that $D^\nu F$ is a holomorphic function on $\mathbb{C}^+$. Moreover, (1.1) implies that $D^n F = (-1)^n F^{(n)}$ when $\nu = n = 0, 1, 2, \dots$.

Proposition 2.1. *Let $\nu \geqslant 0$. Then $D^\nu F \in \mathcal{U}$ for every $F \in \mathcal{U}$ and the linear mapping $D^\nu : \mathcal{U} \to \mathcal{U}$ is continuous.*

Proof. The case $\nu = 0$ is clear by (1.1) since D^0 is the identity operator. Let $\nu > 0$, and take $0 < a < b$. For z, w such that $a \leqslant \Re z \leqslant b$ and $\Re w = a/2$ or $\Re w = 2b$ we get $|\Re(w-z)| \geqslant \frac{a}{2}$ or $|\Re(w-z)| \geqslant b$, respectively. Then using the Fubini's rule, an elementary estimate gives us that $\|D^\nu F\|_{a,b} \leqslant C_{a,b}^{(\nu)}\Gamma(\nu+1)\|F\|_{\frac{a}{2},2b}$ for all $F \in \mathcal{U}$, where $C_{a,b}(\nu) = \int_{-\infty}^{\infty}[((a^2/4) + y^2)^{-(\nu+1)/2} + (b^2+y^2)^{-(\nu+1)/2}]\,dy$. This implies the two assertions of the statement. $\square$

By applying the residues theorem one finds

$$\frac{1}{2\pi i}\int_{\Re\zeta = \rho} \frac{e^{\tau\zeta}}{\zeta^{s+1}}\,d\zeta = \frac{\tau_+^s}{\Gamma(s+1)}, \quad \tau \in \mathbb{R},\ s, \rho > 0 \tag{2.2}$$

(see for instance [7, p. 329] for a quick calculation).

Take $0 < a < c < b$ and $s, \lambda > 0$. For $\Re w = a$ or $\Re w = b$, put

$$L(w, \lambda, s) := \frac{1}{2\pi i}\int_{\Re z = c} \frac{\lambda^{-z}}{(w-z)^{s+1}}\,dz.$$

If $\Re w = a$ we get

$$\begin{aligned} L(w, \lambda, s) &= \frac{\lambda^{-w}}{2\pi i}\int_{\Re\zeta = a-c} \frac{\lambda^\zeta}{\zeta^{s+1}}\,d\zeta = \frac{\lambda^{-w}}{2\pi i}e^{-i(s+1)\pi}\int_{\Re z = c-a} \frac{(\lambda^{-1})^z}{z^{s+1}}\,dz = \frac{\lambda^{-w}}{\Gamma(s+1)}\frac{(\log\lambda^{-1})_+^s}{e^{i(s+1)\pi}} \\ &= -\frac{e^{-2s\pi i}}{\Gamma(s+1)}(\log\lambda)^s\lambda^{-w}\chi_{(0,1)}(\lambda), \end{aligned}$$

where we have used in the second equality that $\arg(\zeta) = \arg(-\zeta) + \pi$ for all $\zeta \in \mathbb{C} \setminus i\mathbb{R}$ with $\Re\zeta < 0$.

Analogously, if $\Re w = b$ then

$$L(w, \lambda, s) = \frac{\lambda^{-w}}{2\pi i}\int_{\Re\zeta = b-c} \frac{\lambda^\zeta}{\zeta^{s+1}}\,d\zeta = \frac{\lambda^{-w}}{\Gamma(s+1)}(\log\lambda)^s\chi_{(1,\infty)}(\lambda).$$

Proposition 2.2. *The family $(D^\nu)_{\nu>0}$ given by formula (2.1) is a strongly continuous semigroup in $\mathcal{B}(\mathcal{U})$. Moreover, each D^ν is a multiplier of $\mathcal{U}$ with symbol $(\log\lambda)^\nu$; that is,*

$$\widehat{D^\nu F}(\lambda) = (\log\lambda)^\nu \widehat{F}(\lambda), \quad F \in \mathcal{U},\ \lambda > 0.$$

Proof. First we show the formula of the symbol. So, let $F \in \mathcal{U}$ and $c, s, \lambda > 0$. By applying the remark prior to the proposition one obtains

$$\begin{aligned}
\mathcal{G}\big(D^s F\big) = \widehat{D^s F}(\lambda) &= \frac{1}{2\pi i}\int_{\Re z=c} \lambda^{-z} D^s F(z)\,dz \\
&= \frac{\Gamma(s+1)}{2\pi i}\bigg(\int_{\Re w=b} L(w,\lambda,s)F(w)\,dw - e^{2\pi si}\int_{\Re w=a} L(w,\lambda,s)F(w)\,dw\bigg) \\
&= (\log\lambda)^s\bigg(\chi_{(1,\infty)}(\lambda)\int_{\Re w=b} \lambda^{-w}F(w)\frac{dw}{2\pi i} + \chi_{(0,1)}(\lambda)\int_{\Re w=a} \lambda^{-w}F(w)\frac{dw}{2\pi i}\bigg) \\
&= (\log\lambda)^s\big(\chi_{(1,\infty)}(\lambda) + \chi_{(0,1)}(\lambda)\big)\widehat{F}(\lambda) = (\log\lambda)^s\widehat{F}(\lambda) = (\log\lambda)^s\mathcal{G}(F)(\lambda)
\end{aligned}$$

for every $F \in \mathcal{U}$. Then the semigroup property follows readily: for $F \in \mathcal{U}$ and $s, t, \lambda > 0$ we have $\mathcal{G}(D^s(D^t F))(\lambda) = (\log\lambda)^s\mathcal{G}(D^t F)(\lambda) = (\log\lambda)^{s+t}\mathcal{G}(F)(\lambda) = \mathcal{G}(D^{s+t}F)(\lambda)$, whence $D^sD^t = D^{s+t}$ on $\mathcal{U}$, since the Gelfand transform $\mathcal{G}$ of $\mathcal{U}$ is injective.

Finally, the semigroup $(D^s)_{s>0}$ is strongly continuous, which is to say, the mappings $s \mapsto D^s F$, $(0,\infty) \to \mathcal{U}$ are continuous for every $F \in \mathcal{U}$. This is a simple consequence of the Fubini's theorem and the dominated convergence theorem. □

Translation semigroup. For $z \in \overline{\mathbb{C}^+}$, let $T_z : \mathcal{U} \to \mathcal{U}$ be defined by

$$(T_z F)(w) = F(w+z), \quad w \in \mathbb{C}^+.$$

Clearly, $T_{z+w} = T_z \circ T_w$, where the symbol $\circ$ means composition, and $\|T_z F\|_{a,b} = \|F\|_{a+\Re z, b+\Re z}$ for every $z, w \in \mathbb{C}^+$ and for all $b \geqslant a > 0$. By [4, Proposition 1.3] $\mathcal{U}_\tau$ is dense in $\mathcal{U}$ (see also Section 1 above), hence we have that $\lim_{\epsilon\to 0^+} T_\epsilon F = F$ $(F \in \mathcal{U})$. Also, $\lim_{y\to 0} T_{iy}F = F$ $(F \in \mathcal{U})$; see [4, Proof of Theorem 2.4]. Therefore the family $(T_z)_{\Re z\geqslant 0}$ is a strongly continuous semigroup in $\mathcal{B}(\mathcal{U})$. Actually, since every $F \in \mathcal{U}$ is holomorphic in $\mathbb{C}^+$, Morera's theorem implies that $(T_z)_{\Re z>0}$ is holomorphic.

Even though we are not concerned in this paper with any detailed analysis of infinitesimal generators of semigroups, it seems appropriate at this place to describe the generator of the semigroup $(T_t)_{t\geqslant 0}$. By definition, the generator A of $(T_t)_{t\geqslant 0}$ is given by

$$AF := \lim_{t\to 0^+} \frac{T_t F - F}{t}$$

in $\mathcal{U}$, when the limit exists. A direct calculation tells us that $AF(z) = F'(z)$ for all $F \in \mathcal{U}$ and $z \in \mathbb{C}^+$. In particular we have that A is defined on all of $\mathcal{U}$ and it is continuous (recall Proposition 2.1). In fact, $A = D := D^1$.

Proposition 2.3. *The translation semigroup $(T_z)_{\Re z\geqslant 0} \subseteq \mathcal{B}(\mathcal{U})$ is strongly continuous on $\mathcal{U}$ and holomorphic in $\Re z > 0$, and its infinitesimal generator coincides with the (continuous) derivation $D = d/dz$ on $\mathcal{U}$. Moreover, each T_t is a multiplier of $\mathcal{U}$ with symbol given by*

$$\widehat{T_t F}(\lambda) = \lambda^t \widehat{F}(\lambda), \quad F \in \mathcal{U},\ t, \lambda > 0.$$

Proof. The only thing to prove is the identity involving the symbol. Take $F \in \mathcal{U}$ and $\lambda, t > 0$. Then for $c > 0$

$$\widehat{T_t F}(\lambda) = \frac{1}{2\pi i}\int_{\Re z=c} \lambda^{-z}F(z+t)\,dz = \frac{1}{2\pi i}\int_{\Re w=c+t} \lambda^t\lambda^{-w}F(w)\,dw = \lambda^t\widehat{F}(\lambda),$$

as we claimed. □

Remark 2.4. Despite that $D = d/dz$ is a continuous operator on $\mathcal{U}$ one cannot deduce from this fact that the formal expression e^{zD} has a sense as a convergent power series $e^{zD} \equiv \sum_{n=0}^\infty \frac{z^n D^n}{n!}$ in $\mathcal{B}(\mathcal{U})$ for all $z \in \mathbb{C}^+$, since for arbitrary functions $F \in \mathcal{U}$ the best estimate that one can get is $\|D^n F\|_{a,b} \leqslant (n!/k^n)C_{a,b}\|F\|_{\frac{a}{2},2b}$, $0 < k < 1$, with $k \to 0$ as $a \to 0$, for every

$b \geqslant a > 0$. Actually, if that series were convergent then the semigroup $T(z) \equiv e^{zD}$ would extend as a holomorphic function on the whole $\mathbb{C}$. Hence, e^{-zD} would exist as the inverse operator of $T(z)$, for every $z \in \mathbb{C}^+$. But then it should be $e^{-zD} = T(\cdot - z)$ for $z \in \mathbb{C}^+$, which obviously is not defined on all of $\mathcal{U}$. However, the semigroup $(T_z)_{\Re z>0}$ is locally equicontinuous and therefore it admits a sort of exponential formula; see [10, Theorem 3.1].

On the other hand, the joint application of the description of symbols obtained in Proposition 2.2 and Proposition 2.3 gives us the equality

$$\left(\frac{d}{dt}T_tF\right)^{\wedge}(\lambda) = \left[F'(\cdot + t)\right]^{\wedge}(\lambda) = \widehat{T_tF'}(\lambda) = \lambda^t \widehat{F'}(\lambda) = \lambda^t(\log\lambda)\widehat{F}(\lambda) = \left(T^tDF\right)^{\wedge}(\lambda),$$

for every $F \in \mathcal{U}$ and $t, \lambda > 0$. This corresponds to the fact that $\frac{d}{dt}T_t = DT_t = T_tD$.

2.2. Semigroups in $\mathcal{U}$

It is shown in [4, Proposition 3.1] that for $\alpha > 0$ the function

$$\gamma_\alpha^t(z) := \frac{1}{\alpha}t^{-z/\alpha}\Gamma(z/\alpha), \quad z \in \mathbb{C}^+,\ t > 0, \tag{2.3}$$

defines a continuous semigroup $(\gamma_\alpha^t)_{t>0}$ in $\mathcal{U}$ with Gelfand transform

$$\widehat{\gamma_\alpha^t}(\lambda) = e^{-t\lambda^\alpha} \quad (\lambda > 0), \tag{2.4}$$

and generator T_α in the sense that

$$\frac{d}{dt}\gamma_\alpha^t = -T_\alpha\gamma_\alpha^t \quad (t > 0). \tag{2.5}$$

Note that (2.5) is coherent with (2.4) and the fact that the symbol of T_α is $\widehat{T_\alpha}(\lambda) = \lambda^\alpha$, $\lambda > 0$ (Proposition 2.3). However, it is not clear to us whether or not $\lim_{t\to 0^+}\gamma_\alpha^t * F = F$ in $\mathcal{U}$, for any $F \in \mathcal{U}$.

In fact, the semigroup γ_α^t corresponds to the fundamental semigroup of the umbral calculus $n \mapsto t^n/n!$ via the two equalities

$$\frac{1}{2\pi i}\int_{\Re z=c}\left(\lambda^\alpha\right)^{-z}t^{-z}\Gamma(z)\,dz = e^{-t\lambda} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\left(-\lambda^\alpha\right)^n, \quad \lambda, t > 0; \tag{2.6}$$

see [4, Introduction]. Variants of functions γ_α^t provide continuous semigroups in the convolution Banach algebra $L^1(\mathbb{R})$; for example, we have that the family $(\Gamma_\alpha^t)_{t>0}$ defined by

$$\Gamma_\alpha^t(y) := \frac{1}{2\pi\alpha}t^{-(t+iy)/\alpha}\Gamma\left(\frac{t+iy}{\alpha}\right), \quad y \in \mathbb{R}, \tag{2.7}$$

is such a semigroup; see [4, Remark 3.2].

Let us mention that the reflection formula, see below, of the Gamma function may be interpreted as a consequence of the semigroup property of $\gamma^t := \gamma_1^t$: Let $b > 0$. Putting $t = 1/b$ in the convolution identity

$$\gamma^{t+1} = \gamma^t * \gamma^1, \quad t > 0,$$

one gets

$$\left(\frac{b}{b+1}\right)^p\Gamma(p) = \gamma^{(b+1)/b}(p) = \gamma^1 * \gamma^{1/b}(p) = \frac{1}{2\pi i}\int_{\Re q=c}\Gamma(p-q)\Gamma(p)b^q\,dq \quad (c > \Re p > 0)$$

and then one can deduce the reflection property for the Gamma function,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z} \quad (0 < \Re z < 1),$$

as in [13, p. 251].

Another (continuous) semigroup in $\mathcal{U}$ related with the Gamma function is $(\beta^t)_{t>0}$, where

$$\beta^t(z) := B(z, t+1), \quad z \in \mathbb{C}^+,\ t > 0,$$

and B is the Beta function. Its Gelfand transform is given by $\widehat{\beta^t}(\lambda) = (1-\lambda)_+^t$, $\lambda > 0$; see [4, Proposition 3.3].

There are indeed many other semigroups in $\mathcal{U}$. Next we consider the example induced by the Gaussian function.

Gaussian semigroup in $\mathcal{U}$. Recall that $g^t(y) := e^{-y^2/(4t)}/\sqrt{4\pi t}$, $y \in \mathbb{R}$, $t > 0$, defines a bounded strongly continuous C_0-semigroup g^t in $L^1(\mathbb{R})$. Moreover, for $t > 0$ the Fourier transform of g^t is $(\mathcal{F}g^t)(\xi) = e^{-t\xi^2}$ for every $\xi \in \mathbb{R}$ [12, p. 25]. Put

$$G^t(z) = \frac{1}{\sqrt{4\pi t}} e^{z^2/(4t)} \quad (\Re z \geqslant 0;\ t > 0).$$

Then $G^t \in \mathcal{U}$, $G^t(iy) = g^t(y)$ for $y \in \mathbb{R}$ and, for every $t, \lambda > 0$,

$$\widehat{G^t}(\lambda) = \frac{1}{2\pi i} \int_{\Re z=0} \lambda^{-z} G^t(z)\,dz = \big(\mathcal{F}g^t\big)(\log\lambda) = e^{-t(\log\lambda)^2}.$$

Take now $F \in \mathcal{U}$. For $z = x + iy \in \mathbb{C}^+$, set $F_x(y) := F(x + iy)$. Then

$$F * G^t(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F\big(x + i(y - v)\big) G^t(iv)\,dv = \big(F_x * g^t\big)(y).$$

By Proposition 2.3 the mapping $(0, \infty) \to \mathcal{U}, r \mapsto F(r + \cdot)$ is continuous, so we have that, for $0 < a < b$, the family $(F_x)_{a \leqslant x \leqslant b}$ is a compact subset of $L^1(\mathbb{R})$. A straightforward argument of compactness jointly with the uniform bound $\sup_{t>0} \|g^t\|_1 < \infty$ gives us that

$$\lim_{t \to 0^+} \sup_{x \in [a,b]} \big\| F_x * g^t - F_x \big\|_1 = 0,$$

and this means that $\lim_{t \to 0^+} \|F * G^t - F\|_{a,b} = 0$. Thus we have proved the following result.

Proposition 2.5. *$(G^t)_{t>0}$ is a strongly continuous semigroup in $\mathcal{U}$ with Gelfand transform*

$$\widehat{G^t}(\lambda) = e^{-t(\log\lambda)^2}, \quad t, \lambda > 0,$$

and infinitesimal generator $-D^2$; that is, $\frac{dG^t}{dt} = -D^2 G^t$ $(t > 0)$.

Proof. It only remains to prove the equality with derivatives. To see this, note that $\partial g^t/\partial t = \partial^2 g^t/\partial y^2$; see [12, p. 25] for instance. Then, for $t > 0$, $z = x + iy \in \mathbb{C}^+$ and every $F \in \mathcal{U}$,

$$\frac{d}{dt}\big(G^t * F\big)(z) = \frac{\partial}{\partial t}\big(g^t * F_x\big)(y) = \frac{\partial^2}{\partial y^2}\big(g^t * F_x\big)(y) = \frac{\partial^2}{\partial y^2}\big(G^t * F\big)(x + iy) = -\frac{d^2}{dz^2}\big(G^t * F\big)(z). \qquad \square$$

3. Sheffer and Appell families in $\mathcal{U}$

We say that a family of functions $(S_t)_{t>0} \subseteq \mathcal{U}$ is a *Sheffer* family if there is a semigroup $(F^t)_{t>0} \subseteq \mathcal{U}$ such that

$$S_{s+t} = F^s * S_t \quad \text{for all } s, t > 0.$$

The above concept has been introduced in [4]. Each semigroup in $\mathcal{U}$ is a Sheffer family. A family of functions in $\mathcal{U}$ which is Sheffer with respect to the semigroup $(\gamma^t)_{t>0}$ is called here an *Appell* family. In the umbral calculus, the Appell sequences are the Sheffer sequences associated to the semigroup $n \mapsto t^n/n!$. One has that a sequence $(\widetilde{a}_n(x))_n$ is Appell if and only if $(d/dx)\widetilde{a}_n(x) = \widetilde{a}_{n-1}(x)$ for every n; see [5, p. 8]. All the examples of Appell families $(\mathcal{A}_t)_{t>0} \subseteq \mathcal{U}$ which we give below are such that there exist functions ϕ with $\mathcal{A}_t = \phi * \gamma^t$, $t > 0$. Thus, similarly to the discrete case, $(d/dt)\mathcal{A}_t = \phi * (d/dt)\gamma^t = \phi * (-T_1\gamma^t) = -\mathcal{A}_{t+1}$, where we have used (2.5) in the second equality.

It is also to be noticed that our examples fit into the following general framework. Suppose that we have a function $F(n, t)$ with associated generating function

$$\sum_{n=0}^{\infty} \frac{F(n, t)}{n!} x^n = \varphi(x) e^{t\psi(x)}, \tag{3.1}$$

for some functions φ and ψ. Then it is to be expected that, under suitable conditions on φ and ψ, the Mellin transform

$$S_t(z) = \int_0^{\infty} \lambda^{z-1} \varphi(-\lambda) e^{t\psi(-\lambda)}\,d\lambda,$$

exists for every $t, \Re z > 0$, and, moreover, is an element of $\mathcal{U}$. If this is the case, then $(S_t)_{t>0}$ is in fact an Appell family. The method has been illustrated in [4, Section 4] with two important examples of special functions $F(n,t)$; namely, the Hermite polynomials, and the Apostol–Bernoulli and Apostol–Euler polynomials defined by the Lerch function.

Next we show more examples of Appell families in $\mathcal{U}$ induced by classical special functions.

Charlier function. As above, let B denote the Beta function $B(z,w) = \Gamma(z)\Gamma(w)/\Gamma(z+w)$, where $z, w, z+w \neq 0, 1, \ldots.$

Let us consider the formula

$$(1+x)^{-t} = \frac{1}{2\pi i} \int_{\Re z = c} x^{-z} B(z, t-z)\, dz, \tag{3.2}$$

where $x, t > 0$ and $0 < c < t$; see [2, p. 85]. Incidentally, we notice that the formula can be immediately obtained from the semigroup property of the function γ^t: The identity $\gamma^{1+x}(t) = \gamma^x * \gamma^1(t)$, for $x, t > 0$, means exactly that

$$(1+x)^{-t}\Gamma(t) = \frac{1}{2\pi i} \int_{\Re z = c} x^{-z}\Gamma(z)\Gamma(t-z)\, dz, \quad t > c > 0, x > 0.$$

The function $z \mapsto B(z, t-z)$ is not defined for all $z \in \mathbb{C}^+$ but only for $z \in \mathbb{C}^+$ such that $0 < \Re z < t$. Thus the function does not belong to $\mathcal{U}$. For our aims here, we modify the function $x \mapsto (1+x)^{-t}$, $x \in (0,\infty)$, to be $x \mapsto e^{-tx}(1+x)^{-t}$.

Proposition 3.1. *For $s, t > 0$ define*

$$(\gamma B)^{s,t}(z) := \frac{s^{-z}}{2\pi i} \int_{\Re w = c} s^w \Gamma(z-w) B(w, t-w)\, dw,$$

where $0 < c < \min\{\Re z, t\}$. Then $(\gamma B)^{s,t}$ does not depend on c, and it is a double-parameter continuous semigroup in $\mathcal{U}$; that is,

$$(\gamma B)^{s+\sigma, t+\tau} = (\gamma B)^{s,t} * (\gamma B)^{\sigma,\tau}$$

for all $s, \sigma, t, \tau > 0$. Moreover, its Gelfand transform is

$$\left[(\gamma B)^{s,t}\right]^{\wedge}(\lambda) = e^{-s\lambda}(1+\lambda)^{-t}, \quad \lambda > 0.$$

From here, the function $(\gamma B)^{s,t}$, for fixed t, induces an Appell family in the parameter s:

$$(\gamma B)^{s+\sigma,t} = \gamma^s * (\gamma B)^{\sigma,t} \quad (s, \sigma > 0).$$

Proof. It can be readily seen that $(\gamma B)^{s,t} \in \mathcal{U}$ for all $s, t > 0$. Moreover, for $\lambda > 0$, and b, c, t such that $t, b > c > 0$, we have

$$\begin{aligned}
\left[(\gamma B)^{s,t}\right]^{\wedge}(\lambda) &= \iint_{\Re w = c, \Re z = b} \lambda^{-z} s^{w-z} \Gamma(z-w) B(w, t-w) \frac{dz\, dw}{(2\pi i)^2} \\
&= \int_{\Re z = b-a} (\lambda s)^{-z} \Gamma(z) \frac{dz}{2\pi i} \int_{\Re w = c} \lambda^{-w} B(w, t-w) \frac{dw}{2\pi i} \\
&= e^{-s\lambda}(1+\lambda)^{-t}
\end{aligned}$$

by (2.6) and (3.2). The above identity shows also that $(\gamma B)^{s,t}$ is independent of c, $0 < c < t$. The continuity in s and t follows readily.

Finally, the equality which gives the Gelfand transform implies immediately that $(\gamma B)^{s+\sigma,t} = \gamma^s * (\gamma B)^{\sigma,t}$ for all $s, \sigma > 0$ and then $(\gamma B)^{s,t}$ is an Appell family in the parameter s. $\square$

Corollary 3.2. *The family $((\gamma B)^{t,t})_{t>0}$ is a continuous semigroup in $\mathcal{U}$ with symbol $[(\gamma B)^{t,t}]^{\wedge}(\lambda) = e^{-t\lambda}(1+\lambda)^{-t}$, for every $\lambda > 0$ and $t > 0$.*

Recall that the so-called Charlier polynomials $C_n(t; s)$ are defined as the coefficients of the power series

$$e^x\left(1 - s^{-1}x\right)^t = \sum_{n=0}^{\infty} \frac{C_n(t,s)}{n!} x^n;$$

see [8, p. 177]. Thus we have

$$e^{-x}\left(1+s^{-1}x\right)^{-t}=\sum_{n=0}^{\infty}\frac{C_n(-t;s)}{n!}(-1)^n x^n$$

and then it is natural to regard the Mellin transform, say $C^{s,t}$, of the function $x \mapsto e^{-x}(1+s^{-1}x)^{-t}$ as the continuous counterpart of the Charlier polynomials (see Introduction). In fact, $C^{s,t}$ is a slight modification of the function $(\gamma B)^{s,t}$:

For $z \in \mathbb{C}$ and $s, t > 0$,

$$C^{s,t}(z)=\int_0^\infty x^{z-1}e^{-x}\left(1+s^{-1}x\right)^{-t}dx=s^z\int_0^\infty \lambda^{z-1}e^{-s\lambda}(1+\lambda)^{-t}d\lambda=s^z(\gamma B)^{s,t}(z).$$

We are tempted to call

$$C^{s,t}(z):=\frac{1}{2\pi i}\int_{\Re w=c} s^w\Gamma(z-w)B(w,t-w)\,dw,\quad 0<c<\Re z,$$

the *Charlier function*.

Gegenbauer function. Let $C_z^\nu(\xi)$ be the Gegenbauer function given for $\xi, z, \nu \in \mathbb{C}$ by

$$C_z^\nu(\xi)=\frac{\Gamma(2\nu+z)\Gamma(\frac{1}{2}+\nu)}{\Gamma(2\nu)\Gamma(z+1)}\left[\left(\xi^2-1\right)/4\right]^{\frac{1}{4}-\frac{\nu}{2}}P_{z+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\xi)$$

where $P_{z+\nu-\frac{1}{2}}^{\frac{1}{2}-\nu}$ is the generalized Legendre function, see [6, p. 66]. Then we have

$$\left(1+2x\xi+x^2\right)^{-\nu}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-z}\Gamma(z)\Gamma(1-z)C_{-z}^\nu(\xi)\,dz, \tag{3.3}$$

for all $x, \xi > 0$, $0 < c < 1$ and $\Re\nu > c/2$ [6, p. 67].

Put, for $\xi, s, t > 0$,

$$(bC)^{s,t}(z,\xi):=\int_0^\infty x^{z-1}\left(1+2x\xi+x^2\right)^{-t}e^{-sx}\,dx\quad\left(z\in\mathbb{C}^+\right).$$

Integrating by parts twice in the above integral it is readily seen that $\sup_{\Re z>0}|(bC)^{s,t}(z,\xi)| < \infty$, and therefore $(bC)^{s,t}(\cdot,\xi) \in \mathcal{U}$. Then, by (3.3) and keeping in mind that $\gamma^s(z)=\int_0^\infty x^{z-1}e^{-sx}\,dx$, we obtain that

$$(bC)^{s,t}(z,\xi):=\frac{s^{-z}}{2\pi i}\int_{\Re w=c}s^w\Gamma(z-w)\Gamma(w)\Gamma(1-w)C_{-w}^t(\xi)\,dw$$

for every $z \in \mathbb{C}^+$, where $0 < c < \min\{1, \Re z\}$.

In conclusion, $z \mapsto (bC)^{s,t}(z,\xi)$ is, for fixed ξ, a continuous double-parametrized semigroup in $\mathcal{U}$ with symbol

$$\left[(bC)^{s,t}(\cdot,\xi)\right]^\wedge(\lambda)=e^{-s\lambda}\left(1+2\lambda\xi+\lambda^2\right)^{-t},\quad \lambda>0.$$

In particular, for fixed ξ and t, $z \mapsto (bC)^{s,t}(z,\xi)$ is an Appell function in $\mathcal{U}$. Note that taking $\xi = 1$ we have $C_{-z}^t(1) = B(z, 2t-z)$ (compare (3.3) with (3.2)).

Abel function. Let $a > 0$. The Abel polynomials $a_n(t)$ are defined as the coefficients of the generating function

$$e^{tf^{-1}(x)}=\sum_{n=0}^\infty\frac{a_n(t)}{n!}x^n$$

where f^{-1} is the inverse function of the bijective function f defined by

$$f: y\mapsto ye^{ay},\quad [0,\infty)\to[0,\infty);$$

see [11, p. 163]. The explicit expression of $a_n(t)$ is

$$a_n(t)=t(t-an)^{n-1}.$$

Since

$$e^{-tf^{-1}(x)} = \sum_{n=0}^{\infty} t(t+an)^{n-1} \frac{(-1)^n}{n!} x^n$$

we look for $A_t(z)$ such that

$$e^{-tf^{-1}(x)} = \frac{1}{2\pi i} \int_{\Re z = c} x^{-z} \Gamma(z) A_t(z)\, dz$$

with $t > 0$ and $c < t/a$ (see the Introduction). So it must be

$$\Gamma(z)A_t(z) := \int_0^\infty x^{z-1} e^{-tf^{-1}(x)}\, dx = \int_0^\infty (1+ay) y^{z-1} e^{-(t-az)y}\, dy = \frac{\Gamma(z)}{(t-az)^z} + a\frac{\Gamma(z+1)}{(t-az)^{z+1}} = \frac{t\Gamma(z)}{(t-az)^{z+1}}$$

and therefore $A_t(z) = t(t-az)^{-(z+1)}$, $0 < \Re z < t/a$. Thus we define the function $(\gamma A)^{s,t}$ given, for $z \in \mathbb{C}^+$, by

$$(\gamma A)^{s,t}(z) = \frac{t}{2\pi i} \int_{\Re w = c} s^{(z-w)} \Gamma(z-w)\Gamma(w)(t-aw)^{-(w+1)}\, dw,$$

with $c < \min\{t/a, \Re z\}$. Integrating by parts twice in the integral $\int_0^\infty \lambda^{z-1} e^{-s\lambda} e^{-tf^{-1}(\lambda)}\, d\lambda$ one gets that the function $(\gamma A)^{s,t}$ belongs to $\mathcal{U}$ with symbol

$$\left[(\gamma A)^{s,t}\right]^\wedge(\lambda) = e^{-s\lambda} e^{-tf^{-1}(\lambda)}, \quad \lambda > 0.$$

This shows that $z \mapsto (\gamma A)^{s,t}(z)$ is an Appell family in $\mathcal{U}$.

Similar assertions can be stated for polynomial sequences whose coefficients are the Stirling numbers of the first and second kind, see [11, pp. 162, 163].

Laguerre function. For $\alpha > -1$, the Laguerre polynomials $L_n^{(\alpha)}(t)$ are defined by

$$L_n^{(\alpha)}(t) = \frac{e^t t^{-\alpha}}{n!} \frac{d^n}{dt^n}\left(e^{-t} t^{n+\alpha}\right), \quad n \in \mathbb{N},\ t \in \mathbb{R},$$

and its generating function is

$$(1-x)^{-(\alpha+1)} \exp\left(-t\frac{x}{1-x}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(t) x^n, \quad |x| < 1,\ t \in \mathbb{R}; \tag{3.4}$$

see [2, p. 283].

Replacing x with $-x$ and putting $s = \alpha + 1$ in (3.4) one encounters that the continuous version of $(-1)^n L_n^{s-1}(t)$ is

$$\int_0^\infty x^{z-1}(1+x)^{-s} \exp\left(t\frac{x}{1+x}\right) dx = \int_0^1 y^{z-1}(1-y)^{s-z-1} e^{ty}\, dy = B(z, s-z)\, {}_1F_1(z, s, t)$$

where ${}_1F_1(z, s, t)$ is the Kummer function [9, p. 274]. Note that this example includes, for $t = 0$, the situation referred to in Proposition 3.1 and Corollary 3.2. For the present case $t > 0$, we leave the corresponding statements to the reader.

On the other hand, for $s \geqslant 0$, $t > 0$ and $z \in \mathbb{C}^+$ one has

$$\int_0^1 x^{z-1}(1-x)^{-s} \exp\left(-t\frac{x}{1-x}\right) dx = \int_0^\infty \frac{y^{z-1}}{(1+y)^{s+z+1}} e^{-ty}\, dy = (\gamma B)^{t,s+z}(z),$$

see Proposition 3.1. This implies that the function $z \mapsto (\gamma B)^{t,s+z}(z)$ is a double-parametrized continuous semigroup in $\mathcal{U}$ with symbol $\lambda \mapsto (1-\lambda)_+^{-s} \exp(-t\frac{\lambda}{1-\lambda})$, $\lambda > 0$. Since $\widehat{\beta^s}(\lambda) = (1-\lambda)_+^{-s}$, $\lambda > 0$, we obtain

$$(\gamma B)^{t,s+z}(z) = \left(w \mapsto (\gamma B)^{t,w}(w)\right) * \beta^s(z), \quad t, s > 0,\ z \in \mathbb{C}^+.$$

Jacobi function. The Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ are defined by

$$P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{2^n n!}(1-t)^{-\alpha}(1+t)^{-\beta} \frac{d^n}{dt^n}\left[(1-t)^{\alpha+n}(1+t)^{\beta+n}\right].$$

These polynomials and the hypergeometric function ${}_3F_1$ are related by the equality

$$\begin{aligned}\frac{(\alpha+1)_n}{n!}\,{}_3F_1(-n,z,n+\alpha+\beta+1;\alpha+1;r)\Gamma(z) &= \int_0^\infty P_n^{(\alpha,\beta)}(1-2rx)x^{z-1}e^{-x}\,dx \\ &= r^{-z}\int_0^\infty P_n^{(\alpha,\beta)}(1-2y)y^{z-1}e^{-y/r}\,dy, \quad r>0;\end{aligned}$$

see [3, p. L38].

This entails that the mapping

$$z \mapsto \frac{(\alpha+1)_n}{n!}t^{-z}\,{}_3F_1\big(-n,z,n+\alpha+\beta+1;\alpha+1;t^{-1}\big)\Gamma(z)$$

is an Appell family in $\mathcal{U}$ with symbol $P_n^{(\alpha,\beta)}(1-2\lambda)e^{-t\lambda}$, $\lambda>0$. Thus one gets, for $z\in\mathbb{C}^+$, $s,t>0$ and $0<c<\Re z$,

$$\begin{aligned}&\left(\frac{t}{s+t}\right)^z {}_3F_1\left(-n,z,n+\alpha+\beta+1;\alpha+1;\frac{1}{s+t}\right) \\ &\quad = \frac{1}{2\pi i}\int_{\Re w=c}\left(\frac{t}{s}\right)^w {}_3F_1\big(-n,w,n+\alpha+\beta+1;\alpha+1;s^{-1}\big)\Gamma(z-w)\,dw.\end{aligned}$$

Remark 3.3. Similar discussions can be done for the Meisner and Hahn polynomials, see [3, pp. L38, L39].

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