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# Continuous Sheffer families I

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ABSTRACT

A convolution locally convex algebra $\mathcal{U}$ of holomorphic functions is introduced as a natural setting where to place special functions, which are continuously indexed counterparts to sequences of the classical orthogonal polynomials arising in the umbral calculus. In this way, such functions become semigroups, or Sheffer-type classes associated with semigroups, in the algebra $\mathcal{U}$. A central role in this approach is to be played by the Gamma function. We also discuss the Hermite and Lerch functions to illustrate the theory.



## 0. Introduction

This paper originates from two basic observations about special functions. Orthogonal sequences of classical polynomials are associated with corresponding systems of special functions which are labeled by continuous parameters. Both discrete and continuous families may be obtained as eigenfunctions of the same differential equation (in each case) in suitable different intervals. The link between those sequences of polynomials and their continuously indexed counterparts is provided by the Mellin transform acting on the generating functions of the polynomial families; see [8, p. 61], for instance.

We are interested in polynomial sequences which appear in the umbral calculus; see [6,14]. Let $\mathcal{S}$ be the algebra of all complex sequences $a = (a(n))_{n=0}^{\infty}$ endowed with the convolution product defined by

$$(a * b)(n) := \sum_{k=0}^{n} a(k)b(n-k), \quad n \geq 0, a, b \in \mathcal{S}.$$

A sequence of polynomials $(p_n(t))_{n=0}^{\infty}$ is said to be of binomial type, or binomial for short, if it satisfies the rule $p_n(s+t) = \sum_{k=0}^{n} \binom{n}{k} p_k(t)p_{n-k}(s)$. If we put $\widetilde{p}_n(t) = \frac{p_n(t)}{n!}$ then we have

$$\widetilde{p}_n(t+s) = \sum_{k=0}^{n} \widetilde{p}_k(t)\widetilde{p}_{n-k}(s),$$

so that one can regard the sequence $\widetilde{p}^t = (\widetilde{p}_n(t))_{n=0}^{\infty}$ as a semigroup in $t > 0$ (or a group if $t$ runs over $\mathbb{R}$) in $\mathcal{S}$, that is, $\widetilde{p}^{s+t} = \widetilde{p}^s * \widetilde{p}^t$ holds for all $s, t > 0$. Alternatively, $\widetilde{p}^t$ is to be identified with the semigroup $\widetilde{p}^t = \sum_{n=0}^{\infty} \widetilde{p}_n(t)X^n$ in the algebra

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of formal power series $\mathbb{C}[\![X]\!]$ endowed with multiplication

$$\left(\sum_{n=0}^{\infty} a_n X^n\right)\left(\sum_{n=0}^{\infty} b_n X^n\right) = \sum_{n=0}^{\infty} c_n X^n,$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all $n \geq 0$.

Thus the umbral calculus may well be considered as the study of semigroups – with polynomial coefficients – and associated families in $\mathbb{C}[\![X]\!]$ (by duality methods) [6,14]. A family $\widetilde{p}_n(t)$ as above will be called here a normalized binomial sequence. Among other families of polynomials naturally related with binomial sequences, the most important seems to be the one formed by Sheffer sequences. Let $(p_n)_n$ be a binomial sequence. A Sheffer (polynomial) sequence for $(p_n)_n$ is another polynomial sequence $(\sigma_n)_n$ such that, for all $n \in \mathbb{N}$,

$$\sigma_n(s+t) = \sum_{k=0}^{n} \binom{n}{k} \sigma_k(s) p_{n-k}(t).$$

Putting $\widetilde{\sigma}_n = \frac{\sigma_n}{n!}, \widetilde{p}_n = \frac{p_n}{n!}$, one has

$$\widetilde{\sigma}_n(s+t) = \sum_{k=0}^{n} \widetilde{\sigma}_k(s)\widetilde{p}_{n-k}(t),$$

that is to say, $\widetilde{\sigma}_{s+t} = \widetilde{\sigma}_s * \widetilde{p}^t$ $(s, t \in \mathbb{R})$, where $\widetilde{\sigma}_s(n) := \widetilde{\sigma}_n(s)$. When $(p_n(t))_n = (t^n)_n$, a family $(\sigma_n)_n$ as above (and, by extension, its normalized $(\widetilde{\sigma}_n)_n$) is called an Appell sequence; see [6, p. 8].

In the present article we introduce a metrizable complete locally convex algebra $\mathcal{U}$ which is formed by holomorphic functions and is endowed with a convolution product. This algebra $\mathcal{U}$ contains Mellin transforms of generating functions of Sheffer sequences, so it is a suitable version of the pre-umbral algebra of complex sequences in a continuous framework. The Gamma and Beta functions are examples of semigroups, and the Hermite and Lerch functions are examples of Sheffer-type families, in $\mathcal{U}$; see Sections 3 and 4 below.

The leading idea to construct the algebra $\mathcal{U}$ is simple. Let us assume for a moment that $F: \mathbb{C} \to \mathbb{C}$ and $f: \mathbb{R} \to \mathbb{C}$ are two functions related by the identity

$$\sum_{n=0}^{\infty} \frac{F(n)}{n!} x^n = f(x), \tag{0.1}$$

so that $f$ is the generating function of the sequence $(F(n))$. Under suitable conditions, which should entail in particular analyticity of $F$ in $\Re z < 0$, we would have

$$\Gamma(z)F(-z) = \mathcal{M}[f(-\cdot)](z) := \int_0^{\infty} f(-x)x^{z-1}\,dx, \quad \Re z > 0,$$

where $\mathcal{M}$ is the Mellin transform, or, conversely,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} F(n)x^n \;(= f(-x)) = \frac{1}{2\pi i}\int_{\Re z = c} \Gamma(z)F(-z)x^{-z}\,dz, \tag{0.2}$$

for every $x, c > 0$.

Our aim is to define a vector space $\mathcal{U}$ where functions $\Gamma(z)F(-z)$ are to be contained. Moreover, in analogy with the convolution $*$ of sequences $(F(n)/n!) \in \mathcal{S}$, the space $\mathcal{U}$ should be endowed with a convolution, also denoted by "$*$", for which the product of functions $(\Gamma(\cdot)\,F(-\cdot) * \Gamma(\cdot)\,G(-\cdot))(z)$ would correspond to pointwise multiplication of functions $f(-x)g(-x)$. For such a convolution it is simple to get semigroups (and Sheffer families) in $\mathcal{U}$. It is enough to take a family $(F_t(z))_{t>0}$ of entire functions in $z$ such that the generating function $f^t$ of $(F_t(n))$ in (0.1) has the form $f^t(x) = e^{t\psi(x)}$ (note that it means that $(F_t(n)/n!)_{t>0}$ is a semigroup in the algebra $\mathcal{S}$). By (0.2), the function $\widetilde{F}^t(z) := \Gamma(z)\,F_t(-z)$, which under mild assumptions on $F_t$ is expected to be in $\mathcal{U}$, is a semigroup, that is, $\widetilde{F}^{s+t} = \widetilde{F}^s * \widetilde{F}^t$.

An important example (for $F_t(z) = t^z$) of the above situation involves the exponential and Gamma functions. From the integral formula for the Gamma function,

$$\Gamma(z) = \int_0^{\infty} x^{z-1}\,e^{-x}\,dx \quad (\Re z > 0),$$

one obtains, for any $c > 0$, the well known converse relation

$$e^{-tx} = \frac{1}{2\pi i}\int_{\Re z = c} t^{-z}\,\Gamma(z)\,x^{-z}\,dz \quad (x > 0, t > 0)$$

(which reflects the above discussion by passing to $-x$ in the identity $\sum_{n=0}^{\infty}(t^n x^n)/n! = e^{tx}$). Hence we have that the function $\gamma^t: z \mapsto t^{-z}\Gamma(z)$ should become a semigroup in our virtual algebra $\mathcal{U}$. Since this semigroup is the analog to the fundamental semigroup $n \mapsto t^n/n!$ of the umbral calculus, the algebra $\mathcal{U}$ must be sufficiently large (but not very big, to be useful) to contain the function $\gamma^t$.

In summary, we propose a framework where to find special functions arising as continuous versions of classical polynomial sequences of the umbral calculus. More precisely, we pay attention on the underlined convolution structures. We do not develop here the duality of $\mathcal{U}$. This is a much more involved question than in the discrete case, due to the lack of appropriate bases in $\mathcal{U}$. For relationships between umbral calculus and special functions more general than polynomials, there is the work [15]. The approach followed in [15] is very different from ours. Papers [9,10], where the umbral calculus is considered in an ambient of analytic functions (convergent series), must also be cited.

The paper is divided into four sections. In Section 1 we introduce the Banach space $\mathcal{U}$ giving its definition and first properties. Section 2 is devoted to define a convolution in $\mathcal{U}$ and to prove that for this convolution $\mathcal{U}$ is a locally convex algebra in the sense defined in [1]. We also study the characters of $\mathcal{U}$ and its Gelfand transform. The semigroup $\gamma^t$ in $\mathcal{U}$ defined by the Gamma function $\Gamma$ is discussed in Section 3. It is to be noticed that on the basis of this semigroup one can construct other semigroups in $L^1(\mathbb{R})$ different from those usually considered in the literature. Finally, in Section 4 we show that the Hermite and Lerch functions are examples of Appell families in $\mathcal{U}$, that is, they are Sheffer families with respect to the semigroup $\gamma^t$. In fact, quite a number of classical special functions are Appell families in $\mathcal{U}$. For space reasons, we leave the discussion of such examples to a forthcoming paper.

## 1. A continuous pre-umbral space

Put $\mathbb{C}^+ := \{z \in \mathbb{C} : \Re z > 0\}$. Let $\mathrm{Hol}(\mathbb{C}^+)$ denote the usual topological algebra of holomorphic functions on $\mathbb{C}^+$ endowed with the compact convergence topology $\tau_c$. We define $\mathcal{U}$ as the space of functions $F \in \mathrm{Hol}(\mathbb{C}^+)$ such that

$$\|F\|_{a,b} := \sup_{a\le x\le b} \int_{-\infty}^{\infty} |F(x+iy)|dy < \infty \quad (0 < a \le b),$$

endowed with the locally convex vector space topology generated by the system of seminorms $\{\|\cdot\|_{a,b}\}_{a\le b}$. Note that each seminorm $\|\cdot\|_{a,b}$ is in fact a norm by the continuation principle for analytic functions. Clearly, the system $\{\|\cdot\|_{\frac{1}{n},n}\}_{n=1}^{\infty}$ is a fundamental subfamily of norms which generates the topology of $\mathcal{U}$. Therefore $\mathcal{U}$ is metrizable.

The proof of the following lemma follows the pattern of [11, p. 125]; see also [7]. For $x > 0$, let $\gamma_x$ be the vertical line $x + i\mathbb{R}$ parameterized from $-\infty$ to $\infty$.

**Lemma 1.1** (*Representation Lemma*). *For $F \in \mathcal{U}$ and $z$ such that $0 < a < \Re z < b$, we have*

$$F(z) = \frac{1}{2\pi i}\int_{\gamma_{a,b}} \frac{F(w)}{w-z}dw$$

*where $\gamma_{a,b} = \{-\gamma_a\} \cup \gamma_b$.*

**Proof.** Take $y > 0$ such that $|\Im z| < y$. Put for $x = a, b$ and $u = \pm y$,

$$I(x,y) := \int_{-y}^{y} \frac{F(x+it)}{x+it-z}dt, \qquad J(u) := \int_a^b \frac{F(s+iu)}{s+iu-z}ds.$$

Take now $Y > 2y$. Then

$$\begin{aligned}2\pi i F(z) &= \frac{1}{Y}\int_Y^{2Y} 2\pi i F(z)dy \\ &= \frac{i}{Y}\int_Y^{2Y}(I(b,y)-I(a,y))dy - \frac{1}{Y}\int_Y^{2Y}(J(y)-J(-y))dy,\end{aligned}$$

where we have used in the second equality Cauchy's formula on the closed path formed by the union of segments $[a-iy, b-iy] \cup [b-iy, b+iy] \cup [b+iy, a+iy] \cup [a+iy, a-iy]$.

Since

$$\frac{1}{Y}\int_Y^{2Y} |J(\pm y)|dy \le \frac{2}{Y^2}\int_a^b\int_Y^{2Y} |F(s\pm iy)|dyds \le \frac{4\pi(b-a)}{Y^2}\|F\|_{a,b}$$

the integrals $\frac{1}{Y}\int_Y^{2Y} |J(\pm y)|dy$ vanish as $Y \to +\infty$ and then it follows that

$$2\pi i F(z) = \lim_{Y\to\infty} \frac{i}{Y}\int_Y^{2Y}(I(b,y)-I(a,y))dy.$$

Moreover, for $x = a, b$, we have

$$\lim_{Y\to\infty} \frac{1}{Y}\int_Y^{2Y} I(x,y)dy = \lim_{Y\to\infty}\big(2I(x,2Y) - I(x,Y)\big)$$
$$= \lim_{Y\to\infty}\left\{\left(2\int_{-2Y}^{2Y} - \int_{-Y}^{Y}\right)\frac{F(x+it)}{x+it-z}dt\right\}$$
$$= \int_{-\infty}^{\infty}\frac{F(x+it)}{x+it-z}dt,$$

and the result follows. □

The above representation formula is a key tool in this paper, for it implies almost immediately several structural consequences. It is also possible to obtain a representation formula of Poisson type for functions in $\mathcal{U}$. For horizontal strips, such an explicit formula is given in [3].

**Proposition 1.2.** *For $0 < a < b$ put $\Omega_{a,b} := \{z \in \mathbb{C} : a < \Re z < b\}$.*

(i) *For every $F \in \mathcal{U}$,*

$$\lim_{|y|\to\infty}\left(\sup_{x\in\overline{\Omega}_{a,b}} |F(x+iy)|\right) = 0; \quad \textit{hence} \int_{\Re z=a} F = \int_{\Re z=b} F.$$

(ii) *For every compact subset $K \subseteq \Omega_{a,b}$,*

$$\sup_{z\in K}|F(z)| \le \frac{2}{\mu_K}\|F\|_{a,b} \quad (F \in \mathcal{U}),$$

*where $\mu_K := \min\{d(b+i\mathbb{R},K), d(a+i\mathbb{R},K)\}$. Hence the inclusion map $\mathcal{U} \hookrightarrow \mathrm{Hol}(\mathbb{C}^+)$ is continuous. Moreover, $\mathcal{U}$ is dense in $\mathrm{Hol}(\mathbb{C}^+)$.*

(iii) *The metrizable locally convex space $\mathcal{U}$ is complete.*

**Proof.** (i) Take $a_0, b_0$ with $0 < a_0 < a < b < b_0$. Then, by Lemma 1.1,

$$\sup_{a\le x\le b}|F(x+iy)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{|F(b_0+it)|}{[(b_0-b)^2+(t-y)^2]^{1/2}} + \frac{|F(a_0+it)|}{[(a-a_0)^2+(t-y)^2]^{1/2}}\right)dt$$

and the integral tends to 0 as $|y| \to \infty$ by the dominated convergence theorem. In particular one deduces that $F$ is bounded on every vertical strip. Then a standard application of Cauchy's formula gives us the identity $\int_{\Re z=a} F = \int_{\Re z=b} F$.

(ii) The inequality for the supremum follows readily by Lemma 1.1, similarly to the estimate obtained in part (i) above. Then the continuity of the mapping $\mathcal{U} \hookrightarrow \mathrm{Hol}(\mathbb{C}^+)$ follows automatically.

To show the density of $\mathcal{U}$ in $\mathrm{Hol}(\mathbb{C}^+)$, note first that the family of functions $h_{t,N}(z) := e^{tz^2}((z-1)/(z+1))^N$, $z \in \mathbb{C}^+$, belongs to $\mathcal{U}$ for all $t > 0$ and $N = 0, 1, 2, \ldots$. Set $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$. For $|w| \le r < 1$ we have $|(1+w)/(1-w)| \le |(1+r)/(1-r)|$ and therefore $\lim_{t\to 0^+}\exp\left[t\left(\frac{1+w}{1-w}\right)^2\right] = 1$ in $\mathrm{Hol}(\mathbb{D})$ for $\tau_c$. Then, since the analytic polynomials are dense in $\mathrm{Hol}(\mathbb{D})$ we have that the span of the family of functions $\left\{\exp\left[t\left(\frac{1+w}{1-w}\right)^2\right]\cdot w^N : t > 0, N = 0, 1, 2, \ldots\right\}$ is dense in $\mathrm{Hol}(\mathbb{D})$. So, through the isomorphism between $\mathrm{Hol}(\mathbb{D})$ and $\mathrm{Hol}(\mathbb{C}^+)$ induced by the Cayley transform $z = (1+w)(1-w)^{-1}$ we obtain that $\mathrm{Hol}(\mathbb{C}^+) = \overline{\mathrm{span}}\left\{h_{t,N} : t > 0, N = 0, 1, 2, \ldots\right\}$. This implies that $\mathcal{U}$ is dense in $\mathrm{Hol}(\mathbb{C}^+)$ as well.

(iii) Let $a, b$ such that $0 < a \le b$. We denote by $\ell^\infty([a,b], L^1(\mathbb{R}))$ the Banach space of all families $g = (g_x)_{x\in[a,b]}$ such that $g_x \in L^1(\mathbb{R})$ for all $x \in [a,b]$ and $\|g\|_{L^1,\infty} := \sup_{x\in[a,b]}\|g_x\|_{L^1(\mathbb{R})} < \infty$.

Let $(F_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{U}$. By part (ii) of this proposition $(F_n)_n$ is also a Cauchy sequence in $\mathrm{Hol}(\mathbb{C}^+)$. Since $\mathrm{Hol}(\mathbb{C}^+)$ is complete for $\tau_c$, there exists $F \in \mathrm{Hol}(\mathbb{C}^+)$ such that $\lim_{n\to\infty} F_n = F$ in $\tau_c$ on $\mathbb{C}^+$. On the other hand, the fact that $\lim_{m,n}\|F_m - F_n\|_{a,b} = 0$ means that $F_n = (F_n(x+i\cdot))_{x\in[a,b]}$ is a Cauchy sequence in $\ell^\infty([a,b], L^1(\mathbb{R}))$, for all $0 < a \le b$. Then there exists $g \in \ell^\infty([a,b], L^1(\mathbb{R}))$ such that $\lim_{n\to\infty} F_n = g$ in $\|\cdot\|_{L^1,\infty}$. For fixed $x$ in $[a,b]$, there is a subsequence $F_{n_k}$ of $F_n$ such that $\lim_{k\to\infty} F_{n_k}(x+iy) = g_x(y)$ for every $y \in \mathbb{R}$ a.e. Since $\lim_n F_n = F$ in $\mathrm{Hol}(\mathbb{C}^+)$ we obtain that $g_x(y) = F(x+iy)$ for every $x, y \in \mathbb{R}$, $x > 0$. Finally, we notice again that $F = g = \lim_n F_n$ in $\ell^\infty([a,b], L^1(\mathbb{R}))$ means that $F = \lim_n F_n$ in $\|\cdot\|_{a,b}$. Since this holds true for all $a, b$ such that $0 < a \le b$ we have that $F = \lim_n F_n$ on $\mathcal{U}$. Thus $\mathcal{U}$ is complete. □

We finish this section with a result about density by translates. Put $\mathcal{U}_\tau := \{G \in \mathcal{U} : \ G = F(\cdot + z); F \in \mathcal{U}, z \in \mathbb{C}^+\}$. Clearly, $\mathcal{U}_\tau$ is a vector subspace of $\mathcal{U}$, the elements of $\mathcal{U}_\tau$ are functions defined in particular on $\overline{\mathbb{C}^+} = \{z \in \mathbb{C} : \Re z \ge 0\}$ and their restrictions on $i\mathbb{R}$ are integrable on $i\mathbb{R}$.

**Proposition 1.3.** *$\mathcal{U}_\tau$ is dense in $\mathcal{U}$.*

**Proof.** Take $a, b, \epsilon > 0$ such that $a \leq b$ and $\epsilon < b/2$. For $x \in [a, b]$ and $y, u \in \mathbb{R}$ we have $|w - x - \epsilon - iy|^2 |w - x - iy|^2 \geq (a^2/4) + (u - y)^2$, if $w = (a/2) + iu$; and $|w - x - \epsilon - iu|^2 \geq (b^2/4) + (u - y)^2$, $|w - x - iy|^2 \geq b^2 + (u - y)^2$, if $w = 2b + iu$.

On the other hand, by Lemma 1.1 we have

$$F(x + \epsilon + iy) - F(x + iy) = \int_{\gamma_{\frac{a}{2},2b}} \frac{\epsilon F(w)}{(w - x - \epsilon - iy)(w - x - iy)} \frac{dw}{2\pi i},$$

and therefore, by the preceding estimates,

$$\int_{-\infty}^{\infty} |F(x + \epsilon + iy) - F(x + iy)| dy \leq \left( \int_{-\infty}^{\infty} \frac{dy}{y^2 + (a^2/4)} \right) \|F\|_{a/2,2b}\, \epsilon = 2\pi \frac{\|F\|_{a/2,2b}}{a} \epsilon.$$

Hence $\lim_{\epsilon \to 0^+} \|F(\cdot + \epsilon) - F\|_{a,b} = 0$. Since it holds for all $0 < a \leq b$ we have that $\lim_{\epsilon \to 0^+} F(\cdot + \epsilon) = F$ in $\mathcal{U}$. □

## 2. Convolution and characters

As it has been explained in the Introduction, one needs to define a convolution product in $\mathcal{U}$ which corresponds to the pointwise product of inverse Mellin transforms of the elements of $\mathcal{U}$.

**Lemma 2.1.** *Let $F, G \in \mathcal{U}$ and $z \in \mathbb{C}^+$. Then the integral*

$$\int_{\Re w = c} F(z - w) G(w) dw$$

*exists for, and it is independent of $c$ such that $0 < c < \Re z$.*

**Proof.** The existence of the integral is a consequence of the fact that $G$ is bounded on $\Re w = c$ and $F \in \mathcal{U}$. The independence of $c$ follows by the Cauchy formula as in former arguments. □

By Lemma 2.1 the following definition is well posed.

**Definition 2.2.** Let $z \in \mathbb{C}^+$ and let $F, G \in \mathcal{U}$. Put

$$F * G(z) := \frac{1}{2\pi i} \int_{\Re w = c} F(z - w) G(w) dw$$

for any $c$ such that $0 < c < \Re z$. Obviously, $F * G = G * F$.

**Proposition 2.3.** *Let $\alpha, a, b$ be such that $0 < \alpha < a < b$. Then, for every $F, G \in \mathcal{U}$,*

$$\|F * G\|_{a,b} \leq \|F\|_{a-\alpha, b-\alpha} \|G\|_{\alpha,\alpha}.$$

*So in particular $F * G \in \mathcal{U}$.*

**Proof.** For any $F, G \in \mathcal{U}$ we have that $F * G$ is continuous in $\mathbb{C}^+$ (by the dominated convergence theorem). Furthermore, using Fubini's theorem one gets $\int_{\partial\triangle} F * G(z) dz = 0$ for the boundary $\partial\triangle$ of any closed triangle in $\mathbb{C}^+$. Hence by Morera's theorem $F * G$ is holomorphic in $\mathbb{C}^+$. Finally, for $0 < \alpha < a \leq x \leq b$ and $y \in \mathbb{R}$,

$$\begin{aligned}\int_{-\infty}^{\infty} |F * G(x + iy)| dy &\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(x - \alpha + i(y - u))| \cdot |G(\alpha + iu)| du dy \\ &= \left( \int_{-\infty}^{\infty} |G(\alpha + iu)| du \right) \left( \int_{-\infty}^{\infty} |F(x - \alpha + ir)| dr \right).\end{aligned}$$

Thus taking the supremum for $x$ running over $[a, b]$, in both members of the inequality, we obtain $\|F * G\|_{a,b} \leq \|F\|_{a-\alpha,b-\alpha} \|G\|_{\alpha,\alpha}$, as we wanted to show. □

The inequality obtained in Proposition 2.3(ii) means that $\mathcal{U}$ is a (complete and metrizable) *locally convex algebra* in the sense defined in [1]. Next, we describe the continuous characters and Gelfand transform of $\mathcal{U}$. A *character* $\varphi$ of $\mathcal{U}$ is by definition a complex algebra homomorphism $\varphi \colon \mathcal{U} \longrightarrow \mathbb{T}$.

**Theorem 2.4.** *All non-zero continuous characters of $\mathcal{U}$ are of the form $\varphi = \varphi_\lambda$, $\lambda > 0$, where $\varphi_\lambda(F) := \frac{1}{2\pi i} \int_{\Re z = c} \lambda^{-z} F(z) dz$ ($F \in \mathcal{U}$) is independent of $c > 0$.*

**Proof.** First we show that the functional $\varphi_\lambda$ is a continuous character of $\mathcal{U}$ for every $\lambda > 0$. Note that for every $F \in \mathcal{U}$ the function $z \to \lambda^{-z}F(z)$ is also in $\mathcal{U}$. Hence $\varphi_\lambda$ is well defined and independent of $c > 0$. Moreover, using Fubini's theorem, we easily obtain that $\varphi_\lambda(F * G) = \varphi_\lambda(F)\varphi_\lambda(G)$ for every $F, G \in \mathcal{U}$. Thus $\varphi_\lambda$ is a character of $\mathcal{U}$, and $\varphi_\lambda$ is certainly non null because $e^{-\lambda} = \varphi_\lambda(\Gamma)$ [2, p. 85]. Note that $\Gamma \in \mathcal{U}$; see [2, Corollary 1.4.4]. Finally, the continuity of the mapping $\varphi_\lambda : \mathcal{U} \longrightarrow \mathbb{C}$ follows from the bound $|\varphi_\lambda(F)| \le (2\pi)^{-1}\lambda^{-c}\|F\|_{c,c}$ $(F \in \mathcal{U};\ c > 0)$.

Conversely, let $\varphi$ be a non-zero continuous character of $\mathcal{U}$. Then there exists $F \in \mathcal{U}_\tau$ such that $\varphi(F) \neq 0$. For $y \in \mathbb{R}$, put $\delta_y F := F(\cdot - iy)$ and $\widetilde{\varphi}(y) := \varphi(\delta_y F)\varphi(F)^{-1}$. If $G$ is another function in $\mathcal{U}_\tau$ with $\varphi(G) \neq 0$ then $(\delta_y G) * F = G * (\delta_y F)$ and therefore we get $\varphi(\delta_y G)\varphi(G)^{-1} = \varphi(\delta_y F)\varphi(F)^{-1}$. Thus the definition of $\widetilde{\varphi}$ does not depend on the choice of $F$. It is readily seen that $\widetilde{\varphi} : \mathbb{R} \longrightarrow \mathbb{T}$ is a group homomorphism. Further, $\widetilde{\varphi}$ is continuous: similarly to the proof of Proposition 1.3, we have $\|\delta_y F - F\|_{a,b} \le (2\pi|y|/a)\|F\|_{a/2,2b}$ $(0 < a \le b)$, which implies that $\lim_{y\to 0}\delta_y F = F$ and from here the continuity of $\widetilde{\varphi}$ follows immediately.

It is well known that every continuous group homomorphism from $\mathbb{R}$ into $\mathbb{T}$ must be an exponential function; therefore, there exists $\xi \in \mathbb{C}$ such that $\widetilde{\varphi}(y) = e^{\xi y}$ $(y \in \mathbb{R})$. Now, since $\varphi$ is continuous, we have

$$|\widetilde{\varphi}(y)| \le |\varphi(F)|^{-1} C \max_{1\le j\le m} \|\delta_y F\|_{a_j,b_j} = C|\varphi(F)|^{-1} \max_{1\le j\le m} \|F\|_{a_j,b_j}$$

for some $0 < a_j \le b_j$ $(j = 1, 2, \ldots, m)$, constant $C$, and all $y \in \mathbb{R}$. Hence $\xi = ic$ with $c \in \mathbb{R}$. Set $t = e^{-c} > 0$, so that $\widetilde{\varphi}(y) = t^{-iy}$ $(y \in \mathbb{R})$.

Let $G$ be an arbitrary element of $\mathcal{U}_\tau$. It is straightforward to see that

$$F * G = \frac{1}{2\pi i}\int_{\Re w = 0} G(w)\delta_{-iw}F\,dw$$

in the topology of the algebra $\mathcal{U}$ for every $F \in \mathcal{U}$. Then, since $\varphi$ is continuous,

$$\begin{aligned}\varphi(F)\varphi(G) &= \varphi(F * G) = \frac{1}{2\pi i}\int_{\Re w=0} G(w)\varphi(\delta_{-iw}F)\,dw \\ &= \left(\frac{1}{2\pi i}\int_{\Re w=0} G(w)\widetilde{\varphi}(-iw)\,dw\right)\varphi(F),\end{aligned}$$

whence one obtains that

$$\varphi(G) = \frac{1}{2\pi i}\int_{\Re w=0} G(w)\lambda^{-w}\,dw = \frac{1}{2\pi i}\int_{\Re z=c} G(z)\lambda^{-z}\,dz,$$

for every $c > 0$. In conclusion we have shown that $\varphi = \varphi_\lambda$ on $\mathcal{U}_\tau$. Then, by continuity of $\varphi$ and $\varphi_\lambda$ and the density of $\mathcal{U}_\tau$ in $\mathcal{U}$, we obtain that $\varphi = \varphi_\lambda$ on $\mathcal{U}$, as it was claimed. $\square$

Put $\mathbb{R}^+ := (0, \infty)$, and let $C_0(\mathbb{R}^+; \lambda^c)$ denote the space of continuous functions $f : \mathbb{R}^+ \to \mathbb{C}$ such that

$$\lim_{\lambda\to 0^+} \lambda^c f(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda\to\infty} \lambda^c f(\lambda) = 0 \quad (c > 0).$$

Endowed with the pointwise multiplication and the locally convex topology defined by the family of seminorms

$$\|f\|_{\infty,c} := \sup_{\lambda>0} |\lambda^c f(\lambda)|, \quad f \in C_0(\mathbb{R}^+; \lambda^c),$$

the space $C_0(\mathbb{R}^+; \lambda^c)$ is a metrizable complete locally convex algebra.

**Proposition 2.5.** *For $F \in \mathcal{U}$ and $\lambda > 0$, set $\widehat{F}(\lambda) := \varphi_\lambda(F)$ and $\mathcal{G}(F) := \widehat{F}$. Then $\widehat{F} \in C_0(\mathbb{R}^+; \lambda^c)$ for every $F \in \mathcal{U}$. Moreover, the mapping $\mathcal{G} : \mathcal{U} \to C_0(\mathbb{R}^+; \lambda^c)$ is an injective and continuous algebra homomorphism with dense range.*

**Proof.** (i) Let $F \in \mathcal{U}$. The continuity of $\widehat{F}$ is an obvious consequence of the dominated convergence theorem. Regarding its behavior at 0 and infinity, it is enough to notice that for any $c > 0$ we have

$$\lambda^c \widehat{F}(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \lambda^{-iy} F(c + iy)\,dy \quad (\lambda > 0),$$

where $F(c + i\cdot) \in L^1(\mathbb{R})$.

(ii) Clearly, $\mathcal{G}$ is linear and the homomorphism property follows by the identity $\varphi_\lambda(F * G) = \varphi_\lambda(F)\varphi_\lambda(G)$, for every $F, G \in \mathcal{U}$ and $\lambda > 0$. The injectivity of $\mathcal{G}$ follows from the identity

$$\widehat{F}(\lambda) = \frac{\lambda^{-c}}{2\pi}(\mathcal{F}F_c)(\log\lambda), \quad \forall c, \lambda > 0,$$

and the injectivity of $\mathcal{F}$, where $F_c = F(c + i\cdot) \in L^1(\mathbb{R})$ and $\mathcal{F}$ is the Fourier transform. Finally, note that for all $\lambda > 0$ we have $|\lambda^c \widehat{F}(\lambda)| \le \|F\|_{c,c}$ for any $c > 0$. This implies the continuity of $\mathcal{G}$.

It remains to show the density of $\mathcal{G}(\mathcal{U})$ in $C_0(\mathbb{R}^+, \lambda^c)$. By straightforward methods, one proves that the space of the test functions $C_c^\infty(0,\infty)$ is dense in $C_0(\mathbb{R}^+, \lambda^c)$. For $f \in C_c^\infty(0,\infty)$, set $F(z) = \int_0^\infty f(\lambda)\lambda^{z-1}d\lambda$ $(z \in \mathbb{C}^+)$. Integrating by parts twice one obtains that $|F(z)| \le C(|z \| z+1|)^{-1}$ for each $z \in \mathbb{C}^+$, which implies that $F \in \mathcal{U}$. Clearly, $\mathcal{G}(F) = f$. Since it holds for every $f \in C_c^\infty(0,\infty)$, we have that $\mathcal{G}(\mathcal{U})$ is dense in $C_0(\mathbb{R}^+, \lambda^c)$. □

We call $\mathcal{G}$ the *Gelfand transform* of $\mathcal{U}$.

## 3. Semigroups in the algebra $\mathcal{U}$

The Gamma function $\Gamma$ and Beta function $B$ are essential in the study of special functions [2, p. xiv]. Both functions yield corresponding semigroups in $\mathcal{U}$.

*Gamma semigroups families.* For $t, \alpha > 0$ let $\gamma_\alpha^t$ be the function in $\mathcal{U}$ defined by

$$\gamma_\alpha^t(z) := \frac{1}{\alpha} t^{-z/\alpha}\Gamma(z/\alpha), \quad z \in \mathbb{C}^+. \tag{3.1}$$

**Proposition 3.1.** *The family $(\gamma_\alpha^t)_{t>0}$ is a semigroup in $\mathcal{U}$ with Gelfand transform given by*

$$\widehat{\gamma_\alpha^t}(\lambda) = e^{-t\lambda^\alpha} \quad (\lambda > 0). \tag{3.2}$$

*Moreover, the vector-valued mapping $t \mapsto \gamma_\alpha^t$, $(0,\infty) \to \mathcal{U}$ is a $C^\infty$ function with derivatives*

$$\frac{d^n}{dt^n}\gamma_\alpha^t(z) = (-1)^n \gamma_\alpha^t(z+n\alpha), \quad z \in \mathbb{C}^+, n \ge 1.$$

**Proof.** The integral formula $\Gamma(z) = \int_0^\infty \lambda^{z-1}e^{-\lambda}d\lambda$ for the Gamma function $\Gamma$ gives rise to the identity

$$\int_0^\infty x^{z-1}e^{-tx^\alpha}dx = \frac{1}{\alpha}t^{-z/\alpha}\Gamma(z/\alpha),$$

for $z \in \mathbb{C}^+$, $t, \alpha > 0$. In turn, the last equality can be expressed in inverse form by

$$e^{-tx^\alpha} = \frac{1}{2\pi i}\int_{\Re z = c} x^{-z}\gamma_\alpha^t(z)dz, \quad x, t > 0,$$

for any $c > 0$. This gives us the first part of the statement.

The $C^\infty$ differentiability of $t \mapsto \gamma_\alpha^t$ as well as the formula for the derivatives is a consequence of the dominated convergence theorem. □

We call $(\gamma_\alpha^t)_{t>0}$ the $\alpha$-*Gamma semigroup* in $\mathcal{U}$. Actually, the semigroup property reads

$$(t+s)^{-z/\alpha}\Gamma(z/\alpha) = \frac{t^{-z/\alpha}}{2\pi\alpha i}\int_{\Re w = c}\left(\frac{t}{s}\right)^{w/\alpha}\Gamma\left(\frac{z-w}{\alpha}\right)\Gamma\left(\frac{w}{\alpha}\right)dw,$$

for every $z \in \mathbb{C}^+$ and $t, s, \alpha > 0$.

**Remark 3.2.** Restrictions of $\gamma_\alpha^t$ on vertical lines of $\mathbb{C}^+$ provide us with examples of semigroups on $L^1(\mathbb{R})$ (for other semigroups in $L^1(\mathbb{R})$ involving special functions; see [4]). Define for $a, t > 0$ and $y \in \mathbb{R}$,

$$\gamma_\alpha^{a,t}(y) = \frac{1}{2\pi\alpha}t^{-a/\alpha}t^{-iy/\alpha}\Gamma\left(\frac{a+iy}{\alpha}\right). \tag{3.3}$$

We have that $\gamma_\alpha^{a,t} \in L^1(\mathbb{R})$ with Fourier transform

$$\begin{aligned}\mathcal{F}(\gamma_a^{a,t})(\xi) &= \frac{1}{2\pi\alpha}\int_{-\infty}^\infty t^{-(a+iy)/\alpha}\Gamma\left(\frac{a+iy}{\alpha}\right)e^{-i\xi y}dy \\ &= \frac{e^{a\xi}}{2\pi i\alpha}\int_{\Re z = a}(e^\xi)^{-z}t^{-z/\alpha}\Gamma(z/\alpha)dz = e^{a\xi}\,\widehat{\gamma_\alpha^t}(\lambda) \\ &= \exp(a\xi)\exp(-te^{\alpha\xi}) = \exp(a\xi - te^{\alpha\xi}), \quad \xi \in \mathbb{R}.\end{aligned}$$

Hence it follows that $\gamma_\alpha^{a,t} * \gamma_\alpha^{b,s} = \gamma_\alpha^{a+b,t+s}$ for every $a, b, t, s > 0$. This means that $(\gamma_\alpha^{a,t})_{a,t>0}$ is a (continuous) bi-parameter semigroup in $L^1(\mathbb{R})$. Taking $a = t$ in (3.3) and setting

$$\Gamma_\alpha^t(y) := \frac{1}{2\pi\alpha}t^{-(t+iy)/\alpha}\Gamma\left(\frac{t+iy}{\alpha}\right), \quad y \in \mathbb{R}, \tag{3.4}$$

one obtains that the family $(\Gamma_\alpha^t)_{t>0}$ is a continuous semigroup in $L^1(\mathbb{R})$ with Fourier transform $(\mathcal{F}\Gamma_\alpha^t)(\xi) = \exp(-t(e^{\alpha\xi} - \xi))$, $\xi \in \mathbb{R}$.

*Beta semigroup.* It is known that

$$(1-x)_+^{s-1} = \frac{\Gamma(s)}{2\pi i}\int_{\Re z=c} x^{-z}\frac{\Gamma(z)}{\Gamma(z+s)}dz \quad (x, s, c > 0); \tag{3.5}$$

see [2, p. 85], for instance. This formula gives rise to the following proposition. Recall that the Beta function $B$ is given by $B(z,w) = \Gamma(z+w)^{-1}\Gamma(z)\Gamma(w)$ for $z, w, z+w \notin \{0, -n : n \in \mathbb{N}\}$. Let us define

$$\beta^t(z) := B(z, t+1), \quad z \in \mathbb{C}^+, t > 0.$$

**Proposition 3.3.** *The family $(\beta^t)_{t>0}$ is a continuous semigroup in $\mathcal{U}$ with Gelfand transform $\widehat{\beta^t}(\lambda) = (1-\lambda)_+^t$, $\lambda > 0$.*

**Proof.** Take $t > 0$. Putting $s = t+1$ in (3.5) one obtains

$$(1-x)_+^t = \frac{1}{2\pi i}\int_{\Re z=c} x^{-z}B(z, t+1)dz \quad (x > 0). \tag{3.6}$$

From the estimate of the Gamma function on vertical lines (see [2, p. 21]), one has that the latter integral is absolutely convergent and, moreover, that $\beta^t \in \mathcal{U}$ for all $t > 0$. The continuity in $t$ follows then readily and we get that $(\beta^t)_{t>0}$ is a continuous semigroup in $\mathcal{U}$. Finally, from (3.6) is evident that $\widehat{\beta^t}(\lambda) = (1-\lambda)_+^t$ for every $\lambda, t > 0$. □

**Remark 3.4.** There are indeed many semigroups in $\mathcal{U}$, other than $\gamma^t$ or $\beta^t$. Since the Gelfand transform $\mathcal{G}$ of Proposition 2.5 is an injective algebra homomorphism with the inverse Mellin transform $\mathcal{M}$, it is enough to choose Borel functions $\psi: (0,\infty) \to (0,\infty)$ making the integral $\Psi^t(z) := \int_0^\infty \lambda^{z-1}e^{t\psi(\lambda)}d\lambda$ convergent for $z \in \mathbb{C}^+$ and such that $\Psi^t \in \mathcal{U}$ for every $t > 0$, to get a semigroup $(F^t)_{t>0}$ in $\mathcal{U}$ (see the Introduction). So is the case for the Gaussian function $G^t(z) := e^{z^2/4t}/\sqrt{4\pi t}$ ($\Re z \geq 0$; $t > 0$), for which $\psi(\lambda) = -\log(\lambda)^2$. This and other examples will be discussed in a forthcoming paper.

## 4. Sheffer families for $\mathcal{U}$

In this section we introduce the notion of Sheffer families associated with semigroups in the algebra $\mathcal{U}$.

**Definition 4.1.** A family $(S_t)_{t>0}$ of functions $S_t: \mathbb{C}^+ \to \mathbb{C}$ is called a *Sheffer* family in $\mathcal{U}$ if there is a semigroup $(F^t)_{t>0}$ in $\mathcal{U}$ such that, in the notation of the introduction,

$$S_{s+t} = F^s * S_t, \quad s, t > 0.$$

Evidently, each semigroup in $\mathcal{U}$ is a Sheffer family. Another important subclass of Sheffer families is the following one: $(S)_{t>0}$ is said to be an *Appell* family if it is a Sheffer family for the Gamma semigroup $(\gamma^t)_{t>0}$ given by $\gamma^t := \gamma_1^t$, $t > 0$. There are a number of special functions which are Appell families in $\mathcal{U}$. Here we give two examples.

*Hermite function.* For $n \in \mathbb{N}$, the Hermite polynomial $H_n$ is defined by

$$H_n(t) := (-1)^n e^{t^2}\frac{d^n}{dt^n}e^{-t^2}$$

with generating function

$$e^{-x^2}e^{2tx} = \sum_{n=0}^{\infty} H_n(t)\frac{x^n}{n!}; \tag{4.1}$$

see [2, p. 278], for instance. Among other properties, one has

$$\frac{H_n(s+t)}{n!\,2^n} = \sum_{k=0}^{n}\frac{H_k(s)}{k!\,2^k}\frac{t^{n-k}}{(n-k)!} \quad (s, t > 0; n \in \mathbb{N}); \tag{4.2}$$

see [6, p. 36], so $(H_n(t)/(n!\,2^n))_{n=1}^{\infty}$ is an Appell sequence.

It is well known that the sequence $(H_n)_{n=1}^{\infty}$ can be interpolated to an entire function $H_{-z}$ which for $z \in \mathbb{C}^+$, and $t > 0$, is given by

$$\begin{aligned} H_{-z}(t) &:= \frac{1}{\Gamma(z)}\int_0^\infty x^{z-1}e^{-x^2}e^{-2tx}dx \\ &= \frac{2^{-z}}{\Gamma(z)}\int_0^\infty \lambda^{z-1}e^{-\lambda^2/4}e^{-t\lambda}d\lambda; \end{aligned} \tag{4.3}$$

see [5, p. 347]. Suggested by the Appell relation expressed in (4.2), we define

$$h_t(z) := 2^z \Gamma(z) H_{-z}(t), \quad t > 0.$$

By (4.3) the Gelfand transform of $h_t$ is $\widehat{h_t}(\lambda) = e^{-\lambda^2/4} e^{-t\lambda}$, and so

$$(h_s * \gamma^t)^\wedge(\lambda) = \widehat{h_s}(\lambda)\widehat{\gamma^t}(\lambda) = e^{-\lambda^2/4} e^{-(s+t)\lambda},$$

for every $s, t, \lambda > 0$. This implies that

$$h_{s+t}(z) = 2^z \Gamma(z) H_{-z}(s+t) = (h_s * \gamma^t)(z) \quad (z \in \mathbb{C}^+),$$

whence it follows that $h_t$ is an Appell family in $\mathcal{U}$.

**Remark 4.2.** The subordination formula

$$\frac{H_n(t)}{n!} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{2^n (t+is)^n}{n!} e^{-s^2} ds,$$

see for example [12, p. 254] (note that the Hermite polynomials considered in [12] are given in a slightly different form from here), is interesting because it enables us to express the Appell polynomial $H_n(t)/2^n n!$ as the convolution of the sequence $t^n/n!$ and the sequence

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{i^n s^n}{n!} e^{-s^2} ds = \frac{i^n(1+(-1)^n)}{2\sqrt{\pi} n!} \Gamma\left(\frac{n+1}{2}\right).$$

Analogously, one has the equality

$$H_{-z}(t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2^{-z} (t+is)^{-z} e^{-s^2} ds \quad (z \in \mathbb{C}^+, t > 0), \tag{4.4}$$

which can be proved taking the inverse Mellin transform of the function $z \mapsto (1/\sqrt{\pi}) \int_{-\infty}^{\infty} 2^{-z}(t+is)^{-z} \Gamma(z) e^{-s^2} ds$ and then applying (4.3).

Then, by (4.4),

$$\begin{aligned} 2^z \Gamma(z) H_{-z}(t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (t+is)^{-z} \Gamma(z) e^{-s^2} ds \\ &= \left( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \gamma^{t+is} e^{-s^2} ds \right)(z), \end{aligned}$$

where $\gamma^{t+is}$ is defined by $\gamma^{t+is}(z) := (t+is)^{-z}\Gamma(z)$, $\Re z > 0$. Thus one can write the function $z \mapsto 2^z \Gamma(z) H_{-z}(t)$ as the formal convolution

$$\left( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \gamma^{is} e^{-s^2} ds \right) * \gamma^t \quad (t > 0)$$

where with the formal integral $\int_{-\infty}^{\infty} \gamma^{is} e^{-s^2} ds$ we denote the function

$$\phi(z) := \Gamma(z) \int_{-\infty}^{\infty} e^{-z \log(is)} e^{-s^2} ds \quad (z \in \mathbb{C}^+).$$

Here, log is the branch of the logarithm with argument lying in $[-\pi, \pi)$. This integral is absolutely convergent if $0 < \Re z < 1$ since $\left|e^{-z\log(is)}\right| = |s|^{-\Re z} e^{-\mathrm{sgn}(s)(\Im z)\frac{\pi}{2}}$, $s \neq 0$. For such values of $z$, we have

$$\phi(z) = \Gamma(z)\, 2\cos\left(\frac{\pi z}{2}\right) \int_0^{\infty} s^{-z} e^{-s^2} ds = \cos\left(\frac{\pi z}{2}\right) \Gamma\left(\frac{1-z}{2}\right) \Gamma(z),$$

whence it follows that $\phi$ extends to $\mathbb{C} \setminus \{0, -2, -4, \ldots\}$.

*Lerch function*. For $t \in \mathbb{C} \setminus \{0, -1, \ldots\}$ and $z \in \mathbb{C}$ if $|\alpha| < 1$, or $\Re z > 1$ if $|\alpha| = 1$, the transcendent Lerch function $\Phi$ is defined by

$$\Phi(\alpha, z, t) = \sum_{n=0}^{\infty} \frac{\alpha^n}{(n+t)^z};$$

see for instance [12, p. 32]. Special cases of the transcendent Lerch function are the Lerch zeta function $\Phi(2\pi i w, z, t)$, the Hurwitz zeta function $\Phi(1, z, t)$ and zeta function $\zeta(z) := \Phi(1, z, 1)$, and others.

Let now take $t > 0$ for simplicity. Then, for $\Re z > 0$ if $|\alpha| \le 1$ with $\alpha \ne 1$, or $\Re z > 1$ if $\alpha = 1$, one gets the representation

$$\Phi(\alpha, z, t) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{x^{z-1}e^{-tx}}{1-\alpha e^{-x}} dx, \tag{4.5}$$

just developing $(1-\alpha e^{-x})^{-1}$ in powers of $x$ in the integral; see [12, p. 34]. For $t > 0$ and $|\alpha| \le 1$, let us define the function $\widetilde{\mathcal{B}}_{t,\alpha} : \mathbb{C}^+ \to \mathbb{C}$ by

$$\widetilde{\mathcal{B}}_{t,\alpha}(z) := \Gamma(z+1)\Phi(\alpha, z+1, t), \quad \Re z > 0.$$

Equality (4.5) implies that the function $\widetilde{\mathcal{B}}_{t,\alpha}$ defines an Appell family $(\widetilde{\mathcal{B}}_{t,\alpha})_{t>0}$ in $\mathcal{U}$ such that $\widehat{\widetilde{\mathcal{B}}_{t,\alpha}}(\lambda) = \lambda e^{-t\lambda}(1-\alpha e^{-\lambda})^{-1}$, $\lambda > 0$.

The above facts obey the general pattern that we are considering in the paper: the family $\mathcal{B}_n(t,\alpha)$ of the so-called Apostol–Bernoulli polynomials is defined by

$$\frac{xe^{tx}}{\alpha e^x - 1} = \sum_{n=0}^\infty \mathcal{B}_n(t,\alpha)\frac{x^n}{n!}. \tag{4.6}$$

Then $\mathcal{B}_n(t,\alpha) = -n\Phi(\alpha, 1-n, t)$ on various domains of analytic continuation; see [13, Introduction]. On the other hand, $(\mathcal{B}_n(t,\alpha))_n$ is an Appell sequence, see for instance [13, proof in Theorem 3], and so the normalized sequence $\widetilde{\mathcal{B}}_n(t,\alpha) := \frac{1}{n!}\mathcal{B}_n(t,\alpha)$ satisfies

$$\widetilde{\mathcal{B}}_n(s+t,\alpha) = \sum_{k=0}^\infty \widetilde{\mathcal{B}}_n(s,\alpha)\,\frac{t^{n-k}}{(n-k)!}.$$

Thus in accordance with our general procedure, applying the Mellin transform to the generating function of (4.6) evaluated at $-x$ one recaptures (4.5),

$$\int_0^\infty x^{z-1}\frac{xe^{-tx}}{1-\alpha e^{-x}}dx = \widetilde{\mathcal{B}}_{t,\alpha}(z),$$

for $|\alpha| \le 1$ and $\Re z > 0$, so that $\widetilde{\mathcal{B}}_{t,\alpha}(z)$ can be regarded as the continuous version of the normalized Bernoulli sequence, satisfying the Appell property

$$\widetilde{\mathcal{B}}_{s+t,\alpha} = \widetilde{\mathcal{B}}_{s,\alpha} * \gamma^t \quad s, t > 0.$$

Note that the case $\alpha = 1$ corresponds to the Bernoulli polynomial sequence; see [6, p. 36].

The family of Apostol–Euler polynomials $\mathcal{E}_n(t,\alpha)$ is defined by

$$\frac{2e^{tx}}{\alpha e^x + 1} = \sum_{n=0}^\infty \frac{\mathcal{E}_n(t,\alpha)}{n!}x^n, \quad \alpha \ne -1,$$

and the discussion carried out for the Apostol–Bernoulli polynomials can be done for the Apostol–Euler polynomials in a similar manner.

However, there is an alternative way to deal with Euler polynomials or functions: it is noticed in [13, Lemma 2] that one can transfer results between the Apostol–Bernoulli and Apostol–Euler polynomials using the relation $(n+1)\mathcal{E}_n(t,\alpha) = -2\mathcal{B}_{n+1}(t,-\alpha)$, that is,

$$\widetilde{\mathcal{E}}_n(t,\alpha)(z) = -2\widetilde{\mathcal{B}}_n(t,-\alpha), \quad \alpha \ne -1. \tag{4.7}$$

In our setting, if we put, for $|\alpha| \le 1$ with $\alpha \ne -1$,

$$\widetilde{\mathcal{E}}_{t,\alpha}(z) := \int_0^\infty \frac{2e^{-tx}}{1+\alpha x}\, x^{z-1}\, dx, \quad \Re z > 0,$$

then by simple substitution we get the equality

$$\widetilde{\mathcal{E}}_{t,\alpha}(z+1) = 2\widetilde{\mathcal{B}}_{t,\alpha}(z), \quad \Re z > 0,$$

which is the continuous counterpart to (4.7).

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