ADVANCES IN APPLIED MATHEMATICS 16, 275–296 (1995)

Further Triangles of Seidel–Arnold Type and Continued Fractions Related to Euler and Springer Numbers

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We introduce several new triangles of numbers of Seidel type and pairs of triangles of Arnold type. We show their relations with representations as continued fractions provided for the ordinary generating functions of classical numbers such as Euler numbers, Springer numbers, and Genocchi numbers.

1. INTRODUCTION

Let us define the Euler numbers E_n and the Springer numbers S_n through their exponential generating functions (egf), respectively $E(x)$ and $S(x)$:

$$E(x) = \frac{1}{\cosh x} + \tanh x = \sum_{n\geq 0} (-1)^n \left(E_{2n} \frac{x^{2n}}{(2n)!} + E_{2n+1} \frac{x^{2n+1}}{(2n+1)!} \right)$$

$$= 1 + x - \frac{x^2}{2!} - 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} - 61\frac{x^6}{6!} - 272\frac{x^7}{7!} + \cdots$$

$$S(x) = \frac{\cosh x}{\cosh 2x} + \frac{\sinh x}{\cosh 2x}$$

$$= \sum_{n\geq 0} (-1)^n \left(S_{2n} \frac{x^{2n}}{(2n)!} + S_{2n+1} \frac{x^{2n+1}}{(2n+1)!} \right)$$

$$= 1 + x - 3\frac{x^2}{2!} - 11\frac{x^3}{3!} + 57\frac{x^4}{4!} + 361\frac{x^5}{5!}$$

$$- 2763\frac{x^6}{6!} - 24611\frac{x^7}{7!} + \cdots .$$

* Supported in part by the EEC Programme on Algebraic Combinatorics, while the author revised the paper in Strasbourg.

0196-8858/95 $12.00

The even (resp. odd) part of the Euler numbers is the secant (resp. tangent) numbers. The even (resp. odd) part of the Springer numbers is what Glaisher called the numbers P_n (resp. Q_n). In Glaisher's lengthy papers [G1] [G2], these numbers appear among many other sequences related to Bernoulli numbers. In fact, Euler examined the P_n numbers [E], and they were also more recently studied by Shanks [Sh], but we shall follow Arnold's denomination for the entire sequence as Springer numbers. Springer [Sp] introduced these numbers for a problem about root systems, and recently Arnold [A2] used them in the theory of singularities of smooth functions.

In [A3] Arnold introduces two pairs of triangles of numbers that lead to simple algorithms for computing Springer numbers. In this article we give short and elementary proofs of both pairs of triangles based on Seidel's method [Se], using only their exponential generating functions (egf), not their combinatorial properties.

Then we transpose the method to the ordinary generating functions (ogf) of the same numbers, and give their continued fraction representations. In particular we find a new continued fraction representation for the ogf of Euler numbers, deduced from Arnold's pairs of triangles. In addition we introduce a new pair of triangles for Euler numbers and define the "median Euler numbers."

2. ARNOLD'S PAIRS OF TRIANGLES

Like Arnold, we think that the reader can discover the algorithm governing the following two pairs of triangles (we add arrows to make clear the left–right/right–left reading):

				0										1				
			1		1				←				1		0			
		0		1		2			→			2		3		3		
	11		11		10		8		← →		8		6		3		0	
0		11		22		32		40		40		48		54		57		57

Arnold's pair $L(b)$ and $R(b)$

				1										1				
			2		1				←				1		0			
		0		2		3			→			3		4		4		
	16		16		14		11		← →		11		8		4		0	
0		16		32		46		57		57		68		76		80		80

Arnold's pair $L(\beta)$ and $R(\beta)$

To be more concise, let us call LL the left side of the left triangle, LR its right side. From left to right, the sides of a pair are LL, LR, RL, RR. On the first values of the first pair, we can see:

—On the LL side, zeroes alternating with the Springer numbers S_{2n+1}

—On the LR and RL sides, the numbers $2^{n-1}E_n$

—On the RR side, zeroes alternating with the Springer numbers S_{2n}.

On the first values of the second pair, one can see:

—On the LL side, zeroes alternating with the numbers $2^{2n}E_{2n}$

—On the LR and RL sides the Springer numbers S_n

—On the RR side, zeroes alternating with the numbers $2^{2n+1}E_{2n+1}$.

This will be proved in the sequel. We recall the elementary facts of the Seidel algorithmic method [Se] [D] [DV]. Starting with an arbitrary sequence $(a_n)_{n \geq 0}$, called the initial sequence, we construct the *Seidel matrix* $(a_n^k)_{n,k \geq 0}$ associated with (a_n) as follows:

(R1) The first line a_n^0 of the matrix is the initial sequence a_n.

(R2) Each entry a_n^k of the kth line is the sum of the entry immediately above and of the entry above and to the right of it:

$$a_n^k = a_n^{k-1} + a_{n+1}^{k-1}.$$

The first column a_0^n of the matrix will be called the final sequence.

PROPOSITION 1 (Seidel). *The egf $A(x)$ of the initial sequence and the egf $\bar{A}(x)$ of the final sequence satisfy the relation*

$$\bar{A}(x) = e^x A(x).$$

The proof is straightforward. By induction on k, the entries on the kth line can be expressed in terms of those on the 0th line:

$$a_n^k = \sum_{i=0}^{i=k} \binom{k}{i} a_{n+i}^0.$$

The result follows when taking $n = 0$ in this identity.

Now impose the following auxiliary conditions:

(E1) $A(x)$ is an even function and $A(0) = 1$.

(E2) $\bar{A}(x) - 1$ is an odd function (Note that condition (E1) implies $\bar{A}(0) = 1$).

Under these assumptions, the Seidel matrix has the form

$$\begin{array}{ccccccc} 1 & 0 & * & 0 & * & 0 & \cdots \\ * & * & * & * & * & * & \cdots \\ 0 & * & * & * & * & * & \cdots \\ * & * & * & * & * & * & \cdots \\ 0 & * & * & * & * & * & \cdots \\ * & * & * & * & * & * & \cdots \\ 0 & * & * & * & * & * & \cdots \end{array} \quad \Rightarrow \quad \begin{array}{cccccccc} 1 & 0 & -1 & 0 & 5 & 0 & -61 & \cdots \\ 1 & -1 & -1 & 5 & 5 & -61 & \cdots & \\ 0 & -2 & 4 & 10 & -56 & \cdots & & \\ -2 & 2 & 14 & -46 & \cdots & & & \\ 0 & 16 & -32 & \cdots & & & & \\ 16 & -16 & \cdots & & & & & \\ 0 & \cdots & & & & & & \end{array}$$

At the beginning the stars are unknown entries. In fact, the matrix is uniquely determined by rule (R2) and conditions (E1) and (E2), and the second step consists of computing the successive obliques. The first oblique is, from N-E to S-W, 0 and 1; the second one is, from S-W to N-E, 0, -1, and -1; the next one is, from N-E to S-W, 0, -1, -2, and -2. This algorithm gives rise to what Arnold calls the "Euler–Bernoulli triangle," which we prefer to call the Seidel triangle for the Euler numbers, first discovered by Seidel [Se], then rediscovered by Entringer, Arnold [A1], and others:

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & \leftarrow & 0 & & & & & \\ & & & & 0 & \rightarrow & 1 & \rightarrow & 1 & & & & \\ & & & 2 & \leftarrow & 2 & \leftarrow & 1 & \leftarrow & 0 & & & \\ & & 0 & \rightarrow & 2 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & 5 & & \\ & 16 & \leftarrow & 16 & \leftarrow & 14 & \leftarrow & 10 & \leftarrow & 5 & \leftarrow & 0 & \\ 0 & \rightarrow & 16 & \rightarrow & 32 & \rightarrow & 46 & \rightarrow & 56 & \rightarrow & 61 & \rightarrow & 61 \end{array}$$

Seidel–Entringer–Arnold triangle for the Euler numbers

On the other hand, the only function $A(x)$ satisfying (E1) and (E2) is the function $A(x) = 1/\cosh x$ (because $A(x)$ is even and $e^x A(x) - 1$ is odd, so $e^{-x}A(x) - 1 = 1 - e^x A(x)$), and then $\bar{A}(x) = 1 + \tanh x$. Therefore the secant and tangent numbers appear on the sides of the triangle.

Now we proceed along the same lines for Arnold's pairs of triangles generating the Springer numbers. To obtain the second pair we first note that the generating function of the Springer numbers is

$$S(x) = e^x \frac{1}{\cosh 2x}.$$

Note that $e^{-x}S(x)$ is an even function and that $e^x S(x) - 1 = \tanh 2x$ is an odd function. Conversely, let $A(x)$ be a function having the following two properties:

(S1) $e^{-x}A(x)$ is even, so that $e^x A(-x) = e^{-x}A(x)$.

(S2) $e^x A(x) - 1$ is odd, so that $e^x A(x) - 1 = -e^{-x}A(-x) + 1$.

From (S1) we have $A(-x) = e^{-2x}A(x)$, and $A(x) = S(x)$ by (S2).

We now construct two Seidel matrices related to the sequence of Springer numbers:

—on the left, the Seidel matrix obtained when the initial sequence is (S_n),

—on the right, the Seidel matrix obtained when the *final* sequence is (S_n).

1	S_1	S_2	S_3	S_4	S_5	⋯		1	0	*	0	*	0	*	⋯
*	*	*	*	*	*	⋯		S_1	*	*	*	*	*	*	⋯
0	*	*	*	*	*	⋯		S_2	*	*	*	*	*	*	⋯
*	*	*	*	*	*	⋯		S_3	*	*	*	*	*	*	⋯
0	*	*	*	*	*	⋯		S_4	*	*	*	*	*	*	⋯
*	*	*	*	*	*	⋯		S_5	*	*	*	*	*	*	⋯

On the right matrix, we find that $S_1 = 1$. On the left matrix we deduce that the first oblique is, from N-E to S-W, 1 and 2, and the next one is, from S-W to N-E, 0, -2, $S_2 = -3$. Knowing S_2, we compute the corresponding oblique on the right matrix, $S_2 = -3$, -4, -4, and deduce the next oblique from N-E to S-W, 0, -4, -8, $S_3 = -11$; then we go back to the right matrix, and so on:

1	1	−3	−11	57	361	⋯		1	0	−4	0	80	0	⋯
2	−2	−14	46	418	⋯			1	−4	−4	80	80	⋯	
0	−16	32	464	⋯				−3	−8	76	160	⋯		
−16	16	496	⋯					−11	68	236	⋯			
0	512	⋯						57	294	⋯				
512	⋯							361	⋯					

This leads exactly to the algorithm of the second pair of triangles.

Similarly, the first Arnold's pair of triangles correspond to the following pair of Seidel matrices:

0	1	0	−11	0	361	⋯		1	−1	−2	8	40	−256	⋯
1	1	−11	−11	361	⋯			0	−3	6	48	−216	⋯	
2	−10	−22	350	⋯				−3	3	54	−168	⋯		
−8	−32	328	⋯					0	57	−114	⋯			
−40	296	⋯						57	−57	⋯				
256	⋯							0	⋯					

On the left, the egf's of the initial and final sequences are

$$A_1(x) = \frac{\sinh x}{\cosh 2x}$$

and

$$\bar{A}_1(x) = e^x A_1(x) = \frac{1}{2}\left(1 - \frac{1}{\cosh 2x} + \tanh 2x\right).$$

On the right the egf's are

$$A_2(x) = \frac{1}{2}\left(1 + \frac{1}{\cosh 2x} - \tanh 2x\right)$$

and

$$\bar{A}_2(x) = e^x A_2(x) = \frac{\cosh x}{\cosh 2x}.$$

The algorithm of computation is now based on the facts that $A_1(x)$ is odd, $A_2(x) = 1 - \bar{A}_1(x)$, and $\bar{A}_2(x)$ is even, and these conditions lead to the first pair of triangles.

3. CONTINUED FRACTIONS EXPRESSING THE ORDINARY GENERATING FUNCTIONS

In this section we deal with the ordinary generating functions (ogf) of the initial and final sequences of a Seidel matrix; more precisely we define

$$a(x) = \sum_{n \geq 0} a_n^0 x^{n+1} \qquad \text{and} \qquad \bar{a}(x) = \sum_{n \geq 0} a_0^n x^{n+1}.$$

These series are, in a formal sense, the Laplace transforms of $A(x)$ and $\bar{A}(x)$, and the analogue to Proposition 1 is

PROPOSITION 2. *The ogf $a(x)$ of the initial sequence and the ogf $\bar{a}(x)$ of the final sequence satisfy the relations*

$$\bar{a}(x) = a\left(\frac{x}{1-x}\right) \qquad \text{and} \qquad a(x) = \bar{a}\left(\frac{x}{1+x}\right).$$

For a proof, identify the coefficients of x^{n+1} on both sides.

In the following we assume that the reader is acquainted with the formal theory of analytic continued fractions (see Henrici [H]). We need the following lemmas:

LEMMA 1 (lemma for contractions). *The following representations of a series $f(x)$ are equivalent*:

$$f(x) = \cfrac{x}{1 + \cfrac{c_1 x}{1 + \cfrac{c_2 x}{1 + \cfrac{c_3 x}{1 + \cfrac{c_4 x}{1 + \cfrac{c_5 x}{\ddots}}}}}}$$

$$f(x) = \cfrac{x}{1 + c_1 x - \cfrac{c_1 c_2 x^2}{1 + (c_2 + c_3)x - \cfrac{c_3 c_4 x^2}{\ddots}}} \qquad (\mathrm{C}_1)$$

$$f(x) = x - \cfrac{c_1 x^2}{1 + (c_1 + c_2)x - \cfrac{c_2 c_3 x^2}{1 + (c_3 + c_4)x - \cfrac{c_4 c_5 x^2}{\ddots}}}. \qquad (\mathrm{C}_2)$$

Proof. Note that the identity

$$\cfrac{x}{1 + \cfrac{c_1 x}{1 + c_2 A(x)}} = x - \frac{c_1 x^2}{1 + c_1 x + c_2 A(x)}$$

holds for each formal series $A(x)$. It follows by induction on n that the nth convergent of the first contraction (C_1) (resp. of the second contraction (C_2)) is the $(2n - 1)$st (resp. the $(2n)$th) convergent of the original continued fraction.

LEMMA 2. *The two equalities*

$$f(x) = \cfrac{x}{1 + \cfrac{c_1 x}{1 + \cfrac{c_2 x}{1 + \cfrac{c_3 x}{\ddots}}}}, \qquad f\left(\frac{x}{1 + Cx}\right) = \cfrac{x}{1 + \cfrac{d_1 x}{1 + \cfrac{d_2 x}{1 + \cfrac{d_3 x}{\ddots}}}}$$

are equivalent to the system

$$d_1 = c_1 + C, \qquad d_1 d_2 = c_1 c_2, \qquad d_2 + d_3 = c_2 + c_3 + C, \dots$$
$$d_{2n-1} d_{2n} = c_{2n-1} c_{2n}, \qquad d_{2n} + d_{2n+1} = c_{2n} + c_{2n+1} + C, \dots .$$

Proof. Replace x by $(x/1 + Cx)$ in (C_1) and make the identifications.

These preliminaries allow us to find continued fractions for the ogf of Euler numbers and Springer numbers. First consider the series

$$s(x) = \sum_{n \geq 0} (-1)^n E_{2n} x^{2n+1} + \cdots = x - x^3 + 5x^5 - 61x^7 + \cdots$$

$$t(x) = x + \sum_{n \geq 1} (-1)^n E_{2n+1} x^{2n+2} + \cdots = x + x^2 - 2x^4 + 16x^6 - \cdots .$$

The next proposition (in fact, its corollary, Corollary 3.1) is a classical result. Propositions 4 and 5 seem to be new.

PROPOSITION 3. *The series $t(x)$ has the continued fraction representation*

$$t(x) = x + x^2 - 2x^4 + 16x^6 - 272x^8 + \cdots$$
$$= \cfrac{x}{1 - \cfrac{x}{1 + \cfrac{x}{1 - \cfrac{2x}{1 + \cfrac{2x}{1 - \cfrac{3x}{1 + \cfrac{3x}{\ddots}}}}}}}.$$

Proof. Consider the Seidel matrix for the Euler numbers. According to Proposition 2 we have $\bar{a}(x) = t(x)$ and $a(x) = s(x) = t(x/(1 + x))$. From the parity characterizations (E_1) and (E_2) in the preceding section, we know that $t(x)$ is the only series $f(x)$ such that $f(x) - x$ is even and $f(x/(1 + x))$ is odd. Assume that $f(x)$ is a continued fraction as in Lemma

1. The first condition ($f(x) - x$ is even) yields

$$c_1 = -1, \quad c_1 + c_2 = 0, \quad c_3 + c_4 = 0, \ldots, \quad c_{2n-1} + c_{2n} = 0, \ldots .$$

According to Lemma 2, the second condition says that

$$c_2 + c_3 + 1 = 0, \quad c_4 + c_5 + 1 = 0, \ldots, \quad c_{2n} + c_{2n+1} + 1 = 0.$$

We deduce by induction that $c_{2n-1} = -n$ and $c_{2n} = n$. This gives the continued fraction for $t(x)$.

COROLLARY 3.1. *The series $t(x)$ and $s(x)$ have the continued fraction representation*

$$t(x) = \cfrac{x}{1 - x + \cfrac{x^2}{1 - x + \cfrac{4x^2}{\ddots \, 1 - x + \cfrac{n^2x^2}{\ddots}}}}$$

$$= x + \cfrac{x^2}{1 + \cfrac{2x^2}{1 + \cfrac{6x^2}{\ddots \, 1 + \cfrac{n(n+1)x^2}{\ddots}}}}$$

$$s(x) = x - x^3 + 5x^5 - 61x^7 + \cdots = \cfrac{x}{1 + \cfrac{x^2}{1 + \cfrac{4x^2}{\ddots \, 1 + \cfrac{n^2x^2}{\ddots}}}}.$$

The continued fractions for $t(x)$ are the contractions of that of Proposition 3. Replacing x by $x/(1 + x)$ in the first contraction, we obtain the continued fraction for $s(x)$. Historically these results are due independently to Rogers and Stieltjes, but they are derived in a different way. For a recent combinatorial proof, see [F1].

COROLLARY 3.2. *The ogf of the Springer numbers has the continued fraction representation*

$$x + x^2 - 3x^3 - 11x^4 + 57x^5 + 361x^6 - \cdots = \cfrac{x}{1 - x + \cfrac{4x^2}{1 - x + \cfrac{16x^2}{\ddots \; 1 - x + \cfrac{4n^2x^2}{\ddots}}}}.$$

Proof. In the right matrix of the second pair for Springer numbers, the ogf of the initial sequence is $a(x) = \frac{1}{2}s(2x)$. The ogf of the final sequence is then

$$\bar{a}(x) = \frac{1}{2}s\left(\frac{2x}{1-x}\right) = x + x^2 - 3x^3 - 11x^4 + 57x^5 + 361x^6 - \cdots.$$

The result follows from the continued fraction for $s(x)$.

Now consider the first Arnold pair of Seidel matrices related to the Springer numbers. The series $a_2(x) = x - \bar{a}_1(x)$ is characterized by the two properties:

(S_1) $\bar{a}_1(x/1 + x) = a_1(x) = \Sigma(-1)^n S_{2n+1} x^{2n+2} = x^2 - 11x^4 + 361x^6 - \cdots$ is an even series;

(S_2) $a_2(x/1 - x) = \Sigma(-1)^n S_{2n} x^{2n+1} = x - 3x^3 + 57x^5 - \cdots$ is an odd series.

Let $e(x)$ be the series

$$e(x) = t(x) - s(x) = \sum_{n \geq 1} (-1)^{n-1}\left(E_{2n-1}x^{2n} + E_{2n}x^{2n+1}\right)$$
$$= x^2 + x^3 - 2x^4 - 5x^5 + 16x^6 + 61x^7 - \cdots$$

so that $a_2(x) = x - \frac{1}{4}e(2x)$.

Assume that $a_2(x)$ has a continued fraction representation as in Lemma 1:

$$a_2(x) = x - x^2 - 2x^3 + 8x^4 + 40x^5 - 256x^6 - \cdots$$

$$= \cfrac{x}{1+\cfrac{c_1x}{1+\cfrac{c_2x}{1+\cfrac{c_3x}{1+\cfrac{c_4x}{1+\cfrac{c_5x}{\ddots}}}}}}.$$

In the first contraction, replace x by $(x/1-x)$. This gives

$$a_2(x/1-x) = \cfrac{x}{1+(c_1-1)x-\cfrac{c_1c_2x^2}{1+(c_2+c_3-1)x-\cfrac{c_3c_4x^2}{\ddots}}}.$$

This series is odd, if and only if

$$c_1 = 1, \qquad c_2+c_3 = 1, \qquad c_4+c_5 = 1,\ldots, \qquad c_{2n}+c_{2n+1} = 1,\ldots.$$

Taking the second contraction, we get

$$\bar{a}_1(x) = \cfrac{c_1x^2}{1+(c_1+c_2)x-\cfrac{c_2c_3x^2}{1+(c_3+c_4)x-\cfrac{c_4c_5x^2}{1+(c_5+c_6)x-\cfrac{c_6c_7x^2}{\ddots}}}}.$$

If x is replaced by $x/(1+x)$, we obtain

$$\cfrac{c_1x^2}{(1+x)^2+(c_1+c_2)x(1+x)-\cfrac{c_2c_3x^2}{1+(c_3+c_4)\dfrac{x}{1+x}-\cfrac{c_4c_5x^2}{(1+x)^2+(c_5+c_6)x(1+x)-\cdots}}}.$$

This last series is even, if and only if

$$c_1+c_2 = -2, \qquad c_3+c_4 = 0, \qquad c_5+c_6 = -2,\ldots,$$
$$c_{4n-1}+c_{4n} = 0, \qquad c_{4n+1}+c_{4n+2} = -2,\ldots.$$

The coefficients c_n are therefore uniquely determined, and we come to the following proposition and its corollary, which seem to be new:

PROPOSITION 4. *The series $x - \frac{1}{4}e(2x)$ admits the continued fraction representation*

$$x - \tfrac{1}{4}e(2x) = x - x^2 - 2x^3 + 8x^4 + 40x^5 - 256x^6 \cdots = \cfrac{x}{1 + \cfrac{x}{1 - \cfrac{3x}{1 + \cfrac{4x}{1 - \cfrac{4x}{1 + \cfrac{5x}{1 - \cfrac{7x}{\ddots}}}}}}},$$

where the four successive generic coefficients are

$$c_{4n-1} = 4n, \qquad c_{4n} = -4n, \qquad c_{4n+1} = 4n+1, \qquad c_{4n+2} = -4n-3.$$

Following Glaisher's notations [G1] [G2], denote by $p(x)$ and $q(x)$ the series

$$p(x) = \sum_{n \geq 0} (-1)^n S_{2n} x^{2n+1} = x - 3x^3 + 57x^5 - \cdots$$

$$q(x) = \sum_{n \geq 0} (-1)^n S_{2n+1} x^{2n+2} = x^2 - 11x^4 + 361x^6 - \cdots.$$

COROLLARY 3.3. *The series $p(x)$ and $q(x)$ have the continued fraction representations*

$$p(x) = x - 3x^3 + 57x^5 - \cdots = \cfrac{x}{1 + \cfrac{3x^2}{1 + \cfrac{16x^2}{1 + \cfrac{35x^2}{1 + \cfrac{64x^2}{\ddots}}}}}$$

$$= \cfrac{x}{1 - x^2 + \cfrac{4x^2}{1 + \cfrac{12x^2}{1 - x^2 + \cfrac{40x^2}{1 + \cfrac{56x^2}{\ddots}}}}},$$

where the two generic successive coefficients are $16n^2$ *and* $(4n + 1)(4n + 3)$ *in the first continued fraction,* $4n(4n - 3)$ *and* $4n(4n - 1)$ *in the second,*

$$q(x) = x^2 - 11x^4 + 361x^6 - \cdots = \cfrac{x^2}{1 - x^2 + \cfrac{12x^2}{1 + \cfrac{20x^2}{1 - x^2 + \cfrac{56x^2}{1 + \cfrac{72x^2}{\ddots}}}}},$$

where the two generic successive coefficients are $(4n - 1)4n$ *and* $4n(4n + 1)$.

Proof. It remains only to prove the second continued fraction for $p(x)$, which will be useful for Proposition 7. In fact, the two continued fractions for $p(x)$ are equivalent by Lemma 2, as one can verify the two identities

$$(4n - 3)(4n - 1) \cdot 16n^2 = 4n(4n - 3) \cdot 4n(4n - 1),$$

$$16n^2 + (4n + 1)(4n + 3) + 1 = 4n(4n - 1) + (4n + 4)(4n + 1).$$

4. A PAIR OF TRIANGLES OF ARNOLD TYPE FOR EULER NUMBERS. THE MEDIAN EULER NUMBERS

We start with a corollary of our last result:

PROPOSITION 5. *The ogf* $e(x)$ *of the Euler numbers admits the continued fraction representation*

$$e(x) = t(x) - s(x) = x^2 + x^3 - 2x^4 - 5x^5 + 16x^6 + 61x^7 - \cdots$$

$$= \cfrac{x^2}{1 - x + \cfrac{3x^2}{1 + \cfrac{5x^2}{1 - x + \cfrac{14x^2}{1 + \cfrac{18x^2}{\ddots}}}}},$$

where the two generic successive coefficients are $n(4n - 1)$ *and* $n(4n + 1)$.

Proof. Take the contraction (C_2) of the continued fraction of Proposition 4 and replace x by $(x/2)$.

Now consider the pair of Seidel matrices obtained by taking the same (signed) sequence of Euler numbers first as initial sequence, then as final sequence:

0	1	1	−2	−5	16	61	⋯
1	2	−1	−7	11	77	⋯	
3	1	−8	4	88	⋯		
4	−7	−4	92	⋯			
−3	−11	88	⋯				
−14	77	⋯					
63	⋯						

0	1	−1	−2	5	16	−61	⋯
1	0	−3	3	21	−45	⋯	
1	−3	0	24	−24	⋯		
−2	−3	24	0	⋯			
−5	21	24	⋯				
16	45	⋯					
61	⋯						

The egf are respectively:

—on the left, $A_1(x) = \tanh x + 1 - 1/\cosh x = (e^x - 1)/\cosh x$ and $\bar{A}_1(x) = (e^{2x} - e^x)/\cosh x$;

—on the right, $\bar{A}_2(x) = (e^x - 1)/\cosh x$ and $A_2(x) = (1 - e^{-x})/\cosh x$, so that $\bar{A}_2(x) = -A_2(-x)$.

Denote by (a_n^k) the matrix on the right. We have $a_0^0 = 0$ and, for $n \geq 1$, $a_n^0 = (-1)^{n+1}a_0^n = E_n$. The matrix is very similar in form to the Seidel matrix for Genocchi numbers [Se] [D] [B] [DV], with obliques alternatively symmetric and antisymmetric; more precisely $a_n^k = (-1)^{n+k+1}a_k^n$, with zeros on the diagonal, $a_n^n = 0$. For an easy proof, see [DV].

Now let us look at the matrix on the left, say (b_n^k). In fact, $\bar{A}_1(x) + A_2(x) = 2(e^x - 1)$, or equivalently

$$(0,1,3,4,-3,-14,63,\dots) + (0,1,-1,-2,5,16,-61,\dots)$$
$$= (0,2,2,2,2,2,2,\dots).$$

A consequence is that $b_1^n = E_n - E_{n+1} = (-1)^{n+1} b_n^1$ for $n \geq 1$. By induction this yields a similar general relation: $b_n^k = (-1)^{n+k} b_k^n$ for $n, k \geq 1$. In particular $b_n^{n+1} = -b_{n+1}^n$, which implies $b_n^n = -2b_n^{n+1}$ for $n \geq 1$. This last relation will be used in the proof of Proposition 6.

Considering now the upper half of the left matrix, and the lower half of the right matrix, we are led to a new pair of triangles of Arnold type:

		1					$\rightarrow$					1
		2	1				$\leftarrow$					1
2 : 2	=	1	2				$\rightarrow$				2	3
		8	7	5			$\leftarrow$				5	3
8 : 2	=	4	11	16			$\rightarrow$			16	21	24
		92	88	77	61		$\leftarrow$			61	45	24
92 : 2	=	46	134	211	272		$\rightarrow$		272	333	378	402
		2048	2002	1868	1657	1385	$\leftarrow$		1385	1113	780	402
2048 : 2	=	1024	3026	4894	6551	7936	$\rightarrow$	7936	9321	10434	11214	11616

Our pair of Arnold type for Euler numbers

The latter triangles are similar to the Seidel triangle of Genocchi numbers (see below, and [Se] [D] [B] [DV] [DR] [DZ] for details). Call *median Euler numbers* the numbers L_n and R_n that appear respectively on the extreme left column ($L_0 = 1$, $L_1 = 1$, $L_2 = 4$, $L_3 = 46, \ldots$) and on the extreme right column ($R_0 = 1$, $R_1 = 3$, $R_2 = 24$, $R_3 = 402, \ldots$). Some of their properties are now presented, similar to those of the median Genocchi numbers.

We need a preliminary general result on Seidel matrices.

PROPOSITION 6. *Given a Seidel matrix* (a_n^k), *then the ogf's of its initial sequence, of its main diagonal, and of its upper diagonal, respectively denoted by*

$$a(x) = \sum_{n \geq 0} a_n^0 x^{n+1}, \qquad d_0(x) = \sum_{n \geq 0} a_n^n x^n, \qquad d_1(x) = \sum_{n \geq 0} a_{n+1}^n x^{n+1},$$

satisfy the identity

$$a(x) = x d_0\left(\frac{x^2}{1+x}\right) + d_1\left(\frac{x^2}{1+x}\right).$$

Proof. Let $y = x^2/(1+x)$, and express the powers of x by replacing x^2 by its value $xy + y$ at each step of the algorithm

$$\begin{aligned} x &= x(1) + 0 \\ x^2 &= x(y) + y \\ x^3 &= x(y + y^2) + y^2 \\ x^4 &= x(2y^2 + y^3) + (y^2 + y^3) \\ \cdots &= \cdots \\ x^{n+1} &= xF_n(y) + yF_{n-1}(y), \end{aligned}$$

where the $F_n(y)$ are the Fibonacci polynomials defined by

$$F_1(y) = 0, \qquad F_0(y) = 1, \qquad F_1(y) = y, \ldots,$$
$$F_n(y) = y(F_{n-1}(y) + F_{n-2}(y)),$$

whose explicit formula is readily found by induction on n to be

$$F_n(y) = \sum_k \binom{k}{n-k} y^k.$$

Then we have

$$a(x) = \sum_{n \geq 0} a_n^0 x^{n+1} = x \sum_{n \geq 0} a_n^0 F_n(y) + y \sum_{n \geq 0} a_n^0 F_{n-1}(y).$$

Seeking the coefficient of y^k in the first term of this expansion, we take all the $F_n(y)$ that contribute to y^k, corresponding to those n such as $k \leq n \leq 2k$, and we find that this coefficient is

$$\sum_{n=k}^{n=2k} \binom{k}{n-k} a_n^0 = \sum_{i=0}^{i=k} \binom{k}{i} a_{k+i}^0 = a_k^k.$$

Similarly, the coefficient of y^{k+1} in the second term is obtained by taking all the $F_{n-1}(y)$ that contribute to y^k, and we find

$$\sum_{n=k+1}^{n=2k+1} \binom{k}{n-1-k} a_n^0 = \sum_{i=0}^{i=k} \binom{k}{i} a_{k+1+i}^0 = a_{k+1}^k.$$

Then $a(x) = xd_0(y) + d_1(y)$.

PROPOSITION 7. *The ogf's of the median Euler numbers satisfy the continued fraction representations*

$$r(x) = \sum(-1)^n R_n x^{n+1} = x - 3x^2 + 24x^3 - 402x^4 + 11616x^5 - \cdots$$

$$= \cfrac{x}{1 + \cfrac{3x}{1 + \cfrac{5x}{1 + \cfrac{2.7x}{1 + \cfrac{2.9x}{1 + \cfrac{3.11x}{1 + \cfrac{3.13x}{1 + \ddots}}}}}}}$$

$$l(x) = \sum(-1)^n L_n x^{n+1} = x - x^2 + 4x^3 - 46x^4 + 1024x^5 - \cdots$$

$$= \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{3x}{1 + \cfrac{2.5x}{1 + \cfrac{2.7x}{1 + \cfrac{3.9x}{1 + \cfrac{3.11x}{1 + \ddots}}}}}}}.$$

Proof. When applying the last proposition to the matrix on the right, we find $d_0(x) = 0$ and $d_1(x) = r(x)$; then

$$r\left(\frac{x^2}{1+x}\right) = e(-x) \qquad \text{or} \qquad r\left(\frac{x^2}{1-x}\right) = e(x).$$

According to Proposition 5, this proves the continued fraction expansion for $r(x)$.

Now take the matrix on the left. There we have $a(x) = e(x)$ and $d_1(x) = l(x)$. On the other hand, we proved that $b_0^0 = 0$ and $b_n^n = -2b_{n+1}^n$ for $n \geq 1$. Then

$$d_0(x) = 2x - 8x^2 + 92x^3 - \cdots = \frac{2}{x}(x - l(x))$$

and Proposition 6 yields

$$e(x) = 2x - \left(1 + \frac{2}{x}\right) l\left(\frac{x^2}{1+x}\right).$$

Now we turn back to the second continued fraction for $p(x)$ given in Proposition 4, and replace x by $(x/1+x)$. This yields another continued fraction for the numbers $2^{n-1}E_n$, precisely

$$x - \tfrac{1}{4}e(2x) = x - x^2 - 2x^3 + 8x^4 + 40x^5 - \cdots$$
$$= \cfrac{x(1+x)}{(1+x)^2 - x^2 + \cfrac{4x^2}{1 + \cfrac{12x^2}{(1+x)^2 - x^2 + \cfrac{40x^2}{1 + \cfrac{56x^2}{\ddots}}}}}.$$

Changing $2x$ into x we obtain the formula

$$e(x) = 2x - \cfrac{2x + x^2}{1 + x + \cfrac{x^2}{1 + \cfrac{3x^2}{1 + x + \cfrac{10x^2}{1 + \cfrac{14x^2}{\ddots}}}}}.$$

If we define $f(x)$ as the continued fraction

$$f(x) = \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{3x}{1 + \cfrac{2.5x}{1 + \cfrac{2.7x}{\ddots}}}}}$$

we find that

$$e(x) = 2x - \left(1 + \frac{2}{x}\right) f\left(\frac{x^2}{1+x}\right).$$

This identity is a characterization of $l(x)$, so we have $f(x) = l(x)$, which completes the proof.

PROPOSITION 8. *When the initial sequence of a Seidel matrix is the sequence of Springer numbers (resp. S_{2n}, or S_{2n+1}), then the final sequence is the sequence $2^{2n}L_n$ or $2^{2n}R_n$, as shown in the tables*

$2^0 \cdot 1$	=	1	3	57	2763
$2^2 \cdot 1$	=	4	60	2820	
$2^4 \cdot 4$	=	64	2880		
$2^6 \cdot 46$	=	2944			

$2^0 \cdot 1$	=	1	11	361	24611
$2^2 \cdot 3$	=	12	372	24972	
$2^4 \cdot 24$	=	384	25344		
$2^6 \cdot 402$	=	25728			

Proof. Let us rewrite slightly differently the continued fractions of Propositions 4 and 7:

$$x + 3x^2 + 57x^3 + \cdots = \cfrac{x}{1 + x - \cfrac{4x}{1 - \cfrac{12x}{1 + x - \cfrac{40x}{1 - \cfrac{56x}{\ddots}}}}},$$

$$x + 11x^2 + 361x^3 + \cdots = \cfrac{x}{1 + x - \cfrac{12x}{1 - \cfrac{20x}{1 + x - \cfrac{56x}{1 - \cfrac{72x}{\ddots}}}}},$$

$$x + x^2 + 4x^3 + 46x^4 + \cdots = \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{3x}{1 - \cfrac{10x}{1 - \cfrac{14x}{\ddots}}}}},$$

$$x + 3x^2 + 24x^3 + \cdots = \cfrac{x}{1 - \cfrac{3x}{1 - \cfrac{5x}{1 - \cfrac{14x}{1 - \cfrac{18x}{1 - \ddots}}}}}.$$

When replacing x by $(x/1 - x)$ in the first two, and x by $4x$ in the last two, we obtain the same continued fractions.

5. ANALOGIES WITH GENOCCHI NUMBERS

The Genocchi numbers are defined by

$$G(x) = \frac{2x}{e^x + 1} = x - \frac{x^2}{2!} + \frac{x^4}{4!} - 3\frac{x^6}{6!} + 17\frac{x^8}{8!} - 155\frac{x^{10}}{10!} + \cdots .$$

The relation $e^x G(x) = 2x - G(x)$, combined with the parity of $G(x) - x$, gives rise to the following Seidel matrix, and to the corresponding Seidel triangle [Se] [DV]:

0	1	−1	0	1	0	−3	0	17	0	−155
1	0	−1	1	1	−3	−3	17	17	−155	
1	−1	0	2	−2	−6	14	34	−138		
0	−1	2	0	−8	8	48	−104			
−1	1	2	−8	0	56	−56				
0	3	−6	−8	56	0					
3	−3	−14	48	56						
0	−17	34	104							
−17	17	138								
0	155									
155										

1				
1				
1	1			
2	1			
2	3	3		
8	6	3		
8	14	17	17	
56	48	34	17	
56	104	138	155	155

Seidel matrix and Seidel triangle for the Genocchi numbers

Note the similarity of the above matrix with the right matrix of our pair for the Euler numbers given in the preceding section. In particular, if we call $g(x)$ and $h(x)$ the ogf of the Genocchi numbers and the ogf of the median Genocchi numbers,

$$g(x)x^2 - x^3 + x^5 - 3x^7 + 17x^9 - 155x^{10} + \cdots$$

$$h(x) = x - x^2 + 2x^3 - 8x^4 + 56x^5 - \cdots = \sum_{n \geq 0} (-1)^n H_{2n+1} x^{n+1},$$

we still have $h(x^2/(1+x)) = g(x)$. We can prove the identities (see [DZ])

$$h\left(\frac{x^2}{1+x}\right) + h\left(\frac{x^2}{1-x}\right) = 2x^2$$

$$h(x) = \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{x}{1 + \cfrac{2^2x}{1 + \cfrac{2^2x}{1 + \cfrac{3^2x}{1 + \cfrac{3^2x}{\ddots}}}}}}}$$

and prove also [DZ] that if the initial sequence of a Seidel matrix is $(2n+1)E_{2n}$, then its final sequence is $2^{2n}H_{2n+1}$:

E_{2n}	=	1	1	5	61	1385
		0.1	0.3	0.5	0.7	0.9
$2^0 \cdot 1$	=	1	3	25	427	12465
$2^2 \cdot 1$	=	4	28	452	12892	
$2^4 \cdot 2$	=	32	480	13344		
$2^6 \cdot 8$	=	512	13824			
$2^8 \cdot 56$	=	14336				

6. COMBINATORIAL AND ARITHMETICAL ASPECTS. OPEN PROBLEMS

The combinatorial interpretation of the Seidel triangle for the Euler numbers in terms of alternating permutations (essentially due to Entringer) is very classical. Similar interpretations for both Arnold's pairs (and many further combinatorial results) will be found in [A3]. Two different combinatorial interpretations of the Seidel triangle for the Genocchi numbers are given in [DR], [DV]. The combinatorial interpretation of the median Genocchi numbers can be found in [DR], where it was proved that *the number* H_{2n+1} *is equal to the number of permutations* σ of $\{1,2,3,\dots,2n\}$ *such that, for all* i, $\sigma(2i-1) > 2i-1$, *and* $\sigma(2i) < 2i$.

Combinatorial interpretations for the remaining Seidel matrices introduced in the present paper have not yet been found.

Concerning the congruence properties of the numbers introduced in this article, we mention two related works: Barsky's paper [B], on the divisibility of the median Genocchi numbers by powers of 2, and Flajolet's paper [F2], which relates the properties for certain classical sequences of integers of being periodic modulo given integers, to the existence of continued fractions representing their ogf's.

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