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CAUCHY IDENTITIES FOR UNIVERSAL SCHUBERT POLYNOMIALS

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ABSTRACT

We prove the Cauchy type identities for the universal double Schubert polynomials, introduced recently by W. Fulton. As a corollary, the determinantal formulae for some specializations of the universal double Schubert polynomials corresponding to the Grassmannian permutations are obtained. We also introduce and study the universal Schur functions and multiparameter deformation of Schubert polynomials.

§0. Introduction.

The aim of this note is to study the algebraic properties of universal double Schubert polynomials introduced recently by W. Fulton [F]. More specifically, the main goal of our note is to generalize the results from [K2] on the case of universal double Schubert polynomials and universal Schur functions. Another goal of this note is to introduce and study the multiparameter deformation of classical and quantum Schubert polynomials, which (conjecturally) should be closely related with the second form of universal double Schubert polynomials introduced by W. Fulton [F]. Our main tool is the various generalizations of the quantum Cauchy identity [KM]. Based on a generalized Cauchy's identities technique, we give an affirmative answer on some questions raised in the paper [F]. As another application, we prove the determinantal formulae for some specializations of the universal double Schubert polynomials corresponding to the Grassmannian permutations.

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§1. Schur functions.

In this section we give a brief review of some basic definitions and results of the theory of Schur functions. In exposition we follow to the I. Macdonald book [M1] (see also [M2]) where proofs and more details can be found.

• Schur functions.

Let us recall at first a definition of the Schur function $s_\lambda(X_n)$ as the quotient of two alternates:

$$s_\lambda(X_n)=\frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\le i,j\le n}}{\det\left(x_i^{n-j}\right)_{1\le i,j\le n}}. \tag{1.1}$$

where $X_n=(x_1,\ldots,x_n)$ is the set of independent variables and $\lambda=(\lambda_1,\ldots,\lambda_n)$ is a partition of the length $\le n$. The denominator on the RHS(1.1) is the Vandermond determinant, and is equal to the product $\prod\limits_{1\le i<j\le n}(x_i-x_j)$.

When $\lambda=(r)$, $s_\lambda(X_n)$ is the complete symmetric function $h_r(X_n)$, and when $\lambda=(1^r)$, $s_\lambda(X_n)$ is the elementary symmetric function $e_r(X_n)$. In terms of complete symmetric functions, $s_\lambda(X_n)$ is given by Jacobi–Trudi formula

$$s_\lambda(X_n)=\det\left(h_{\lambda_i-i+j}(X_n)\right)_{1\le i,j\le n}. \tag{1.2}$$

Dually, in terms of the elementary symmetric functions, $s_\lambda(X_n)$ is given by the Nägelsbach–Kostka formula

$$s_\lambda(X_n)=\det\left(e_{\lambda_i'-i+j}(X_n)\right)_{1\le i,j\le m}, \tag{1.3}$$

where $\lambda'=(\lambda_1',\ldots,\lambda_m')$ is the conjugate ([M1], p.2) of the partition λ.

Remark 1. Using the recurrence relation for the elementary symmetric functions

$$e_r(X_n)=e_r(X_{n-1})+x_ne_{r-1}(X_{n-1})$$

it is a matter of simple row transformations to show that (cf. [K2], (3.2) and (4.13))

$$s_\lambda(X_n)=\det\left(h_{\lambda_i-i+j}(X_{n+1-j})\right)_{1\le i,j\le n},$$

and dually,

$$s_\lambda(X_n)=\det\left(e_{\lambda_i'-i+j}(X_{n+j-1})\right)_{1\le i,j\le m}.$$

To complete this Section we should mention the Cauchy identity

$$\sum_\lambda s_\lambda(X_n)s_{\widehat{\lambda'}}(Y_m)=\prod_{1\le i\le n}\prod_{1\le j\le m}(x_i+y_j), \tag{1.4}$$

summed over all partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \leq m$, where $\widehat{\lambda} = (\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n)$ is the complementary partition defined by $\widehat{\lambda}_i = m - \lambda_{n-i+1}$, and $\widehat{\lambda}'$ is the conjugate of $\widehat{\lambda}$.

- Generalized Schur functions.

The generalized Schur functions were introduced by D. Littlewood [L]. Namely, for any formal series $f = \sum_{k \geq 0} z^k s_k$, the generalized skew Schur functions are defined as the minors of the Hankel matrix

$$\mathbf{S}(f) = (s_{j-i})_{i,j \geq 0},$$

putting $s_i = 0$, if $i < 0$.

More precisely, for given partitions λ and μ such that $\mu \subset \lambda$, $l(\lambda) \leq n$, the generalized skew Schur function $s_{\lambda/\mu}(f)$ is defined to be

$$s_{\lambda/\mu}(f) = \det \left(s_{\lambda_i - \mu_j - i + j} \right)_{1 \leq i,j \leq n}.$$

Example 1. Let us take

$$f = \frac{\prod_{y \in Y} (1 - zy)}{\prod_{x \in X} (1 - zx)} = \sum_{k \geq 0} s_k(X - Y) z^k.$$

Then the generalized Schur function $s_{\lambda/\mu}(f) := s_{\lambda/\mu}(X - Y)$ coincides with the so–called super–Schur function [M1], Chapter I, §3, Example 23 .

§2. Quantum elementary and quantum complete homogeneous polynomials.

Let $X_n = (x_1, \ldots, x_n)$ and $Y_n = (y_1, \ldots, y_n)$ be two sets of variables, and $q = (q_1, \ldots, q_{n-1})$ and $q' = (q'_1, \ldots, q'_{n-1})$ be two sets of independent parameters. Follow to [GK], let us define the quantum elementary polynomials $e_i^q(X_k)$ from the decomposition of the Givental-Kim determinant $\displaystyle\sum_{i=0}^{k} e_i^q(X_k) t^{k-i} =$

$$= \det \begin{pmatrix} x_1 + t & q_1 & 0 & \ldots & \ldots & \ldots & 0 \\ -1 & x_2 + t & q_2 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & x_3 + t & q_3 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & x_{k-2} + t & q_{k-2} & 0 \\ 0 & \ldots & \ldots & 0 & -1 & x_{k-1} + t & q_{k-1} \\ 0 & \ldots & \ldots & \ldots & 0 & -1 & x_k + t \end{pmatrix} \qquad (2.1)$$

We define (see [FGP], [KM], or [K2], (3.4)) the quantum complete homogeneous polynomial $h_k^q(X_r)$ by the following formula

$$h_k^q(X_r) = \det \left(e_{1-i+j}^q (X_{r-1+j}) \right)_{1 \le i,j \le k} . \tag{2.2}$$

From the very definition (2.2) of quantum complete homogeneous polynomials, one can deduce, [K2], (3.4), the following "inversion formula":

$$e_k^q(X_r) = \det \left(h_{1-i+j}^q (X_{r+1-j}) \right)_{1 \le i,j \le k} . \tag{2.3}$$

Using the recurrence relations ($0 \le k \le m \le n$) for quantum elementary polynomials $e_k^q(X_m)$:

$$e_k^q(X_m) = e_k^q(X_{m-1}) + x_m e_{k-1}^q(X_{m-1}) + q_{m-1} e_{k-2}^q(X_{m-2})$$

it is a matter of simple row transformations to show that

$$h_k^q(X_r) = \det \left(e_{1-i+j}^q (X_{\min(r-1+j,n-1)}) \right) . \tag{2.4}$$

Now let us define the quantum super elementary polynomials $e_m^{q,q'}(X_k - Y_l)$, and quantum super complete homogeneous polynomials $h_m^{q,q'}(X_k - Y_l)$. Namely, let us put

$$h_m^{q,q'}(X_k - Y_l) = \sum_{j=0}^{m} h_{m-j}^q (X_k) e_j^{q'}(Y_l), \tag{2.5}$$

$$e_m^{q,q'}(X_k - Y_l) = \sum_{j=0}^{m} e_{m-j}^q (X_k) h_j^{q'}(Y_l), \tag{2.6}$$

It follows from (2.5) and (2.6) that the quantum super elementary and quantum super complete homogeneous polynomials satisfy the following duality

$$h_m^{q,q'}(X_k - Y_l) = e_m^{q',q}(Y_l - X_k). \tag{2.7}$$

More generally, let us introduce the "semi–universal" (cf. [F]) analogues of polynomials $h_m^{q,q'}(X_k - Y_l)$ and $e_m^{q,q'}(X_k - Y_l)$. To do this, let us consider two sets of independent variables $c_i(j)$ and $d_i(j)$ for $1 \le i \le j \le n-1$. It is convenient to put $c_i(j) = d_i(j) = 1$ if $i = 0$, and $d_i(j) = c_i(j) = 0$ if $i < 0$ or $i > j$. Then we define

$$e_m^{q'}(k|Y_l) = \sum_{j=0}^{m} c_{m-j}(k) h_j^{q'}(Y_l), \tag{2.8}$$

$$h_m^q(X_r|l) = \sum_{j=0}^{m} h_{m-j}^q(X_r) d_j(l). \tag{2.9}$$

Let us remark that polynomials $e_m^{q'}(k|Y_l)$ (resp. $h_m^q(X_r|k)$) are specialized to the quantum super elementary polynomials $e_m^{q,q'}(X_k - Y_l)$ (resp. to the quantum super complete homogeneous polynomials $h_m^{q,q'}(X_r - Y_l)$) when each $c_i(j)$ is sent to $e_i^q(X_j)$ (resp. each $d_i(j)$ is sent to $e_i^{q'}(Y_j)$). On the other hand, if in (2.8) we take $q' = 0$, then $e_m^{q'}(k|-Y_l)|_{q'=0} = f_m(k,l,0)$, where polynomials $f_m(k,a,b)$ were introduced in [F], (19).

Finally, let us define the quantum super multi–Schur functions $s_{\lambda/\mu}^{q,q'}(\mathcal{X},\mathcal{Y})$, where $\mathcal{X} = (X_{k_1},\dots,X_{k_m})$ and $\mathcal{Y} = (Y_{l_1},\dots,Y_{l_m})$ be two families of flagged sets of variables, and λ,μ be partitions of length $\le m$.

Definition 1. *The quantum super multi–Schur function $s_{\lambda/\mu}^{q,q'}(\mathcal{X},\mathcal{Y})$ is defined to be*

$$s_{\lambda/\mu}^{q,q'}(\mathcal{X},\mathcal{Y}) = \det\left(h^{q,q'}_{\lambda_i-\mu_j-i+j}(X_{k_i} - Y_{l_i})\right)_{1\le i,j\le m}. \tag{2.10}$$

§3. Universal double Schubert polynomials.

Follow to [F], let us define at first the universal Schubert polynomials $\mathfrak{S}_w(c)$. Let $c_i(j)$, $1 \le i \le k \le n$, be set of independent variables. It is convenient to define $c_0(j) = 1$, and $c_i(j) = 0$ if $i < 0$ or $i > j$. We start with definition of the universal Schubert polynomial $\mathfrak{S}_{w_0}(c)$ corresponding to the element of maximal length $w_0 \in S_n$:

$$\mathfrak{S}_{w_0}(c,y) = \prod_{i=1}^{n-1}\left(\sum_{j=0}^{i} y_{n-i}^j c_{i-j}(i)\right). \tag{3.1}$$

The next step is to define the double polynomials $\mathfrak{S}_w(c,y)$ for any permutation $w \in S_n$. Let us put [F], cf [KM],

$$\mathfrak{S}_w(c,y) = \partial^{(y)}_{ww_0}\mathfrak{S}_{w_0}(c,y), \tag{3.2}$$

where divided difference operator $\partial^{(y)}_{ww_0}$ acts on the y variables.

Definition 2 ([F]). *Let $w \in S_n$ be a permutation. The single universal Schubert polynomial $\mathfrak{S}_w(c)$ is defined to be the specialization $y_1 = \dots = y_n = 0$ of the double polynomial $\mathfrak{S}_w(c,y)$:*

$$\mathfrak{S}_w(c) = \mathfrak{S}_w(c,0).$$

Finally, let us define (see [F]) the universal double Schubert polynomials $\mathfrak{S}_w(c,d)$, where c stands for the variables $c_i(j)$ and d stands for another set of variables $d_i(j)$.

Definition 3 ([F]). *The universal double Schubert polynomial $\mathfrak{S}_w(c,d)$ is defined by the following formula*

$$\mathfrak{S}_w(c,d) = \sum_{u\in S_n}\mathfrak{S}_u(c)\mathfrak{S}_{uw^{-1}}(d), \tag{3.3}$$

where the sum is over all $u \in S_n$ such that $l(u) + l(uw^{-1}) = l(w)$.

It is clear from (3.3) that

$$\mathfrak{S}_w(c,d) = \mathfrak{S}_{w^{-1}}(d,c). \tag{3.4}$$

§4. Generalized quantum Cauchy identity.

The quantum Cauchy identity is the quantum analog of the Cauchy formula in the theory of Schubert polynomials ([M1], (5.10)). As it was shown in [KM], Section 4, the quantum Cauchy identity corresponds to the quantization of Cauchy's formula (1.4) with respect to the x variables.

The generalized quantum Cauchy identity corresponds to the quantization of quantum Cauchy identity ([KM], [K1]) with respect to the y variables.

Theorem 1. (Generalized quantum Cauchy identity for quantum Schubert polynomials)

$$\sum_{w \in S_n} \mathfrak{S}_w^q(X_n)\mathfrak{S}_{ww_0}^{q'}(Y_n) = \mathfrak{S}_{w_0}^{q,q'}(X_n, Y_n), \tag{4.1}$$

where $\mathfrak{S}_{w_0}^{q,q'}(X_n,Y_n)$ is given by

$$\mathfrak{S}_{w_0}^{q,q'}(X_n,Y_n) = \det\left(h_{n-2i+j}^{q,q'}(X_i - Y_{n-i})\right)_{1 \le i,j \le n-1}.$$

Theorem 1 follows from the following more general result:

Theorem 2.

$$\sum_{w \in S_n} \mathfrak{S}_w^q(X_n)\mathfrak{S}_{ww_0}(d) = \mathfrak{S}_{w_0}^q(X_n, d), \tag{4.2}$$

where

$$\mathfrak{S}_{w_0}^q(X_n,d) = \det\left(h_{n-2i+j}^q(X_i|n-i)\right)_{1 \le i,j \le n-1},$$

and $h_m^q(X_r|l)$ is defined by (2.9).

Proof of Theorem 2. First of all, if all $q_i = 0$, the formula (4.2) follows from [F], Lemma 2.1. General case follows from the following statement:

Lemma 1. *Let $c_i(j)$ and $d_i(j)$ for $1 \le i \le j \le n-1$ be two sets of independent variables, and let*

$$e_m(k|l) := \sum_{j=0}^{m} c_{m-j}(k)d_j(l)$$

be "universal" elementary polynomial. Then

$$\det\left(e_{n-2i+j}(i|n-i)\right)_{1 \le i,j \le n-1} = \sum_{I \subset \delta_n} s_I d_{\delta_n - I}, \tag{4.3}$$

where

$$s_I := \det\left(c_{i_\alpha-\alpha+\beta}(\alpha)\right)_{1\le\alpha,\beta\le n-1}, \quad if \quad I=(i_1,\ldots,i_{n-1});$$
$$d_J := \prod_{k=1}^{n-1} d_{j_k}(n-k);$$
$$c_0(j)=d_0(j)=1, \quad c_i(j)=d_i(j)=0, \ if \ i<0 \ \ or \ \ i>j.$$

Indeed, using Lemma 1, one can rewrite the formula (4.2) with $q=0$ in the following form

$$\sum_{w\in S_n} \mathfrak{S}_w(X_n)\mathfrak{S}_{ww_0}(d) = \mathfrak{S}_{w_0}(X_n,d) = \sum_{I\subset\delta_n} s_I(X_1,\ldots,X_{n-1})d_{\delta_n-I} = \sum_{I\subset\delta_n} x^I d_{\delta_n-I}. \tag{4.4}$$

We used here the following formula ([M3], (3.5'))

$$x^I = \det\left(h_{i_\alpha-\alpha+\beta}(X_\alpha)\right)_{1\le\alpha,\beta\le n-1} = s_I(X_1,\ldots,X_n).$$

Hence, using the properties of quantization map ([FGP], [KM], [K2]) with respect to the x variables, we obtain

$$\sum_{w\in S_n} \mathfrak{S}_w^q(X_n)\mathfrak{S}_{ww_0}(d) = \sum_{I\subset\delta_n} \widetilde{x}^I d_{\delta_n-I} = \sum_{I\subset\delta_n} s_I^q(X_1,\ldots,X_{n-1})d_{\delta_n-I}$$
$$= \det\left(h^q_{n-2i+j}(X_i|n-i)\right)_{1\le i,j\le n-1} = \mathfrak{S}^q_{w_0}(X_n,d).$$

Here we used the following result ([K2], Corollary 4): if $I\subset\delta_n$, then $\widetilde{x}^I = s_I^q(X_1,\ldots,X_{n-1})$, where $\widetilde{x}^I$ is the quantization of monomial $x^I = x_1^{i_1}\ldots x_{n-1}^{i_{n-1}}$, and

$$s_I^q(X_1,\ldots,X_{n-1}) = \det\left(h^q_{i_\alpha-\alpha+\beta}(X_\alpha)\right)_{1\le\alpha,\beta\le n-1}.$$

■

Corollary 1.

$$\sum_{w\in S_n} \mathfrak{S}_w^{q,q"}(X_n,Z_n)\mathfrak{S}_{ww_0}^{q',q"}(Y_n,-Z_n) = \mathfrak{S}_{w_0}^{q,q'}(X_n,Y_n), \tag{4.5}$$

$$\sum_{\substack{u\in S_n \\ l(u)+l(uw^{-1})=l(w)}} \mathfrak{S}_u^{q,q"}(X_n,Z_n)\mathfrak{S}_{uw^{-1}}^{q',q"}(Y_n,-Z_n) = \mathfrak{S}_w^{q,q'}(X_n,Y_n). \tag{4.6}$$

Here the double quantum Schubert polynomials $\mathfrak{S}_w^{q,q'}(X_n,Y_n)$ is defined as follows (cf. [F])

$$\mathfrak{S}_w^{q.q'}(X_n,Y_n) = \sum_{u\in S_n} \mathfrak{S}_u^q(X_n)\mathfrak{S}_{uw^{-1}}^{q'}(Y_n),$$

where the sum is over all $u \in S_n$ such that $l(u) + l(uw^{-1}) = l(w)$.

Remark 2. It follows from (4.6) that the double quantum Schubert polynomial $\mathfrak{S}_w^{q,q'}(X_n, Y_n)$ can be obtained from the universal double Schubert polynomial $\mathfrak{S}_w(c,d)$ (see [F]) under the specialization

$$c_i(k) \longrightarrow e_i^q(X_k),$$
$$d_j(l) \longrightarrow e_j^{q'}(Y_l).$$

Remark 3. Polynomial $\mathfrak{S}_{w_0}^{q,q'}(X_n, Y_n)$ appears at first in [KM], Remark 11, and corresponds to the dual class of the diagonal in the quantum cohomology ring

$$QH^*(Fl_n, q) \otimes QH^*(Fl_n, q').$$

The specialization $\Phi(x) := \mathfrak{S}_w^{q,q}(X_n, X_n)$ plays an important role in the proof of the Vafa–Intriligator type formula for the quantum cohomology ring of flag variety ([KM], Section 8.1). On the other hand, it follows from (4.2) that $\mathfrak{S}_w^{q,q}(X_n, -X_n) = 0$, if $w \neq \mathrm{id}$.

§5. Cauchy identity for universal Schubert polynomials.

In this Section we are going to formulate the Cauchy identity for the universal Schubert polynomials (see [F], or Section 3). We start with definition of polynomials $\mathfrak{S}_I(c)$, $I \subset \delta_n := (n-1, n-2, \ldots, 1, 0)$. Let us remark, that if $I \subset \delta_n$, then the monomial $x^I := x_1^{i_1} \cdots x_{n-1}^{i_{n-1}}$ is a $\mathbf{Z}$–linear combination of elementary polynomials

$$e_J(X_{n-1}) := \prod_{k=1}^{n-1} e_{j_k}(X_{n-k}), \quad \text{say,}$$
$$x^I = \sum_{J \subset \delta_n} \alpha_{I,J} e_J(X_{n-1}), \tag{5.1}$$

and such representation (5.1) is unique ([LS]).

Let us define

$$\mathfrak{S}_I(c) = \sum_{J \subset \delta_n} \alpha_{I,J} c_J, \tag{5.2}$$

where $c_J = \prod_{k=1}^{n-1} c_{j_k}(n-k)$.

Theorem 3 (Cauchy identity for universal Schubert polynomials).

$$\mathfrak{S}_{w_0}(c,d) = \sum_{I \subset \delta_n} \mathfrak{S}_I(c) d_{\delta_n - I}, \tag{5.3}$$

where $d_J = \prod_{k=1}^{n-1} d_{j_k}(n-k)$.

Proof. Under the specialization $c_i(k) \to e_i^q(X_k)$, the formula (5.3) is reduced to (4.2). General case follows from [F], Lemma 2.1. ■

Corollary 2. *Let* $b = \{b_j(k)\}$ *be the third set of variables such that* $b_0(k) = 1$, *and* $b_j(k) = 0$, *if* $j < 0$ *or* $j > k$, *then*

$$\sum_{w \in S_n} \mathfrak{S}_w(c,b)\mathfrak{S}_{ww_0}(d,\widetilde{b}) = \mathfrak{S}_{w_0}(c,d), \tag{5.4}$$

$$\sum_{u \in S_n} \mathfrak{S}_u(c,b)\mathfrak{S}_{uw^{-1}}(d,\widetilde{b}) = \mathfrak{S}_w(c,d), \tag{5.5}$$

where $\widetilde{b}_j(k) = (-1)^j b_j(k)$.

Corollary 3. $\mathfrak{S}_w(b,\widetilde{b}) = 0$, *if* $w \neq \mathrm{id}$.

Proof. Let us take $b = d$ in (5.5). ■

Remark 4. Corollary 3 was formulated by W. Fulton as a Conjecture (see [F], (30)).

§6. Cauchy identity for universal Schubert polynomials of the second form.

Let us remind [F] the definition of the universal Schubert polynomials of the second form. Follow to [F], let us consider a set of variables $g_i[j]$ for $i \geq 1$ and $j \geq 0$ with $i + j \leq n$; and let us put $\deg(g_i[j]) = j + 1$. Now we are going to define the generalized elementary functions $\square_i(k)$. Let us consider the generalized Givental–Kim determinant (cf. [F]) $\square_k(t,g) = \det(tI_k + A_k)$, where A_k is the $k \times k$ matrix with $g_i[j-i]$ in the (i,j) position for $i \leq j$, and with -1 in position $(i+1,i)$ below the diagonal.

Definition 4. *The generalized elementary functions* $\square_i(k)$ *are defined from the decomposition*

$$\square_k(t,g) = \sum_{j=0}^{k} t^{k-j}\square_j(k). \tag{6.1}$$

Definition 5 ([F]). *The universal Schubert polynomial of the second form, denoted* $\mathfrak{S}_w(g)$, *is obtained from* $\mathfrak{S}_w(c)$ *by replacing each* $c_i(k)$ *by the generalized elementary function* $\square_i(k)$.

Cauchy's type identity for polynomials $\mathfrak{S}_w(g)$, can be obtained as a simple corollary of the Cauchy identity for universal Schubert polynomials (Theorem 3).

Corollary 4. *Let us define* $\mathfrak{S}_{w_0}(g,Y_n) := \prod_{j=1}^{n-1} \square_j(y_{n-j},g)$. *Then*

$$\sum_{w\in S_n} \mathfrak{S}_w(g)\mathfrak{S}_{ww_0}(Y_n) = \mathfrak{S}_{w_0}(g,Y_n), \tag{6.2}$$

$$\sum_{w\in S_n} \mathfrak{S}_w(g,Z_n)\mathfrak{S}_{ww_0}(Y_n,-Z_n) = \mathfrak{S}_{w_0}(g,Y_n), \tag{6.3}$$

where $\mathfrak{S}_w(g,Z_n) := \partial^{(z)}_{ww_0}\mathfrak{S}_{w_0}(g,Z_n)$.

Remark 5. The classical Schubert polynomials can be recovered from $\mathfrak{S}_w(g)$ by setting $g_i[0]=x_i$ and $g_i[j]=0$ for $j\ge 1$, and the quantum Schubert polynomial $\mathfrak{S}^q_w$ can be recovered from $\mathfrak{S}_w(g)$ by setting $g_i[0]=x_i$, $g_i[1]=q_i$, and $g_i[j]=0$ for $j\ge 2$.

§7. Orthogonality.

Let us put $g_i[0]=x_i$, $1\le i\le n$, and consider the polynomial ring $\mathbf{Z}[g]$ in variables $g_i[j]$ for $j>0$ and $i+j\le n$. Follow [F], let us consider a universal ring

$$R_n = \mathbf{Z}[g][x_1,\ldots,x_n]/\mathcal{R},$$

where the ideal $\mathcal{R}\subset \mathbf{Z}[g][x_1,\ldots,x_n]$ is generated by the generalized elementary functions $\square_i(n)$, $1\le i\le n$. Here we consider $\square_i(n)$ as polynomial in x and g variables. There exists a natural pairing $\langle,\rangle_{\mathcal{R}}$ on the ring R_n with values in $\mathbf{Z}[g]$, which is induced by the Grothendieck residue with respect to the ideal $\mathcal{R}$. Namely, if $f\in\mathbf{Z}[g][X_n]$, then consider the image $\overline{f}$ of f in the quotient ring R_n, and then picking off the coefficient of $\mathfrak{S}_{w_0}(g)$, or the coefficient of $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$.

Conjecture 1 (cf. [F], (29)). *Universal Schubert polynomials $\mathfrak{S}_w(g)$ can be obtained as the Gram-Schmidt orthogonalization of the set of lexicographically ordered monomials $\{x^I | I\subset\delta_n\}$ with respect to the residue pairing $\langle,\rangle_{\mathcal{R}}$:*

$$\langle\mathfrak{S}_u(g),\mathfrak{S}_v(g)\rangle_{\mathcal{R}} = \langle\mathfrak{S}_u,\mathfrak{S}_v\rangle = \begin{cases}1, & \text{if } v=w_0u,\\ 0, & \text{otherwise };\end{cases}$$

$$\mathfrak{S}_w(g) = x^{c(w)} + \sum_{I<c(w)} a_I(g)x^I,$$

where $a_I(g)\in\mathbf{Z}[g]$, and $I<c(w)$ means the lexicographical order, and $c(w)$ is the code of a permutation $w\in S_n$, [M3], p.9.

It is clear from (6.1) that

$$\mathfrak{S}_{w_0}(g,Y_n) = \sum_{I\subset\delta_n} \square_I(g)y^{\delta_n-I}, \tag{7.1}$$

where

$$\Box_I(g) = \prod_{k=1}^{n-1} \Box_{i_k}(n-k). \tag{7.2}$$

Conjecture 2. *If $I \subset \delta_n$, and $J \subset \delta_n$, then*

$$\langle \Box_I(g), \Box_J(g) \rangle_{\mathcal{R}} = \langle e_I(x), e_J(x) \rangle, \tag{7.3}$$

where $e_I(x) = \prod_{k=1}^{n-1} e_{i_k}(X_{n-k})$.

As it was shown in [KM], Conjecture 1 follows from Conjecture 2. Conjecture 2 can be proven on the way suggested in [KM], using the recurrence relations for $\Box_I(g)$. Details will appear elsewhere.

§8. Multiparameter deformation of Schubert polynomials.

In the joint paper with S. Fomin [FK] we introduced the quadratic algebra $\mathcal{E}_n^q$ ([FK], Definition 2.1) which is closely related to the small quantum cohomology ring of the flag variety. More precisely, we construct a commutative subalgebra in $\mathcal{E}_n^q$ which appears to be canonically isomorphic to the small quantum cohomology ring of the flag variety ([FK], [P]). This construction admits a natural generalization (see [FK], Section 15) which gives rise to another natural generalization of the classical and quantum Schubert polynomials. Let us briefly explain this construction. Namely, let us consider the set of variables $t = \{t_{ij} | 1 \le i < j \le n\}$, and the polynomial ring $\mathbf{Z}[t][X_n] := \mathbf{Z}[t][x_1, \ldots, x_n]$. Let $\mathcal{T}$ be an ideal in $\mathbf{Z}[t][X_n]$ which is generated by the following generalization of elementary symmetric functions $e_m(X_k)$:

$$e_m(t|X_k) = \sum_l \sum_{\substack{1 \le i_1 < \cdots < i_l \le n \\ j_1 > i_1, \ldots, j_l > i_l}} e_{m-2l}(X_{\overline{I \cup J}}) \prod_{k=1}^{l} t_{i_k j_k}, \tag{8.1}$$

where $i_1, \ldots, i_l, j_1, \ldots, j_l$ should be distinct elements of the set $\{1, \ldots, n\}$, and $X_{\overline{I \cup J}}$ denotes the set of variables x_a for which the subscript a is neither one of the i_k nor one of the j_k.

Let us consider the quotient ring

$$B_n := \mathbf{Z}[t][X_n]/\mathcal{T}.$$

It is easy to see that $\dim B_n = n!$, and there exists a natural pairing $\langle , \rangle_{\mathcal{T}}$ on the ring B_n which is given by the Grothendieck residue with respect to the ideal $\mathcal{T}$ (see Section 7).

Follow to the general strategy of [KM], we define a new multiparameter deformation of the quantum Schubert polynomials:

Definition 6. *Define the multiparameter Schubert polynomials, denoted by* $\mathfrak{S}_w^t$*, as the Gram–Schmidt orthogonalization of the set of lexicographically ordered monomials* $\{x^I | I \subset \delta_n\}$ *with respect to the residue pairing* $\langle , \rangle_{\mathcal{T}}$*:*

$$\langle \mathfrak{S}_u^t, \mathfrak{S}_v^t \rangle_{\mathcal{T}} = \langle \mathfrak{S}_u, \mathfrak{S}_v \rangle = \begin{cases} 1, & \text{if } v = w_0, \\ 0, & \text{otherwise;} \end{cases}$$

$$\mathfrak{S}_w^t = x^{c(w)} + \sum_{I < c(w)} a_I(t) x^I,$$

where $a_i(t) \in \mathbf{Z}[t]$*, and* $I < c(w)$ *means the lexicographical order.*

One can easily check, that if $w \in S_n$, $x^I = x_1^{i_1}, \ldots, x_n^{i_n}$, then

$$\langle w(x^I) \rangle_{\mathcal{T}} = (-1)^{l(w)} w \left(\langle x^I \rangle_{\mathcal{T}} \right). \tag{8.2}$$

Example 2. For the symmetric group S_3 we have

$$\begin{aligned} \langle x_1^3 x_2^2 \rangle_{\mathcal{T}} &= -2t_{12} - t_{13}, \\ \langle x_1^4 x_2 \rangle_{\mathcal{T}} &= t_{12} + 2t_{13} \end{aligned}$$

(Hint: $x_1^3 \equiv t_{12}(2x_1 + x_2) + t_{13}(x_1 - x_2) \bmod \mathcal{T}$).

Consequently,

$$\begin{aligned} \mathfrak{S}_{321}^t &= x_1^2 x_2 + t_{12} x_1 - t_{13} x_2, \\ \mathfrak{S}_{231}^t &= x_1 x_2 + t_{12}, \\ \mathfrak{S}_{321}^t &= x_1^2 - t_{12} - t_{13}, \\ \mathfrak{S}_{132}^t &= x_1 + x_2, \\ \mathfrak{S}_{213}^t &= x_1, \\ \mathfrak{S}_{123}^t &= 1. \end{aligned}$$

Example 3. For the symmetric group S_4 we have

$$\begin{aligned} \langle x_1^5 x_2^2 x_3 \rangle_{\mathcal{T}} &= t_{12} + 2t_{13} + 3t_{14}, \\ \langle x_1^4 x_2^3 x_3 \rangle_{\mathcal{T}} &= -3t_{12} - t_{13} - 2t_{14}, \\ \langle x_1^4 x_2^2 x_3^2 \rangle_{\mathcal{T}} &= 2t_{12} - 2t_{13}, \\ \langle x_1^3 x_2^3 x_3^2 \rangle_{\mathcal{T}} &= 2t_{13} + t_{14} - 2t_{23} - t_{24}, \\ \langle x_1^5 x_2^3 \rangle_{\mathcal{T}} &= t_{14} - t_{13}, \\ \langle x_1^5 x_2^3 x_3^2 \rangle_{\mathcal{T}} &= -3t_{12}^2 + 2t_{13}^2 + t_{14}^2 + 8t_{12}t_{13} + 6t_{13}t_{14} + t_{12}t_{14} - 2t_{12}t_{23} \\ &\quad - t_{12}t_{24} - t_{13}t_{34} - 4t_{13}t_{23} - 6t_{14}t_{23} - 4t_{14}t_{24} + t_{14}t_{34}, \\ &\text{and so on.} \end{aligned}$$

Using the symmetry property (8.2) and residue formulae above, one can find all multiparameter Schubert polynomials $\mathfrak{S}_w^t$, $w \in S_4$. We will give the answer only for $\mathfrak{S}_{w_0}^t$. Namely,

$$\begin{aligned}
\mathfrak{S}_{4321}^t &= x_1^3x_2^2x_3 + 2t_{12}x_1^2x_2x_3 - (t_{13}+2t_{14})x_1x_2^2x_3 + t_{13}x_1^2x_2^2 \\
&+ (t_{14}+t_{23})x_1^3x_2 - t_{24}x_1^3x_3 + (t_{12}t_{14}+t_{12}t_{23}-t_{13}t_{24})x_1^2 \\
&+ (t_{12}t_{13}-t_{12}t_{14}-2t_{13}t_{14}-t_{13}t_{23}-t_{14}^2-2t_{14}t_{23})x_1x_2 \\
&+ (t_{12}^2+t_{12}t_{14}+t_{12}t_{24}+t_{13}t_{24}+t_{14}t_{24})x_1x_3 \\
&+ (-t_{13}^2+t_{14}^2-t_{13}t_{14}-t_{13}t_{24}+t_{14}t_{34})x_2^2 \\
&+ (-t_{12}t_{13}-t_{12}t_{14}-t_{13}t_{14}+t_{14}^2)x_2x_3.
\end{aligned}$$

There exists an alternative way to compute the multiparameter Schubert polynomials, using the algebra $\mathcal{E}_n^t$ from [FK], Section 15. We are going to present detailed exposition in a separate publication. Let us say only, that the multiparameter Schubert polynomials have many nice properties such as stability, orthogonality, Pieri's type formula (cf. [FK], [P]), and so on. However, the structural constants $\alpha_{u,v}^w(t)$:

$$\mathfrak{S}_u^t \cdot \mathfrak{S}_v^t = \sum_w \alpha_{u,v}^w(t)\mathfrak{S}_w^t$$

appear to be the polynomials in t's which may have, in general, some negative coefficients.

It seems an interesting task to understand a connection between universal Schubert polynomials $\mathfrak{S}_w(g)$ [F], and multiparameter Schubert polynomials $\mathfrak{S}_w^t$.

§9. Universal Schur and universal factorial Schur functions.

In this section, we introduce the universal Schur functions $s_\lambda(h)$, and universal factorial Schur functions $s_\lambda(g,h)$, and study their basic properties.

Definition 7. *Let λ be a partition, $\lambda \subset ((n-r)^r)$, $1 \le r < n$. The universal Schur function $s_\lambda(g)$ is defined as the universal Schubert polynomial $\mathfrak{S}_w(g)$, corresponding to the Grassmannian permutation $w \in S_n$ of shape λ and descent at r.*

Definition 8. *Let λ be a partition such that $\lambda_1 \le n-r$ and $l(\lambda) \le r$ for some r. We define a universal factorial Schur function $s_\lambda(g,h)$ to be equal to the universal double Schubert polynomial $\mathfrak{S}_w(g,h)$, where $w \in S_n$ is the Grassmannian permutation of shape λ and descent at r.*

Now we are going to explain a connection between the universal Schur functions and the Macdonald construction of the "9-th variation" of Schur functions, [M1], Chapter 1, §3, Example 21, or [M2], 9-th Variation. Let us remind at first some definitions from [M2].

Let $h_i(k)$ ($i \geq 1$, $k \in \mathbf{Z}$) be independent indeterminates over $\mathbf{Z}$. Also, for convenience, define $h_0(k) = 1$ and $h_i(k) = 0$ for $i < 0$ and all $k \in \mathbf{Z}$. Define an automorphism of the ring R generated by the $h_i(k)$ by

$$\varphi(h_i(k)) = h_i(k+1), \quad \text{for all} \ \ i, k.$$

Thus $h_i(k) = \varphi^k h_i$, where $h_i := h_i(0)$.

Now define [M2], for any two partitions λ, μ of length $\leq n$,

$$s_{\lambda/\mu}(h) = \det\left(\varphi^{\mu_j - j + 1} h_{\lambda_i - \mu_j - i + j}\right)_{1 \leq i,j \leq n}, \tag{9.1}$$

and in particular ($\mu = 0$)

$$s_{\lambda}(h) = \det\left(\varphi^{-j+1} h_{\lambda_i - i + j}\right)_{1 \leq i,j \leq n}. \tag{9.2}$$

From (9.1) it follows that $h_r = s_{(r)}$ ($r \geq 0$), and we define

$$e_r := e_r(h) = s_{(1^r)}(h), \tag{9.3}$$

for all $r \geq 0$, and $e_r = 0$ for $r < 0$.

Schur functions $s_\lambda := s_\lambda(h)$ possess many properties similar to those of the classical Schur functions. For example,

- $s_{\lambda/\mu} = 0$, unless $\lambda \supset \mu$;
- Nägelsbach–Kostka's formula:

$$s_{\lambda/\mu} = \det\left(\varphi^{-\mu'_j + j - 1} e_{\lambda'_i - \mu'_j - i + j}\right)_{1 \leq i,j \leq n}; \tag{9.4}$$

- Giambelli's formula: if $\lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r)$ in Frobenius notation, then

$$s_\lambda = \det\left(s_{(\alpha_i|\beta_j)}\right)_{1 \leq i,j \leq r}. \tag{9.5}$$

See [M1], [M2] where the proofs and additional results are contained. For quantum Schur functions the formulae (9.4) and (9.5) were proven in [K2].

Now we are ready to explain a connection between the universal Schur functions and the Macdonald 9-th variation of Schur functions.

Proposition 1. *Let $w \in S_n$ be a Grassmannian permutation with shape λ and descent at r. Then*

$$\mathfrak{S}_w(c) = \varphi^r s_\lambda(c), \tag{9.6}$$

where the Schur function $s_\lambda(c)$ is obtained from $s_\lambda(h)$ (see (9.1)) by setting $h_i(j) \to c_i(j)$.

Proof follows from [F], Proposition 4.4, or Section 10.

It follows from Proposition 1, that the universal Schur functions $s_\lambda(g)$ satisfy (9.4) and (9.5) with $\mu = 0$ (cf. [K2]).

§10. Determinantal formulae.

In this Section we give a generalization of some determinantal formulae obtained in [K2], Section 4, and [F], Proposition 4.4. To start, let us remind a few definitions:

- $\mathfrak{S}_w^q(X_n, d)$ is the specialization $c_j(k) \to e_j^q(X_k)$ of the universal double Schubert polynomial $\mathfrak{S}_w(c, d)$;
- $\mathfrak{S}_w^{q'}(c, Y_n)$ is the specialization $d_j(k) \to e_j^{q'}(Y_k)$ of $\mathfrak{S}_w(c, d)$;
- if $q' = 0$ then $\mathfrak{S}_w^{q'}(c, Y_n)|_{q'=0} = \mathfrak{S}_w(c, y)$, where double polynomial $\mathfrak{S}_w(c, y)$ is defined in [F], or Section 3, (3.2).

Theorem 4. *Let $w \in S_n$ be a Grassmannian permutation with shape λ and descent at r, and let $\widetilde{\theta} = \theta(w^{-1})$ be the flag of inverse permutation w^{-1}. Then*

$$\mathfrak{S}_w^{q'}(c, Y_n) = \det\left(e^{q'}_{\lambda'_i - i + j}(r - 1 + j | Y_{\widetilde{\theta}_i})\right)_{1 \le i, j \le n-r}. \tag{10.1}$$

Proof. For $q' = 0$, the formula (10.1) coincides with Proposition 4.4 in [F]. For general q', (10.1) can be obtained from the particular case $q' = 0$ by applying the quantization map with respect to the y variables.

■

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