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Hankel determinants of sums of consecutive weighted Schröder numbers[☆]

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ABSTRACT

We consider weighted large and small Schröder paths with up steps $(1, 1)$, down steps $(1, -1)$ assigned the weight of 1 and with level steps $(2, 0)$ assigned the weight of t, where t is a real number. The weight of a path is the product of the weights of all its steps. Let $r_\ell^{(t)}$ and $s_\ell^{(t)}$ be the total weight of all weighted large and small Schröder paths from $(0, 0)$ to $(2\ell, 0)$, respectively. For constants α, β, we derive the generating functions and the explicit formulae for the determinants of the Hankel matrices $(\alpha r^{(t)}_{i+j-2}+\beta r^{(t)}_{i+j-1})^n_{i,j=1}$, $(\alpha r^{(t)}_{i+j-1}+\beta r^{(t)}_{i+j})^n_{i,j=1}$, $(\alpha s^{(t)}_{i+j-2}+\beta s^{(t)}_{i+j-1})^n_{i,j=1}$ and $(\alpha s^{(t)}_{i+j-1}+\beta s^{(t)}_{i+j})^n_{i,j=1}$ combinatorially via suitable lattice path models.

1. Introduction

1.1. Hankel determinants of Catalan, Motzkin, and Schröder numbers

Let $\{a_\ell\}_{\ell\geq 0}$ be a sequence. For a nonnegative integer k, let $A_n^{(k)}$ denote the *Hankel matrix* of order n of the sequence $\{a_\ell\}_{\ell\geq 0}$ of the form

$$A_n^{(k)} = (a_{k+i+j-2})^n_{i,j=1}. \tag{1}$$

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When $\{a_\ell\}_{\ell\geq 0}$ is one of the three classical combinatorial sequences (Catalan, Motzkin and Schröder numbers) arising from lattice path enumeration, the problem to evaluate the determinant $\det(A_n^{(k)})$ has been extensively studied. Readers may be referred to [3,16,18,20] for more examples, especially the comprehensive references listed in [20].

We give a quick introduction. The Catalan number $c_\ell = \frac{1}{\ell+1}\binom{2\ell}{\ell}$ counts the number of Dyck paths of length ℓ, which are the lattice paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,0)$ to $(2\ell,0)$ using *up* steps $\mathsf{U}=(1,1)$ and *down* steps $\mathsf{D}=(1,-1)$ that never pass below the x-axis. It is folklore that $\det_{1\leq i,j\leq n}(c_{i+j-2})=1$, $\det_{1\leq i,j\leq n}(c_{i+j-1})=1$ and $\det_{1\leq i,j\leq n}(c_{i+j})=n+1$. Desainte-Catherine and Viennot [12] proved that $\det_{1\leq i,j\leq n}(c_{i+j+k-2})=\prod_{1\leq i\leq j\leq k-1}\frac{i+j+2n}{i+j}$. Gessel and Viennot [15] gave an evaluation of $\det_{0\leq i,j\leq n-1}(c_{\alpha_i+j})$ for nonnegative integers $\alpha_0,\alpha_1,\ldots,\alpha_{n-1}$. An extension of this study is recently given by Krattenthaler [20].

The Motzkin numbers $\{m_\ell\}_{\ell\geq 0}=\{1,1,2,4,9,21,51,\ldots\}$ count the number of Motzkin paths of length ℓ, which are the lattice paths from $(0,0)$ to $(\ell,0)$ using up steps, down steps and unit level steps $(1,0)$ that never pass below the x-axis. It is known that $\det_{1\leq i,j\leq n}(m_{i+j-2})=1$ for all positive integer n and $\det_{1\leq i,j\leq n}(m_{i+j-1})$ equals 1 if $n\equiv 0,1 \pmod 6$, equals 0 if $n\equiv 2,5 \pmod 6$, and equals -1 if $n\equiv 3,4 \pmod 6$. See for instance [2,24].

The large Schröder numbers $\{r_\ell\}_{\ell\geq 0}=\{1,2,6,22,90,394,1806,\ldots\}$ count the number of large Schröder paths of length ℓ, which are the lattice paths from $(0,0)$ to $(2\ell,0)$ using up steps, down steps and level steps $\mathsf{L}=(2,0)$ that never pass below the x-axis. Furthermore, the small Schröder numbers $\{s_\ell\}_{\ell\geq 0}=\{1,1,3,11,45,197,903,\ldots\}$ count the number of small Schröder paths of length ℓ, which are large Schröder paths of length ℓ without level steps on the x-axis. By applying the Lindström–Gessel–Viennot lemma, Eu and Fu [13] proved that $\det_{1\leq i,j\leq n}(r_{i+j-2})=2^{\binom{n}{2}}$, $\det_{1\leq i,j\leq n}(r_{i+j-1})=2^{\binom{n+1}{2}}$, $\det_{1\leq i,j\leq n}(s_{i+j-2})=2^{\binom{n}{2}}$, and $\det_{1\leq i,j\leq n}(s_{i+j-1})=2^{\binom{n}{2}}$. At the same time, Brualdi and Kirkland also obtained the results in the case of large Schröder numbers via linear algebra [6].

Note that once $\det(A_n^{(0)})$ and $\det(A_n^{(1)})$ are determined, the evaluation of $\det(A_n^{(k)})$ can be obtained for all $k\geq 2$ by the following relation:

$$\det(A_{n+1}^{(k)})\det(A_{n-1}^{(k+2)})=\det(A_n^{(k)})\det(A_n^{(k+2)})-\det(A_n^{(k+1)})^2, \tag{2}$$

for $n\geq 1$, which is known as the *condensation identity*. This statement is due to Desnanot, and the first rigorous proof is given by Jacobi; see [5, Ch. 4,17, Sec. 3,21, pp. 140–142].

1.2. Hankel determinants of sums of two consecutive terms

A variation is to consider the Hankel determinant of the sequence $\{a_\ell+a_{\ell+1}\}_{\ell\geq 0}$, i.e., to evaluate

$$\det_{1\leqslant i,j\leqslant n}(a_{k+i+j-2}+a_{k+i+j-1}).$$

For Catalan numbers, Cvetković, Rajković and Ivković [11] proved algebraically that

$$\det_{1\leq i,j\leq n}(c_{i+j-2}+c_{i+j-1})=f_{2n+1} \quad\text{and}\quad \det_{1\leq i,j\leq n}(c_{i+j-1}+c_{i+j})=f_{2n+2}, \tag{3}$$

where f_n is the nth Fibonacci number. This elegant result stimulated several follow-up papers; see for instance [4,8–10,20,22].

The case for Motzkin numbers was also done by several authors [7,10]. One can generalize to a weighted version. For a real number t, a t-Motzkin path is a Motzkin path in which the up step, down step, and unit level step have weights 1, 1 and t, respectively, and the weight of a path is the product of the weights of all its steps. Let $m_\ell^{(t)}$ be the total weight of all t-Motzkin paths of length ℓ, then the

Hankel determinants $\det_{1\le i,j\le n}(m^{(t)}_{k+i+j-2})$ and $\det_{1\le i,j\le n}(m^{(t)}_{k+i+j-1})$ were computed in [18,23], for example. By using lattice path arguments, Cameron and Yip [7] also obtained recurrence formulae for the determinant

$$\det_{1\le i,j\le n}(m^{(t)}_{k+i+j-2}+m^{(t)}_{k+i+j-1}) \tag{4}$$

in the case where $k=0$ or $k=1$.

In this paper, by a *weighted large* (or *small*) *Schröder path* we mean a large (or small) Schröder path in which the steps U, D, L have weights 1, 1 and t, respectively. Let $r^{(t)}_\ell$ (respectively, $s^{(t)}_\ell$) denote the total weight of all weighted large (respectively, small) Schröder paths of length ℓ. Note that $r^{(t)}_0=s^{(t)}_0$ and $r^{(t)}_\ell=(1+t)s^{(t)}_\ell$, for $\ell\ge 1$. Recently, Sulanke and Xin [23] proved that

$$\det_{1\le i,j\le n}(r^{(t)}_{i+j-2})=(1+t)^{\binom{n}{2}} \quad\text{and}\quad \det_{1\le i,j\le n}(r^{(t)}_{i+j-1})=(1+t)^{\binom{n+1}{2}}. \tag{5}$$

Hence it is natural to consider the Hankel determinants of the sequence of sum of weighted large or small Schröder numbers. Rajković, Petković, and Barry [22] gave the following explicit formula:

$$\begin{aligned}\det_{1\le i,j\le n}\left(r^{(t)}_{i+j-2}+r^{(t)}_{i+j-1}\right)=\frac{L^{\binom{n}{2}}}{2^{n+1}\sqrt{L^2+4}}&\left(\left(\sqrt{L^2+4}+L\right)\left(\sqrt{L^2+4}+L+2\right)^n\right.\\&\left.+\left(\sqrt{L^2+4}-L\right)\left(L+2-\sqrt{L^2+4}\right)^n\right),\end{aligned} \tag{6}$$

where $L=1+t$. Their proof was done algebraically in terms of orthogonal polynomials.

1.3. Main results

In this paper, we will evaluate Hankel determinant of the sequence of linear combinations of two consecutive weighted large (or small) Schröder numbers. For $n\ge 1$, define

$$\begin{aligned}\Theta_n&=(1+t)^{-\binom{n}{2}}\det_{1\le i,j\le n}\left(\alpha r^{(t)}_{i+j-2}+\beta r^{(t)}_{i+j-1}\right),\\ \Phi_n&=(1+t)^{-\binom{n+1}{2}}\det_{1\le i,j\le n}\left(\alpha r^{(t)}_{i+j-1}+\beta r^{(t)}_{i+j}\right),\\ \Psi_n&=(1+t)^{-\binom{n}{2}}\det_{1\le i,j\le n}\left(\alpha s^{(t)}_{i+j-2}+\beta s^{(t)}_{i+j-1}\right),\\ \Gamma_n&=(1+t)^{-\binom{n}{2}}\det_{1\le i,j\le n}\left(\alpha s^{(t)}_{i+j-1}+\beta s^{(t)}_{i+j}\right).\end{aligned}$$

For initial conditions, let $\Theta_0=\Phi_0=\Psi_0=\Gamma_0=1$. Here are our main results.

Theorem 1.1. *We have the following generating functions:*

(i) $$\sum_{n\ge 0}\Theta_n(t)z^n=\frac{1-\beta z}{1-(\alpha+\beta(2+t))z+\beta^2(1+t)z^2}.$$

(ii) $$\sum_{n\ge 0}\Phi_n(t)z^n=\frac{1}{1-(\alpha+\beta(2+t))z+\beta^2(1+t)z^2}.$$

(iii) $$\sum_{n\ge 0}\Psi_n(t)z^n=\frac{1-\beta(1+t)z}{1-(\alpha+\beta(2+t))z+\beta^2(1+t)z^2}.$$

(iv) $\sum_{n\geq 0}\Gamma_n(t)z^n = \dfrac{1}{1-(\alpha+\beta(2+t))z+\beta^2(1+t)z^2}$.

As for the explicit formulae, we have the following.

Theorem 1.2. *For $n \geq 1$ and two constants α, β, let*

$$f_n = \sum_{m=0}^{n}\sum_{k=0}^{2m-n}\sum_{\ell=0}^{k}(-1)^{n-m}\binom{m}{2m-n}\binom{2m-n}{k}\binom{k}{\ell}\alpha^{\ell}\beta^{n-\ell}(1+t)^{m-k}.$$

Then the following identities hold.

(i) $\det_{1\leq i,j\leq n}\left(\alpha r^{(t)}_{i+j-2}+\beta r^{(t)}_{i+j-1}\right) = (1+t)^{\binom{n}{2}}(f_n-\beta f_{n-1})$.

(ii) $\det_{1\leq i,j\leq n}\left(\alpha r^{(t)}_{i+j-1}+\beta r^{(t)}_{i+j}\right) = (1+t)^{\binom{n+1}{2}}f_n$.

(iii) $\det_{1\leq i,j\leq n}\left(\alpha s^{(t)}_{i+j-2}+\beta s^{(t)}_{i+j-1}\right) = (1+t)^{\binom{n}{2}}(f_n-\beta(1+t)f_{n-1})$.

(iv) $\det_{1\leq i,j\leq n}\left(\alpha s^{(t)}_{i+j-1}+\beta s^{(t)}_{i+j}\right) = (1+t)^{\binom{n}{2}}f_n$.

Note that if letting $t = 0$, $\alpha = \beta = 1$ in (i) and (ii) of Theorem 1.2, we obtain the results in Eq. (3). If letting $\alpha = 1$, $\beta = 0$ in (i) and (ii) of Theorem 1.2, we obtain the results in Eq. (5). If letting $\alpha = \beta = 1$ in (i) of Theorem 1.2, we obtain the result in Eq. (6).

We will derive recurrence relations for 'normalized' expressions of the above determinants (see Proposition 6.2). We prove those relations combinatorially by applying the Lindström–Gessel–Viennot lemma on suitable lattice path model. Readers are referred to [1,14,19] for more information.

Here we would like to make some points about the proofs. The proofs are unusual in the sense that, from a conceptual viewpoint, (i), (ii) and (iii) of Theorem 1.2 are proved simultaneously, while (iv) of Theorem 1.2 is merely a direct consequence of (ii). The reason is that, in order to obtain the results on weighted large Schröder numbers, one needs the corresponding results on weighted small Schröder numbers (of smaller size) and vice versa. These 'intertwined' facts are reflected in two lemmas (Key Lemmas I and II) in Section 3 and two lemmas (Lemmas 4.1 and 4.2) in Section 4.

The rest of this paper is organized as follows. We introduce the lattice path model in Section 2. We prove the key lemmas in Section 3. After more intermediate results in Sections 4 and 5, we complete the proof of Theorem 1.2 in Section 6.

2. Lattice path model

Let G denote the directed graph with vertex set $\{(x, y) \in \mathbb{Z}^2 : y \geq 0\}$ and edge set $\{(i, j) \to (i+2, j)\}\cup\{(i, j) \to (i+1, j+1)\}\cup\{(i, j) \to (i+1, j-1)\}$, where the level edges $\{(i, j) \to (i+2, j)\}$ are of weight t and the other edges are of weight 1. Then a weighted large (or small) Schröder path is a directed path on G which starts from and ends at the x-axis. Now, we introduce our lattice path model.

2.1. Lattice path model

We consider the following two classes of path families.

(i) Let $\Pi^{(k)}_n$ (respectively, $\Omega^{(k)}_n$) be the set of n-tuples $(\pi_0, \pi_1, \ldots, \pi_{n-1})$ of weighted large (respectively, small) Schröder paths satisfying the following two conditions (see Fig. 1).
- The path π_j runs from $(-k-2j, 0)$ to $(k+2j, 0)$, for $0 \leq j \leq n-1$.
- Any two paths π_i and π_j do not touch each other.

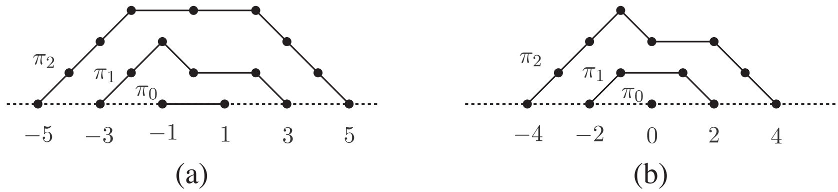

Fig. 1. A triple $(\pi_0, \pi_1, \pi_2) \in \Pi_3^{(1)}$ and a triple $(\pi_0, \pi_1, \pi_2) \in \Pi_3^{(0)}$.

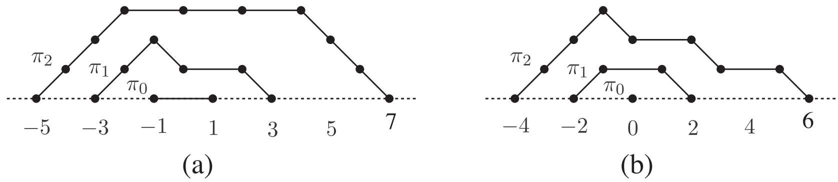

Fig. 2. A triple $(\pi_0, \pi_1, \pi_2) \in \Pi_{3,2}^{(1)}$ and a triple $(\pi_0, \pi_1, \pi_2) \in \Pi_{3,2}^{(0)}$.

(ii) For $0 \leqslant i \leqslant n$, let $\Pi_{n,i}^{(k)}$ (respectively, $\Omega_{n,i}^{(k)}$) be the set of n-tuples $(\pi_0, \pi_1, \ldots, \pi_{n-1})$ of weighted large (respectively, small) Schröder paths satisfying the following three conditions (see Fig. 2).
- The path π_j runs from $(-k-2j, 0)$ to $(k+2j, 0)$, for $0 \le j \le i-1$.
- The path π_j runs from $(-k-2j, 0)$ to $(k+2j+2, 0)$, for $i \le j \le n-1$.
- Any two paths π_i and π_j do not touch each other.

For an n-tuple $\mu = (\pi_0, \pi_1, \ldots, \pi_{n-1})$ of paths, the weight of μ is defined to be the product of the weights of $\pi_0, \pi_1, \ldots, \pi_{n-1}$. For a set X of n-tuples, the wight of X, denoted by $|X|$, is the total weight of all the n-tuples in X.

2.2. Lindström–Gessel–Viennot lemma

A family $(p_1, p_2, \ldots p_n)$ of lattice paths is called *non-intersecting* if no two paths in this family have a point in common. The Lindström–Gessel–Viennot lemma associates determinants with non-intersecting path families in an acyclic directed graph with weighted edges. The following simplified version serves our need.

Lemma 2.1 (Lindström–Gessel–Viennot). *Consider the graph G. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be lattice points on the x-axis. Then the total weight of all families $(p_1, p_2, \ldots, p_n)$ of non-intersecting lattice paths, p_i running from X_i to Y_i, is given by the determinant*

$$\det_{1 \le i,j \le n} (a_{i,j}),$$

where $a_{i,j}$ is the total weight of the lattice paths from X_i to Y_j.

For the weighted large and small Schröder numbers $\{r_\ell^{(t)}\}_{\ell \ge 0}$ and $\{s_\ell^{(t)}\}_{\ell \ge 0}$, we define their Hankel matrices

$$H_n^{(k)} = (r_{k+i+j-2}^{(t)})_{i,j=1}^n \quad \text{and} \quad G_n^{(k)} = (s_{k+i+j-2}^{(t)})_{i,j=1}^n. \tag{7}$$

From the Lindström–Gessel–Viennot lemma, we immediately obtain the weight of the sets $\Pi_n^{(k)}$ and $\Omega_n^{(k)}$.

Lemma 2.2. *For integers* $n, k \geq 0$, *we have*

$$|\Pi_n^{(k)}| = \det(H_n^{(k)}) \quad \textit{and} \quad |\Omega_n^{(k)}| = \det(G_n^{(k)}).$$

For each $0 \leq i \leq n$, we write $A_{n,i}^{(k)}$ for the $n \times n$ matrix obtained from $A_{n+1}^{(k)}$ by deleting the $(n+1)$th row and the $(i+1)$th column, i.e.,

$$A_{n,i}^{(k)} = \begin{pmatrix} a_k & a_{k+1} & \cdots & a_{k+i-1} & a_{k+i+1} & \cdots & a_{k+n} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+i} & a_{k+i+2} & \cdots & a_{k+n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k+n-1} & a_{k+n} & \cdots & a_{k+i+n-2} & a_{k+i+n} & \cdots & a_{k+2n-1} \end{pmatrix}_{n\times n}. \tag{8}$$

The matrices $H_{n,i}^{(k)}$ and $G_{n,i}^{(k)}$ are obtained from $H_{n+1}^{(k)}$ and $G_{n+1}^{(k)}$ accordingly. Similarly, from the Lindström–Gessel–Viennot lemma, we obtain the weight of the sets $\Pi_{n,i}^{(k)}$ and $\Omega_{n,i}^{(k)}$.

Lemma 2.3. *For integers* $n, k \geq 0$, *we have*

(i) $|\Pi_{n,0}^{(k)}| = \det(H_{n,0}^{(k)}) = \det(H_n^{(k+1)}) = |\Pi_n^{(k+1)}|$.

(ii) $|\Omega_{n,0}^{(k)}| = \det(G_{n,0}^{(k)}) = \det(G_n^{(k+1)}) = |\Omega_n^{(k+1)}|$.

(iii) $|\Pi_{n,n}^{(k)}| = \det(H_{n,n}^{(k)}) = \det(H_n^{(k)}) = |\Pi_n^{(k)}|$.

(iv) $|\Omega_{n,n}^{(k)}| = \det(G_{n,n}^{(k)}) = \det(G_n^{(k)}) = |\Omega_n^{(k)}|$.

(v) $|\Pi_{n,i}^{(k)}| = \det(H_{n,i}^{(k)})$, *for* $1 \leq i \leq n-1$.

(vi) $|\Omega_{n,i}^{(k)}| = \det(G_{n,i}^{(k)})$, *for* $1 \leq i \leq n-1$.

3. Two key lemmas

Our proof of the main results is based on two key lemmas. The following lemma relates certain families of weighted large Schröder paths with the determinants of certain weighted small Schröder numbers. Let $\Pi_{n,i}^{(1)*} \subseteq \Pi_{n,i}^{(1)}$ be the set of n-tuples of weighted large Schröder paths in which none of their paths touches the point $(2i+1, 0)$. See Fig. 3(a) for an example.

Lemma 3.1 Key Lemma I. *For* $1 \leqslant i \leqslant n$, *we have*

$$|\Pi_{n,i}^{(1)*}| = (1+t)^n \det(G_{n,i}^{(0)}).$$

Proof. We partition the set $\Pi_{n,i}^{(1)*}$ into two subsets X and Y, each of which can be directly counted. Let X (respectively, Y) be the set of n-tuples $(\pi_0, \pi_1, \ldots, \pi_{n-1})$ with $\pi_0 = \mathsf{L}$ (respectively, $\pi_0 = \mathsf{UD}$).

(i) There is a bijection between X and $\Pi_{n-1,i-1}^{(2)}$, which sends $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in X$ to $(\omega_1, \omega_2, \ldots, \omega_{n-1}) \in \Pi_{n-1,i-1}^{(2)}$, where $\pi_j = \mathsf{U}\omega_j\mathsf{D}$, for $1 \leqslant j \leqslant n-1$. See Fig. 3 for an example.

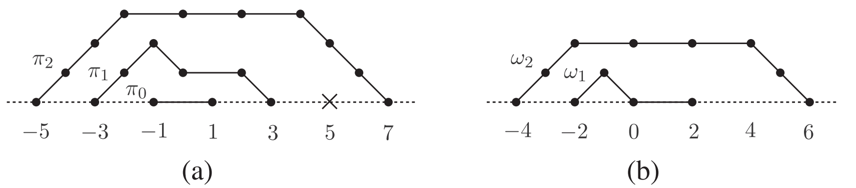

Fig. 3. A triple $(\pi_0, \pi_1, \pi_2) \in X \subseteq \Pi^{(1)*}_{3,2}$ mapped to $(\omega_1, \omega_2) \in \Pi^{(2)}_{2,1}$.

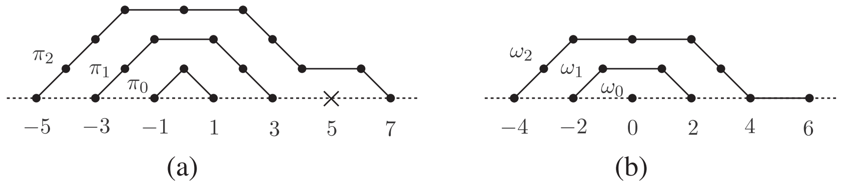

Fig. 4. A triple $(\pi_0, \pi_1, \pi_2) \in Y \subseteq \Pi^{(1)*}_{3,2}$ mapped to $(\omega_0, \omega_1, \omega_2) \in \Pi^{(0)}_{3,2}$.

Hence the weight of $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in X$ equals t times the weight of $(\omega_1, \omega_2, \ldots, \omega_{n-1}) \in \Pi^{(2)}_{n-1,i-1}$. Therefore,

$$|X| = t|\Pi^{(2)}_{n-1,i-1}| = \det \left(\begin{array}{c|cccccccc} t & r_1^{(t)} & r_2^{(t)} & \cdots & r_{i-1}^{(t)} & r_{i+1}^{(t)} & \cdots & r_n^{(t)} \\ \hline 0 & r_2^{(t)} & r_3^{(t)} & \cdots & r_i^{(t)} & r_{i+2}^{(t)} & \cdots & r_{n+1}^{(t)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & r_n^{(t)} & r_{n+1}^{(t)} & \cdots & r_{n+i-2}^{(t)} & r_{n+i}^{(t)} & \cdots & r_{2n-1}^{(t)} \end{array} \right)_{n\times n}. \tag{9}$$

(ii) There is a weight-preserving bijection between Y and $\Pi^{(0)}_{n,i}$, which sends $(\pi_0, \ldots, \pi_{n-1}) \in Y$ to $(\omega_0, \ldots, \omega_{n-1}) \in \Pi^{(0)}_{n,i}$, where $\pi_j = \mathsf{U}\omega_j\mathsf{D}$, for $0 \le j \le n-1$. See Fig. 4 for an example. Hence

$$|Y| = |\Pi^{(0)}_{n,i}| = \det \begin{pmatrix} r_0^{(t)} & r_1^{(t)} & \cdots & r_{i-1}^{(t)} & r_{i+1}^{(t)} & \cdots & r_n^{(t)} \\ r_1^{(t)} & r_2^{(t)} & \cdots & r_i^{(t)} & r_{i+2}^{(t)} & \cdots & r_{n+1}^{(t)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ r_{n-1}^{(t)} & r_n^{(t)} & \cdots & r_{n+i-2}^{(t)} & r_{n+i}^{(t)} & \cdots & r_{2n-1}^{(t)} \end{pmatrix}_{n\times n}. \tag{10}$$

Now, by the fact that $r_0^{(t)} = 1$, $s_0^{(t)} = 1$, $r_m^{(t)} = (1+t)s_m^{(t)}$ for $m \geq 1$ and direct computation with Eqs. (9) and (10), we have

$$|\Pi^{(1)*}_{n,i}| = |X| + |Y| = (1+t)^n \det(G^{(0)}_{n,i})$$

as desired. □

The following lemma relates determinants of certain weighted small Schröder numbers to determinants (of smaller size) of certain other weighted Schröder numbers.

Lemma 3.2 Key Lemma II. *For $1 \leqslant i \leqslant n$, we have*

$$\det(G^{(0)}_{n,i}) = \det(H^{(1)}_{n-1,i-1}) + (1+t)^{n-1} \det(G^{(0)}_{n-1,i}).$$

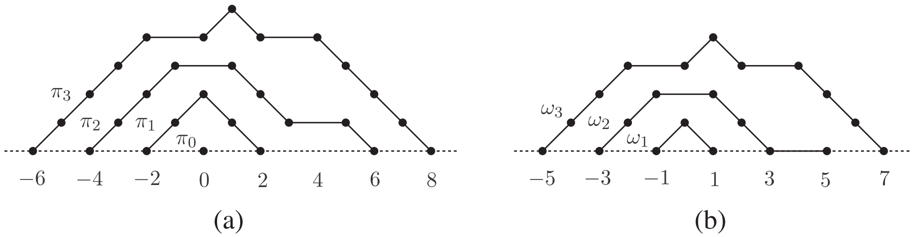

Fig. 5. A quadruple $(\pi_0, \pi_1, \pi_2, \pi_3) \in X \subseteq \Omega^{(0)}_{4,2}$ mapped to $(\omega_1, \omega_2, \omega_3) \in \Pi^{(1)}_{3,1}$.

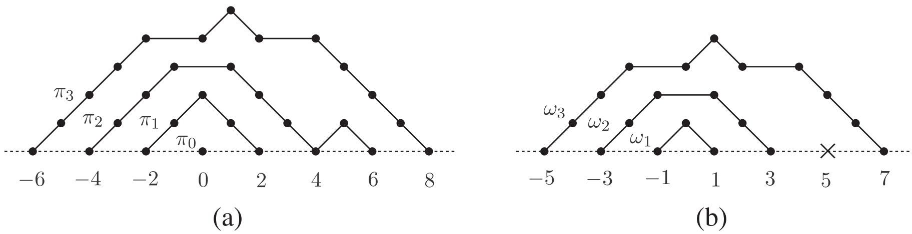

Fig. 6. A quadruple $(\pi_0, \pi_1, \pi_2, \pi_3) \in Y \subseteq \Pi^{(0)}_{4,2}$ mapped to $(\omega_0, \omega_1, \omega_2) \in \Pi^{(1)*}_{3,2}$.

Proof. By applying the Key Lemma I to $\det(G^{(0)}_{n-1,i})$ and by (v),(vi) of Lemma 2.3, we see that it suffices to prove

$$|\Omega^{(0)}_{n,i}| = |\Pi^{(1)}_{n-1,i-1}| + |\Pi^{(1)*}_{n-1,i}|.$$

Fix an i ($1 \le i \le n$). Consider an n-tuple $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in \Omega^{(0)}_{n,i}$. Note that the endpoints of π_{i-1} and π_i are $(2i-2, 0)$ and $(2i+2, 0)$, respectively. We partition $\Omega^{(0)}_{n,i}$ into two subsets X and Y. The set X (respectively, Y) consists of the n-tuples $(\pi_0, \pi_1, \ldots, \pi_{n-1})$ such that π_i does not touch (respectively, π_i touches) the point $(2i, 0)$.

(i) There is a weight-preserving bijection between X and $\Pi^{(1)}_{n-1,i-1}$, which sends $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in X$ to $(\omega_1, \ldots, \omega_{n-1}) \in \Pi^{(1)}_{n-1,i-1}$, where $\pi_j = \mathsf{U}\omega_j\mathsf{D}$ for $1 \le j \le n-1$. See Fig. 5 for an example. Hence, we have

$$|X| = |\Pi^{(1)}_{n-1,i-1}|.$$

(ii) For each n-tuple $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in Y$, the path π_i can be factorized as $\pi_i = \mathsf{U}\omega_i\mathsf{DUD}$ for some large Schröder path ω_i above the line $y = 1$. Thus there is a weight-preserving bijection between Y and $\Pi^{(1)*}_{n-1,i}$, which sends $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in Y$ to $(\omega_1, \ldots, \omega_{n-1}) \in \Pi^{(1)*}_{n-1,i}$, where $\pi_i = \mathsf{U}\omega_i\mathsf{DUD}$ and $\pi_j = \mathsf{U}\omega_j\mathsf{D}$ for $1 \le j \le n-1, j \ne i$. See Fig. 6 for an example. Hence, we have

$$|Y| = |\Pi^{(1)*}_{n-1,i}|,$$

and the proof is completed. □

4. Evaluation of $\det(H^{(1)}_{n,i})$ and $\det(H^{(0)}_{n,i})$

In this section, we use the key lemmas to derive recurrence formulae for $\det(H^{(1)}_{n,i})$ and $\det(H^{(0)}_{n,i})$ combinatorially, which involve weighted small Schröder numbers.

Lemma 4.1. *For* $1 \leqslant i \leqslant n$*, we have*

$$\det(H^{(1)}_{n,i}) = (1+t)^n \det(H^{(1)}_{n-1,i-1}) + (1+t)^{n+1} \det(H^{(1)}_{n-1,i}) + (1+t)^{2n-1} \det(G^{(0)}_{n-1,i}).$$

Proof. Since $\det(H^{(1)}_{n,i}) = (1+t)^n \det(G^{(1)}_{n,i})$, it suffices to prove

$$|\Omega^{(1)}_{n,i}| = |\Pi^{(1)}_{n-1,i-1}| + (1+t)|\Pi^{(1)}_{n-1,i}| + (1+t)^{n-1}|\Omega^{(0)}_{n-1,i}|.$$

The approach is the same as in the proof of Lemma 3.2. Fix an i $(1 \le i \le n)$. Consider an n-tuple $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in \Omega^{(1)}_{n,i}$. Noth that the endpoints of π_{i-1} and π_i are $(2i-1, 0)$ and $(2i+3, 0)$, respectively. We distinguish the n-tuples by three cases depending on the path π_i.

(i) π_i does not touch the point $(2i, 1)$, i.e., the first down step of π_i descending from the line $y = 2$ to the line $y = 1$ occurs from $(2i+1, 2)$ to $(2i+2, 1)$.
(ii) π_i touches the point $(2i, 1)$ but does not touch the point $(2i+1, 2)$.
(iii) π_i touches both of the points $(2i, 1)$ and $(2i+1, 2)$.

We partition $\Omega^{(1)}_{n,i}$ into three subsets X, Y and Z. Let X be the subset consisting of the n-tuples $(\pi_0, \pi_1, \ldots, \pi_{n-1})$ such that the path π_i has property (i), let Y be the subset corresponding to (ii), and let Z be the subset corresponding to (iii).

- There is a weight-preserving bijection between X and $\Pi^{(1)}_{n-1,i-1}$, which sends $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in X$ to $(\omega_1, \ldots, \omega_{n-1}) \in \Pi^{(1)}_{n-1,i-1}$, where $\pi_j = \mathsf{UU}\omega_j\mathsf{DD}$, for $1 \leqslant j \le n-1$. (Note that $\pi_0 = \mathsf{UD}$.) Hence, we have

$$|X| = |\Pi^{(1)}_{n-1,i-1}|.$$

- For each n-tuple $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in Y$, the path π_i can be factorized as $\pi_i = \mathsf{UU}\omega_i\mathsf{DLD}$ or as $\pi_i = \mathsf{UU}\omega_i\mathsf{DDUD}$ for some large Schröder path ω_i. Thus there is a two-to-one correspondence between Y and $\Pi^{(1)}_{n-1,i}$, which sends $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in Y$ to $(\omega_1, \ldots, \omega_{n-1}) \in \Pi^{(1)}_{n-1,i}$, where $\pi_i = \mathsf{UU}\omega_i\mathsf{DLD}$ or $\pi_i = \mathsf{UU}\omega_i\mathsf{DDUD}$, and $\pi_j = \mathsf{UU}\omega_j\mathsf{DD}$ for $1 \le j \le n-1, j \ne i$. Hence, we have

$$|Y| = (1+t)|\Pi^{(1)}_{n-1,i}|.$$

- For each n-tuple $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in Z$, the path π_i can be factorized as $\pi_i = \mathsf{UU}\omega_i\mathsf{DUDD}$. Then the n-tuple corresponds to an $(n-1)$-tuple $(\omega_1, \ldots, \omega_{n-1})$, where $\pi_i = \mathsf{UU}\omega_i\mathsf{DUDD}$ and $\pi_j = \mathsf{UU}\omega_j\mathsf{DD}$, for $1 \le j \le n-1, j \ne i$. Note that none of the paths $\omega_1, \ldots, \omega_{n-1}$ touches point $(2i+1, 0)$, i.e., $(\omega_1, \ldots, \omega_{n-1}) \in \Pi^{(1)*}_{n-1,i}$. By Lemma 3.1, we have

$$|Z| = |\Pi^{(1)*}_{n-1,i}| = (1+t)^{n-1} \det(G^{(0)}_{n-1,i}) = (1+t)^{n-1}|\Omega^{(0)}_{n-1,i}|.$$

The proof is completed. □

Lemma 4.2. *For* $1 \leqslant i \leqslant n$*, we have*

$$\det(H^{(0)}_{n,i}) = \det(H^{(1)}_{n-1,i-1}) + t\det(H^{(1)}_{n-1,i}) + (1+t)^{n-1} \det(G^{(0)}_{n-1,i}).$$

Proof. The approach is the same as above. Fix an i $(1 \le i \le n)$. Consider an n-tuple $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in \Pi^{(0)}_{n,i}$. Noth that the endpoints of π_{i-1} and π_i are $(2i-2, 0)$ and $(2i+2, 0)$, respectively. We distinguish the n-tuples by three cases depending on the path π_i:

(i) π_i does not touch the point $(2i, 0)$, i.e., the first the down step of π_i descending from the line $y = 1$ to the line $y = 0$ occurs from $(2i + 1, 1)$ to $(2i + 2, 0)$.
(ii) π_i touches the point $(2i, 0)$ but does not touch the point $(2i + 1, 1)$.
(iii) π_i touches both of the points $(2i, 0)$ and $(2i + 1, 1)$.

Let X be the subset consisting of the n-tuples $(\pi_0, \pi_1, \ldots, \pi_{n-1})$ such that the path π_i has property (i), let Y be the subset corresponding to (ii), and let Z be the subset corresponding to (iii).

- There is a weight-preserving bijection between X and $\Pi^{(1)}_{n-1,i-1}$, which sends $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in X$ to $(\omega_1, \ldots, \omega_{n-1}) \in \Pi^{(1)}_{n-1,i-1}$, where $\pi_j = \mathsf{U}\omega_j\mathsf{D}$, for $1 \leqslant j \leq n - 1$. Hence, we have
$$|X| = \left|\Pi^{(1)}_{n-1,i-1}\right| = \det(H^{(1)}_{n-1,i-1}).$$
- There is a bijection between Y and $\Pi^{(1)}_{n-1,i}$, which sends $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in Y$ to $(\omega_1, \ldots, \omega_{n-1}) \in \Pi^{(1)}_{n-1,i}$, where $\pi_j = \mathsf{U}\omega_j\mathsf{DL}$, for $1 \leq j \leq n - 1$. Hence, we have
$$|Y| = t\left|\Pi^{(1)}_{n-1,\bar{i}}\right| = t\det(H^{(1)}_{n-1,i}).$$
- There is a bijection between Z and $\Pi^{(1)*}_{n-1,i}$, which sends $(\pi_0, \pi_1, \ldots, \pi_{n-1}) \in Z$ to $(\omega_1, \ldots, \omega_{n-1}) \in \Pi^{(1)*}_{n-1,i}$, where where $\pi_j = \mathsf{U}\omega_j\mathsf{DUD}$, for $1 \leq j \leq n - 1$. Hence, we have
$$|Z| = \left|\Pi^{(1)*}_{n-1,i}\right| = (1 + t)^{n-1}\det(G^{(0)}_{n-1,i}).$$

The proof is completed by combining the three identities. □

5. Two recurrences

In this section, we derive a recurrence formula for $\det(H^{(0)}_{n,i})$ and $\det(H^{(1)}_{n,i})$, respectively. For simplicity, for $0 \leqslant i \leqslant n$, let

$$P_{n,i} = (1 + t)^{-\binom{n}{2}}\det(H^{(0)}_{n,i}),$$
$$Q_{n,i} = (1 + t)^{-\binom{n+1}{2}}\det(H^{(1)}_{n,i}),$$
$$R_{n,i} = (1 + t)^{-\binom{n}{2}}\det(G^{(0)}_{n,i}),$$

with $P_{0,0} = Q_{0,0} = R_{0,0} = 1$ and $P_{i,j} = Q_{i,j} = R_{i,j} = 0$ if $j > i$. The following identities are the direct translations of Lemmas 3.2, 4.1, and 4.2.

Lemma 5.1. *For $1 \leqslant i \leqslant n$, we have*

(i) $R_{n,i} = Q_{n-1,i-1} + R_{n-1,i}$.
(ii) $Q_{n,i} = Q_{n-1,i-1} + (1 + t)Q_{n-1,i} + R_{n-1,i}$.
(iii) $P_{n,i} = Q_{n-1,i-1} + tQ_{n-1,i} + R_{n-1,i}$.

First, we deal with the cases $i = 0$ and $i = n$.

Lemma 5.2. *We have*

(i) $Q_{n,0} = 1 + (1 + t)Q_{n-1,0}$ *and* $Q_{n,n} = 1$.
(ii) $P_{n,0} = (1 + t)P_{n-1,0}$ *and* $P_{n,n} = 1$.
(iii) $R_{n,0} = R_{n,n} = 1$.

Proof. (i) We have $Q_{n,n} = Q_{n-1,n-1} = \cdots = Q_{0,0} = 1$ by Lemma 5.1(ii). By Lemma 2.3, $Q_{n,0} = (1+t)^{-\binom{n+1}{2}} \det(H_{n,0}^{(1)}) = (1+t)^{-\binom{n+1}{2}} \det(H_n^{(2)})$. We use the following condensation identity (obtained from Eq. (2)) to compute $\det(H_n^{(2)})$:

$$\det(H_{n+1}^{(0)}) \det(H_{n-1}^{(2)}) = \det(H_n^{(0)}) \det(H_n^{(2)}) - \det(H_n^{(1)})^2.$$

By applying the known formulae for $\det(H_n^{(0)})$, $\det(H_n^{(1)})$ and some simple calculation, we arrive at

$$Q_{n,0} = Q_{n-1,0} + (1+t)^n.$$

This can be solved to obtain $Q_{n,0} = \sum_{k=0}^{n} (1+t)^k$. Therefore

$$Q_{n,0} = 1 + (1+t)Q_{n-1,0}.$$

(ii) We have $P_{n,n} = Q_{n-1,n-1} = 1$ by (i) and Lemma 5.1(iii). By Lemma 2.3, we have $\det(H_{n,0}^{(0)}) = \det(H_n^{(1)}) = \det(H_{n,n}^{(1)})$. Thus

$$P_{n,0} = (1+t)^{-\binom{n}{2}} \det(H_{n,0}^{(0)}) = (1+t)^{-\binom{n}{2}} \det(H_{n,n}^{(1)}) = (1+t)^n Q_{n,n} = (1+t)^n.$$

Hence, we have

$$P_{n,0} = (1+t)P_{n-1,0}.$$

(iii) We have $R_{n,n} = Q_{n-1,n-1} = 1$ by (i) and Lemma 5.1(i). Besides,

$$R_{n,0} = (1+t)^{-\binom{n}{2}} \det(G_{n,0}^{(0)}) = (1+t)^{-\binom{n}{2}} \det(G_n^{(1)}) = (1+t)^{-\binom{n+1}{2}} \det(H_n^{(1)}).$$

Here, we have used the fact $s_m = (1+t)^{-1} r_m$ for $m \geqslant 1$. Thus

$$R_{n,0} = (1+t)^{-\binom{n+1}{2}} \det(H_n^{(1)}) = (1+t)^{-\binom{n+1}{2}} \det(H_{n,n}^{(1)}) = Q_{n,n} = 1$$

as desired. □

The last pieces we need are the recurrence formulae for $P_{n,i}$ and $Q_{n,i}$, for $1 \le i \le n$.

Lemma 5.3. *For $1 \leqslant i \leqslant n$, we have*

$$Q_{n,i} = (1+t)Q_{n-1,i} + \sum_{k=i}^{n} Q_{k-1,i-1}.$$

Proof. Repeatedly applying Lemma 5.1(i), we obtain

$$R_{n,i} = \sum_{k=i}^{n-1} Q_{k,i-1} + R_{i,i}.$$

Since $R_{i,i} = Q_{i-1,i-1} = 1$, we have

$$R_{n,i} = \sum_{k=i}^{n} Q_{k-1,i-1}.$$

This is substituted into Lemma 5.1(ii) to complete the proof of the lemma. □

Lemma 5.4. *For $1 \leqslant i \leqslant n$,*

$$P_{n,i} = (1+t)P_{n-1,i} + \sum_{k=i}^{n} P_{k-1,i-1}.$$

Proof. We have

$$P_{n,i} = Q_{n-1,i-1} + tQ_{n-1,i} + R_{n-1,i} = Q_{n,i} - Q_{n-1,i}$$

by (ii) and (iii) of Lemma 5.1. Applying the result of Lemma 5.3, we obtain

$$\begin{aligned} P_{n,i} &= \left((1+t)Q_{n-1,i} + \sum_{k=i}^{n} Q_{k-1,i-1}\right) - \left((1+t)Q_{n-2,i} + \sum_{k=i}^{n-1} Q_{k-1,i-1}\right) \\ &= (1+t)\left(Q_{n-1,i} - Q_{n-2,i}\right) + \sum_{k=i+1}^{n} \left(Q_{k-1,i-1} - Q_{k-2,i-1}\right) + Q_{i-1,i-1} \\ &= (1+t)P_{n-1,i} + \sum_{k=i}^{n} P_{k-1,i-1} \end{aligned}$$

as required. $\square$

6. Proof of the main theorem

For a sequence $\{a_\ell\}_{\ell \geq 0}$, recall the Hankel matrix $A_n^{(k)} = (a_{k+i+j-2})_{i,j=1}^{n}$ in Eq. (1) and the matrices $A_{n,i}^{(k)}$ defined in Eq. (8), for $0 \leq i \leq n$. First, we need the following simple fact about Hankel determinants of the sequence $\{\alpha a_\ell + \beta a_{\ell+1}\}_{\ell \geq 0}$. We omit the proof.

Lemma 6.1. *For $n \geq 1$ and constants α, β, we have*

$$\det_{1 \leq i,j \leq n} (\alpha a_{k+i+j-2} + \beta a_{k+i+j-1}) = \sum_{i=0}^{n} \alpha^i \beta^{n-i} \det(A_{n,i}^{(k)}).$$

To prove the main theorem, we derive recurrence relations for Θ_n, Φ_n, Ψ_n, and Γ_n.

Proposition 6.2. *We have the following recurrence relations:*

(i) $\Theta_n = \beta(1+t)\Theta_{n-1} + \alpha \sum_{m=0}^{n-1} \beta^m \Theta_{n-1-m}$.

(ii) $\Phi_n = \beta^n + \beta(1+t)\Phi_{n-1} + \alpha \sum_{m=0}^{n-1} \beta^m \Phi_{n-1-m}$.

(iii) $\Psi_n = -t\beta^n + \beta(1+t)\Psi_{n-1} + \alpha \sum_{m=0}^{n-1} \beta^m \Psi_{n-1-m}$.

(iv) $\Gamma_n = \beta^n + \beta(1+t)\Gamma_{n-1} + \alpha \sum_{m=0}^{n-1} \beta^m \Gamma_{n-1-m}$.

Proof. (i) Expanding Θ_n by Lemma 6.1, we have $\Theta_n = \sum_{i=0}^{n} \alpha^i \beta^{n-i} P_{n,i}$. Splitting the sum into two parts and applying Lemmas 5.2 and 5.4, we have

$$\begin{aligned}
\Theta_n &= \beta^n P_{n,0} + \sum_{i=1}^{n} \alpha^i \beta^{n-i} P_{n,i} \\
&= \beta^n (1+t) P_{n-1,0} + \sum_{i=1}^{n} \alpha^i \beta^{n-i} \left((1+t) P_{n-1,i} + \sum_{k=i}^{n} P_{k-1,i-1} \right) \\
&= \beta (1+t) \sum_{i=0}^{n-1} \alpha^i \beta^{n-1-i} P_{n-1,i} + \sum_{i=1}^{n} \alpha^i \beta^{n-i} \left(\sum_{k=i}^{n} P_{k-1,i-1} \right) \\
&= \beta (1+t) \Theta_{n-1} + \sum_{k=1}^{n} \alpha \beta^{n-k} \left(\sum_{i=1}^{k} \alpha^{i-1} \beta^{k-i} P_{k-1,i-1} \right) \\
&= \beta (1+t) \Theta_{n-1} + \alpha \sum_{m=0}^{n-1} \beta^m \Theta_{n-1-m}
\end{aligned}$$

as desired.

(ii) The proof is similar to above. Expanding Φ_n by Lemma 6.1, we have $\Phi_n = \sum_{i=0}^{n} \alpha^i \beta^{n-i} Q_{n,i}$. Splitting the sum into two parts and applying Lemmas 5.2 and 5.3, we have

$$\begin{aligned}
\Phi_n &= \beta^n Q_{n,0} + \sum_{i=1}^{n} \alpha^i \beta^{n-i} Q_{n,i} \\
&= \beta^n \left(1 + (1+t) Q_{n-1,0}\right) + \sum_{i=1}^{n} \alpha^i \beta^{n-i} \left((1+t) Q_{n-1,i} + \sum_{k=i}^{n} Q_{k-1,i-1} \right) \\
&= \beta^n + \beta (1+t) \Phi_{n-1} + \sum_{k=1}^{n} \alpha \beta^{n-k} \sum_{i=1}^{k} \alpha^{i-1} \beta^{k-i} Q_{k-1,i-1} \\
&= \beta^n + \beta (1+t) \Phi_{n-1} + \alpha \sum_{m=0}^{n-1} \beta^m \Phi_{n-1-m}
\end{aligned}$$

as desired.

(iii) We prove this relation by induction on n. The case $n = 1$ holds trivially. By Lemma 6.1 and the definition of $R_{n,i}$, we can expand Ψ_n into

$$\Psi_n = \alpha^n R_{n,n} + \sum_{i=1}^{n-1} \alpha^i \beta^{n-i} R_{n,i} + \beta^n R_{n,0}.$$

Now, by Lemma 5.1, we have $R_{n,i} = Q_{n-1,i-1} + R_{n-1,i}$, for $1 \le i \le n-1$. Moreover, $R_{n,n} = Q_{n-1,n-1} = 1$ and $R_{n,0} = R_{n-1,0} = 1$ by Lemma 5.2. Substituting these findings in the above identity, we arrive at

$$\begin{aligned}
\Psi_n &= \alpha^n Q_{n-1,n-1} + \sum_{i=1}^{n-1} \alpha^i \beta^{n-i} \left(Q_{n-1,i-1} + R_{n-1,i} \right) + \beta^n R_{n-1,0} \\
&= \alpha \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j} Q_{n-1,j} + \beta \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j} R_{n-1,j} \\
&= \alpha \Phi_{n-1} + \beta \Psi_{n-1}.
\end{aligned}$$

Then by (ii) and the induction hypothesis, we get

$$\begin{aligned}
\Psi_n &= \alpha\Phi_{n-1} + \beta\Psi_{n-1} \\
&= \alpha\left(\beta^{n-1} + \beta(1+t)\Phi_{n-2} + \alpha\sum_{m=0}^{n-2}\beta^m\Phi_{n-2-m}\right) \\
&\quad + \beta\left(-t\beta^{n-1} + \beta(1+t)\Psi_{n-2} + \alpha\sum_{m=0}^{n-2}\beta^m\Psi_{n-2-m}\right) \\
&= -t\beta^n + \alpha\beta^{n-1}\Psi_0 + \beta(1+t)(\alpha\Phi_{n-2} + \beta\Psi_{n-2}) \\
&\quad + \alpha\sum_{m=0}^{n-2}\beta^m(\alpha\Phi_{n-2-m} + \beta\Psi_{n-2-m}) \\
&= -t\beta^n + \beta(1+t)\Psi_{n-1} + \alpha\sum_{m=0}^{n-1}\beta^m\Psi_{n-1-m},
\end{aligned}$$

and the proof is completed.

(iv) By (ii) and the identity $r_n = (1+t)s_n$ for $n \geqslant 1$, we are done. $\square$

Now we are ready to prove the main theorem.

Proof of Theorem 1.1 and Theorem 1.2. From the proposition above it is easy to derive the following recurrence relations:

(i) $\Theta_n = (\alpha + \beta(2+t))\Theta_{n-1} - \beta^2(1+t)\Theta_{n-2}$, with $\Theta_0 = 1$, $\Theta_1 = \alpha + \beta(1+t)$.
(ii) $\Phi_n = (\alpha + \beta(2+t))\Phi_{n-1} - \beta^2(1+t)\Phi_{n-2}$, with $\Phi_0 = 1$, $\Phi_1 = \alpha + \beta(2+t)$.
(iii) $\Psi_n = (\alpha + \beta(2+t))\Psi_{n-1} - \beta^2(1+t)\Psi_{n-2}$, with $\Psi_0 = 1$, $\Psi_1 = \alpha + \beta$.
(iv) $\Gamma_n = (\alpha + \beta(2+t))\Gamma_{n-1} - \beta^2(1+t)\Gamma_{n-2}$, with $\Gamma_0 = 1$, $\Gamma_1 = \alpha + \beta(2+t)$.

The generating functions and the explicit formulae can then be derived routinely. $\square$

7. Concluding notes

Different expansions may lead to other explicit formulae. For example, we can have

$$\det_{1\le i,j\le n}(\alpha r^{(t)}_{i+j-2} + \beta r^{(t)}_{i+j-1}) = (1+t)^{\binom{n}{2}}\sum_{m=0}^{n}\sum_{k=0}^{m}\binom{m}{k}\binom{n-m+k-1}{n-m}\alpha^k\beta^{n-k}(1+t)^{m-k},$$

or

$$\det_{1\le i,j\le n}(\alpha r^{(t)}_{i+j-2} + \beta r^{(t)}_{i+j-1}) = (1+t)^{\binom{n}{2}}\sum_{k=0}^{n}\sum_{\ell=0}^{n-k}\binom{k+\ell}{\ell}\binom{n+k}{n-k-\ell}\alpha^k\beta^{n-k}t^{\ell}.$$

A natural extension is to consider the Hankel determinants in which each entry is the linear combination of more than two consecutive terms of t-large (or small) Schröder numbers. However, an approach using lattice path models turns out to be messy and seems not so attractive. Another natural generalization is to put different weights with respect to heights, or to consider q-analogues. We leave these interesting problems to the readers.

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References

[1] M. Aigner, A Course in Enumeration, Graduate Texts in Mathematics, vol. 238, Springer, Berlin, 2007.
[2] M. Aigner, Motzkin numbers, European J. Combin. 19 (6) (1998) 663–675.
[3] M. Aigner, Catalan-like numbers and determinants, J. Combin. Theory Ser. A 87 (1) (1999) 33–51.
[4] A. Benjamin, N. Cameron, J. Quinn, C. Yerger, Catalan determinants – a combinatorial approach, Congr. Numer. 200 (2010) 27–34.
[5] D.M. Bressoud, Proofs and Confirmations – The Story of the Alternating Sign Matrix Conjecture, Cambridge University Press, Cambridge, 1999.
[6] R.A. Brualdi, S. Kirkland, Aztec diamonds and digraphs, and Hankel determinants of Schröder numbers, J. Combin. Theory Ser. B 94 (2) (2005) 334–351.
[7] N.T. Cameron, A.C.M. Yip, Hankel determinants of sums of consecutive Motzkin numbers, Linear Algebra Appl. 434 (3) (2011) 712–722.
[8] M. Chamberland, C. French, Generalized Catalan numbers and generalized Hankel transformations, J. Integer Seq. 10 (1) (2007), Article 07.1.1, 7 pp (electronic).
[9] J. Ciglerci, Some relations between generalized Fibonacci Catalan numbers, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 211 (2002) 134–154.
[10] J. Cigler, C. Krattenthaler, Some determinants of path generating functions, Adv. Appl. Math. 46 (1–4) (2011) 144–174.
[11] A. Cvetković, P. Rajković, M. Ivković, Catalan numbers, the Hankel transform, and Fibonacci numbers, J. Integer Seq. 5 (1) (2002), Article 02.1.3, 8 pp (electronic).
[12] M. Desainte-Catherine, X. Viennot, Enumeration of certain Young tableaux with bounded height, in: Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985), Lecture Notes in Mathematics, vol. 1234, Springer, Berlin, 1986, pp. 58–67.
[13] S.-P. Eu, T.-S. Fu, A simple proof of the Aztec diamond theorem, Electron. J. Combin. 12 (2005). , #R18, 8 pp (electronic).
[14] I. Gessel, X. Viennot, Binomial determinants, paths, and hook length formulae, Adv. Math. 58 (3) (1985) 300–321.
[15] I. Gessel, X. Viennot, Determinants, paths, and plane partitions, Preprint, 1989. Available at: <http://www.cs.brandeis.edu/ ira>.
[16] M. Ishikawa, H. Tagawa, J. Zeng, A q-analogue of Catalan Hankel determinants, RIMS Kôkyûroku Bessatsu B11 (2009) 19–41.
[17] D.E. Knuth, Overlapping pfaffians, Electron. J. Combin. 3 (no. 2, The Foata Festschrift) (1996) #R5, 13 pp (electronic).
[18] C. Krattenthaler, Advanced determinant calculus: a complement, Linear Algebra Appl. 411 (2005) 68–166.
[19] C. Krattenthaler, Watermelon configurations with wall interaction: exact and asymptotic results, J. Phys. Conf. Ser. 42 (2006) 179–212.
[20] C. Krattenthaler, Determinants of (generalised) Catalan numbers, J. Statist. Plann. Inference 140 (8) (2010) 2260–2270.
[21] T. Muir, The Theory of Determinants in the Historical Order of Development, vol. 4, Macmillan, London, 1906.
[22] P.M. Rajković, M.D. Petković, P. Barry, The Hankel transform of the sum of consecutive generalized Catalan numbers, Integral Transform. Spec. Funct. 18 (3–4) (2007) 285–296.
[23] R.A. Sulanke, G. Xin, Hankel determinants for some common lattice paths, Adv. Appl. Math. 40 (2) (2008) 149–167.
[24] X. Viennot, Une théorie combinatoire des polynômes orthogonaux généraux, UQAM, Montréal, Québec, 1983. Available at: <http://web.mac.com/xgviennot>.