Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 306 (2006) 2234–2240
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Note

A class of combinatorial identities☆

Yingpu Deng

Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China

Received 26 January 2006; received in revised form 9 April 2006; accepted 21 April 2006
Available online 3 July 2006

Abstract

A general theorem for providing a class of combinatorial identities where the sum is over all the partitions of a positive integer is proven. Five examples as the applications of the theorem are given.

Keywords: Combinatorial identities; Partitions; Möbius inversion formula

1. Introduction

If a polynomial has no multiple roots, then it is called separable. For $n \geqslant 1$, let $M_n^{(q)}$ denote the number of monic separable polynomials of degree n over the finite field $\mathbb{F}_q$ with q elements. In 1932, Carlitz [1] showed that $M_1^{(q)} = q$ and $M_n^{(q)} = q^n - q^{n-1}$, for $n > 1$.

The following method for computing the value of $M_n^{(q)}$ is due to Deng [2]. Here we repeat the procedure. Since finite fields are perfect, so any irreducible polynomial over finite fields is separable, and the product of distinct irreducible polynomials is also separable. Let $I_d^{(q)}$ denote the number of monic irreducible polynomials over $\mathbb{F}_q$ of degree d. For a monic separable polynomial over $\mathbb{F}_q$ of degree n, consider its decomposition over $\mathbb{F}_q$ into the product of irreducible factors, we know that each decomposition of the polynomial corresponds to a partition of n. Concretely, suppose

$$\pi(n) = (i_1^{a_1}, \ldots, i_d^{a_d})$$

is a partition of n, where $d > 0$, $i_1 > i_2 > \cdots > i_d > 0$, $a_1 > 0, \ldots, a_d > 0$, and $n = a_1 i_1 + \cdots + a_d i_d$. Let $I_{\pi(n)}^{(q)}$ denote the number of monic separable polynomials over $\mathbb{F}_q$ of degree n which decompose as the product over $\mathbb{F}_q$ of a_j many monic irreducible polynomials of degree i_j, $j = 1, 2, \ldots, d$, then we have

$$I_{\pi(n)}^{(q)} = \prod_{j=1}^{d} \binom{I_{i_j}^{(q)}}{a_j}.$$

☆ Supported by NNSF of China under Grants nos. 10501049, 90304012.
E-mail address: dengyp@amss.ac.cn.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2006.04.021

Hence we have

$$M_n^{(q)} = \sum_{\pi(n)} I_{\pi(n)}^{(q)},$$

where the sum is over all the partitions of n. Thus we get the following.

Proposition (*Deng* [2]). *Suppose n is a positive integer. Let $\pi(n)$ denote a partition of n, usually denoted by $(1^{k_1}, 2^{k_2}, \ldots, n^{k_n})$, with $k_1 + 2k_2 + \cdots + nk_n = n$. Then we have that*

$$M_n^{(q)} = \sum_{\pi(n)} \prod_{j=1}^{n} \binom{I_j^{(q)}}{k_j},$$

where the sum is over all the partitions of n.

In view of this proposition and the results of Carlitz, we have the following identity.

$$\sum_{\pi(n)} \prod_{j=1}^{n} \binom{I_j^{(q)}}{k_j} = \begin{cases} q & \text{if } n = 1, \\ q^n - q^{n-1} & \text{if } n > 1, \end{cases}$$

where the sum is over all the partitions of n.

Combinatorial identities have been studied extensively by various authors, e.g. see [3,7] and the references therein. But combinatorial identities where the sum is over all the partitions of a positive integer are rare. Motivating by the above identity, in this paper we will consider a class of combinatorial identities where the left-hand side is a sum over all the partitions of a positive integer. We get a class of such identities. It is remarkable that these identities have a common profile, that is it involves a parameter in an infinite set.

2. Main theorem and the proof

First let us fix some notations. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of all positive integers and let $\mathbb{R}$ denote the set of all reals. Let μ denote the Möbius function and let log denote the natural logarithm. Define binomial coefficients as follows:

$$\binom{x}{0} = 1, \quad \binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!} \quad \forall n \in \mathbb{N}, \ x \in \mathbb{R}.$$

Theorem. *Suppose Λ is a nonempty subset of $\mathbb{R}$ (called space of parameters). For each $\lambda \in \Lambda$, let $a_1^{(\lambda)}, a_2^{(\lambda)}, \ldots$ be a sequence of reals. The sequence $(b_n^{(\lambda)})$ is defined by*

$$b_n^{(\lambda)} = \sum_{d \mid n} d a_d^{(\lambda)}.$$

Assume that the sequence $(b_n^{(\lambda)})$ satisfies one of the following two conditions:

(i) *there exists a function $f : \Lambda \times \mathbb{N} \longrightarrow \Lambda$ such that*

$$b_{nm}^{(\lambda)} = b_m^{(f(\lambda,n))} \quad \forall n, m \in \mathbb{N}, \lambda \in \Lambda.$$

(ii)

$$b_{nm}^{(\lambda)} = b_n^{(\lambda)} + b_m^{(\lambda)} \quad \forall n, m \in \mathbb{N}, \lambda \in \Lambda.$$

The sequence $(c_n^{(\lambda)})$ *is defined by*

$$c_n^{(\lambda)} = \begin{cases} \dfrac{b_n^{(\lambda)}}{n} & \text{if } n \text{ is odd,} \\ \dfrac{b_n^{(\lambda)}}{n} - \dfrac{b_{n/2}^{(\lambda)}}{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Then we have the following identity

$$\sum_{\pi(n)} \prod_{j=1}^{n} \binom{a_j^{(\lambda)}}{k_j} = \sum_{i=1}^{n} \frac{1}{i!} \sum_{j_1+\ldots+j_i=n} c_{j_1}^{(\lambda)} \cdots c_{j_i}^{(\lambda)} \quad \forall n \in \mathbb{N}, \lambda \in \Lambda,$$

where the sum of the left-hand side is over all the partitions of n.

Remark. The theorem in essence transforms a sum over partitions into another sum. In general, sums over partitions are difficult to handle (see p. 224 of [3] as an open problem 4.2). Using this general theorem, in the next section we can transform several sums over partitions into ordinary sums.

To prove this theorem we need a lemma.

Lemma. *Using the notations as in the theorem. For* $n \geqslant 1$, *it holds that*

$$\frac{1}{n} \sum_{d \mid n} (-1)^{n/d-1} d a_d^{(\lambda)} = \begin{cases} \dfrac{b_n^{(\lambda)}}{n} & \text{if } n \text{ is odd,} \\ \dfrac{b_n^{(\lambda)}}{n} - \dfrac{b_{n/2}^{(\lambda)}}{n/2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. If n is odd, then the result follows from the definition of $b_n^{(\lambda)}$.

Suppose now n is even. We write n as $n = 2^\alpha \beta$, where $\alpha > 0$ and β is odd. Any divisor d of n can be written as $d = 2^i j$, where $0 \leqslant i \leqslant \alpha$ and $j \mid \beta$. Thus we have that

$$\frac{1}{n} \sum_{d \mid n} (-1)^{n/d-1} d a_d^{(\lambda)} = \frac{1}{n} \left(\sum_{j \mid \beta} 2^\alpha j a_{2^\alpha j}^{(\lambda)} - \sum_{\substack{0 \leqslant i < \alpha \\ j \mid \beta}} 2^i j a_{2^i j}^{(\lambda)} \right).$$

Obviously, by the definition of $b_n^{(\lambda)}$ we have that

$$\sum_{\substack{0 \leqslant i < \alpha \\ j \mid \beta}} 2^i j a_{2^i j}^{(\lambda)} = \sum_{d \mid 2^{\alpha-1}\beta} d a_d^{(\lambda)} = \sum_{d \mid \frac{n}{2}} d a_d^{(\lambda)} = b_{n/2}^{(\lambda)}.$$

From the Möbius inversion formula and the properties of Möbius function, we have

$$2^\alpha j a_{2^\alpha j}^{(\lambda)} = \sum_{d \mid 2^\alpha j} \mu(d) b_{2^\alpha j/d}^{(\lambda)} = \sum_{d \mid j} \mu(d) b_{2^\alpha j/d}^{(\lambda)} - \sum_{d \mid j} \mu(d) b_{2^{\alpha-1} j/d}^{(\lambda)}.$$

Assume that the sequence $(b_n^{(\lambda)})$ satisfies Condition (i). Then

$$\begin{aligned} &\sum_{d \mid j} \mu(d) b_{2^\alpha j/d}^{(\lambda)} - \sum_{d \mid j} \mu(d) b_{2^{\alpha-1} j/d}^{(\lambda)} \\ &\quad = \sum_{d \mid j} \mu(d) b_{j/d}^{(f(\lambda,2^\alpha))} - \sum_{d \mid j} \mu(d) b_{j/d}^{(f(\lambda,2^{\alpha-1}))} \\ &\quad = j a_j^{(f(\lambda,2^\alpha))} - j a_j^{(f(\lambda,2^{\alpha-1}))}. \end{aligned}$$

Hence

$$\sum_{j\mid\beta} 2^{\alpha} j a_{2^{\alpha}j}^{(\lambda)} = \sum_{j\mid\beta}\left(ja_j^{(f(\lambda,2^{\alpha}))} - ja_j^{(f(\lambda,2^{\alpha-1}))}\right)$$
$$= b_{\beta}^{(f(\lambda,2^{\alpha}))} - b_{\beta}^{(f(\lambda,2^{\alpha-1}))} = b_{2^{\alpha}\beta}^{(\lambda)} - b_{2^{\alpha-1}\beta}^{(\lambda)} = b_n^{(\lambda)} - b_{n/2}^{(\lambda)}.$$

Therefore

$$\frac{1}{n}\sum_{d\mid n}(-1)^{n/d-1} d a_d^{(\lambda)} = \frac{1}{n}(b_n^{(\lambda)} - b_{n/2}^{(\lambda)} - b_{n/2}^{(\lambda)}) = \frac{b_n^{(\lambda)}}{n} - \frac{b_{n/2}^{(\lambda)}}{n/2}.$$

If the sequence $\left(b_n^{(\lambda)}\right)$ satisfies Condition (ii), similar to the above argument, we can prove the same result. This completes the proof of the lemma. □

Proof of the theorem. The following arguments work for sufficiently small real variable x. Consider the function defined by

$$g(x) = \prod_{j=1}^{\infty}(1+x^j)^{a_j^{(\lambda)}}.$$

Its power series expansion is

$$g(x) = \prod_{j=1}^{\infty}\left(\sum_{k_j=0}^{\infty}\binom{a_j^{(\lambda)}}{k_j} x^{jk_j}\right)$$
$$= 1 + \sum_{n=1}^{\infty}\left(\sum_{\pi(n)}\prod_{j=1}^{n}\binom{a_j^{(\lambda)}}{k_j}\right) x^n.$$

On the other hand, by the lemma we have

$$\log g(x) = \sum_{j=1}^{\infty} a_j^{(\lambda)}\log(1+x^j) = \sum_{j=1}^{\infty} a_j^{(\lambda)}\sum_{l=1}^{\infty}(-1)^{l-1}\frac{x^{jl}}{l}$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{d\mid n}(-1)^{\frac{n}{d}-1} d a_d^{(\lambda)}\right) x^n$$
$$= \sum_{n=1}^{\infty} c_n^{(\lambda)} x^n.$$

Hence

$$g(x) = \exp\left(\sum_{n=1}^{\infty} c_n^{(\lambda)} x^n\right)$$
$$= 1 + \sum_{j=1}^{\infty}\frac{1}{j!}\left(\sum_{n=1}^{\infty} c_n^{(\lambda)} x^n\right)^j$$
$$= 1 + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\frac{1}{i!}\sum_{j_1+\cdots+j_i=n} c_{j_1}^{(\lambda)}\cdots c_{j_i}^{(\lambda)}\right) x^n.$$

Therefore

$$\sum_{\pi(n)}\prod_{j=1}^{n}\binom{a_j^{(\lambda)}}{k_j}=\sum_{i=1}^{n}\frac{1}{i!}\sum_{j_1+\cdots+j_i=n}c_{j_1}^{(\lambda)}\cdots c_{j_i}^{(\lambda)}.$$

This completes the proof of the theorem. □

3. Examples

In this section, we will get some concrete identities by means of the theorem.

Example 1. As a first example, we see how the identity at the beginning of the paper falls into our theorem. Let Λ be the set of all prime powers of $\mathbb{N}$. For $q\in\Lambda$, $a_n^{(q)}=I_n^{(q)}$. From [6], $b_n^{(q)}=q^n$. And $f(q,n)=q^n$. It is easy to see that the condition $b_{nm}^{(q)}=b_m^{(f(q,n))}$ holds.

$$\begin{aligned}\sum_{n=1}^{\infty}c_n^{(q)}x^n&=\sum_{n=1}^{\infty}\frac{q^n}{n}x^n-\sum_{n=1}^{\infty}\frac{q^n}{n}x^{2n}\\&=\log\frac{1-qx^2}{1-qx}.\end{aligned}$$

Thus

$$g(x)=\frac{1-qx^2}{1-qx}=1+qx+\sum_{n=2}^{\infty}(q^n-q^{n-1})x^n.$$

So we get the preceding identity by the theorem.

Example 2. This is an example from combinatorial theory (see p. 12 of [4]). Let Λ be the set of all integers greater than 1. For $r\in\Lambda$, let $M_n^{(r)}$ denote the number of circular sequences of length and period n over the set $\{1,2,\ldots,r\}$. Then

$$\sum_{d\mid n}\mathrm{d}M_d^{(r)}=r^n,$$

$$M_n^{(r)}=\frac{1}{n}\sum_{d\mid n}\mu(d)r^{n/d}.$$

Completely similar to Example 1 we have the following identity.

$$\sum_{\pi(n)}\prod_{j=1}^{n}\binom{M_j^{(r)}}{k_j}=\begin{cases}r & \text{if } n=1,\\ r^n-r^{n-1} & \text{if } n>1.\end{cases}$$

The following three examples need some elementary facts from number theory, one can see Hardy and Wright [5].

Example 3. It is well-known that

$$\sum_{d\mid n}\mu(d)=\delta_{1,n},$$

where $\delta_{1,n}$ is the Kronecker symbol. We put $\Lambda=\mathbb{R}$ and $a_n^{(\lambda)}=\lambda\mu(n)/n$. So $b_n^{(\lambda)}=\lambda\delta_{1,n}$. Define $f(\lambda,n)=\lambda\delta_{1,n}$. Here

$$\sum_{n=1}^{\infty}c_n^{(\lambda)}x^n=\lambda x-\lambda x^2.$$

It is easy to see that

$$\exp(\lambda x-\lambda x^2)=1+\sum_{n=1}^{\infty}\left(\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\frac{\lambda^{n-i}}{(n-i)!}\binom{n-i}{i}\right)x^n.$$

Thus we have the following identity

$$\sum_{\pi(n)}\prod_{j=1}^{n}\binom{\lambda\mu(j)/j}{k_j}=\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\frac{\lambda^{n-i}}{(n-i)!}\binom{n-i}{i}.$$

Example 4. Let ϕ denote Euler phi-function. It is well-known that

$$\sum_{d\mid n}\phi(d)=n.$$

We put $\Lambda=\mathbb{R}$ and $a_n^{(\lambda)}=\lambda\phi(n)/n$. So $b_n^{(\lambda)}=\lambda n$. Define $f(\lambda,n)=\lambda n$. Here

$$\sum_{n=1}^{\infty}c_n^{(\lambda)}x^n=\frac{\lambda x}{1-x^2}.$$

It is easy to see that

$$\exp\left(\frac{\lambda x}{1-x^2}\right)=1+\sum_{n=1}^{\infty}\left(\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\frac{\lambda^{n-2i}}{(n-2i)!}\binom{n-i-1}{i}\right)x^n.$$

Thus we have the following identity

$$\sum_{\pi(n)}\prod_{j=1}^{n}\binom{\lambda\phi(j)/j}{k_j}=\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\frac{\lambda^{n-2i}}{(n-2i)!}\binom{n-i-1}{i}.$$

Example 5. In the above four examples, the sequence $(b_n^{(\lambda)})$ satisfies Condition (i). In this example the sequence $(b_n^{(\lambda)})$ satisfies Condition (ii). Consider von Mangoldt function $\Lambda(n)$. It is defined by

$$\Lambda(n)=\begin{cases}\log p & \text{if } n=p^m,\ m\geqslant 1,\ p \text{ is a prime},\\ 0 & \text{otherwise}.\end{cases}$$

It is well-known that

$$\sum_{d\mid n}\Lambda(d)=\log n.$$

We put the space of parameters be $\mathbb{R}$ and $a_n^{(\lambda)}=\lambda\Lambda(n)/n$. So $b_n^{(\lambda)}=\lambda\log n$ and it satisfies Condition (ii). By the theorem we can get the corresponding identity. Here we omit the details.

Acknowledgement

The author thanks the reviewers for their helpful comments and suggestions.

References

[1] L. Carlitz, The arithmetic of polynomials in a Galois field, Amer. J. Math. 54 (1932) 39–50.
[2] Y. Deng, Isomorphism classes of hyperelliptic curves of genus 3 over finite fields, Finite Fields Appl. 12 (2006) 248–282.
[3] G.P. Egorychev, Integral representation and the computation of combinatorial sums, Translations of Mathematical Monographs, vol. 59, American Mathematical Society, Providence, 1984.
[4] M. Hall, Combinatorial Theory, second ed., Wiley, New York, 1986.
[5] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, fifth ed., The Clarendon Press and Oxford University Press, New York, 1979.
[6] R. Lidl, H. Niederreiter, Finite Fields, second ed., Cambridge University Press, Cambridge, 1997.
[7] J. Riordan, Combinatorial Identities, Wiley, New York, 1968.