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SOME GENERATING RELATIONS CONNECTED WITH A FUCNTION DEFINED BY A GENERALIZED RODRIGUES FORMULA

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In an attempt to provide an elegant unification of several classes of polynomials, we have introduced a sequence of functions $\{V_n^\alpha(x; a, k, s) | n = 0, 1 \ldots\}$ by means of the generalized Rodrigues formula (1.1). The present paper is concerned only with the linear generating relations for $V_n^\alpha(x; a, k, s)$. Some applications of the generating relations have been noticed.

1. INTRODUCTION

Rodrigues formulae have drawn the attention of several workers and a variety of polynomials have been introduced with the help of these formulae from time to time.

For instance employing the operator $(x^{k+1}D_x)$, Srivastava and Singhal (1971) introduced a general class of polynomials $G_n^{(\alpha)}(x, r, p, k)$ defined by means of

$$G_n^{(\alpha)}(x, r, p, k) = \frac{1}{n!} x^{-\alpha-kn} \exp(px^r)(x^{k+1}D)^n [x^\alpha \exp(-px^r)]$$

which contains, as special cases, several polynomial systems studied earlier.

The Rodrigues formula for the generalized Laguerre polynomials $T_{kn}^{(\alpha)}(x)$ due to Mittal (1971a) is given by

$$T_{kn}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} \exp\{p_k(x)\} D_x^n [x^{\alpha+n} \exp\{-p_k(x)\}]$$

where $p_k(x)$ is a polynomial in x of degree k. For these polynomials, Mittal (1971b) proved the following relation

$$T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!} x^{-\alpha-n} \exp\{p_k(x)\} T_s^n [x^\alpha \exp\{-p_k(x)\}]$$

where $\quad T_s \equiv x(s + xD)$.

Subsequently, Patil and Thakare (1975) have obtained several operational formulae and generating relations for

$$P_n^{(\alpha)}(x, r, \lambda, p, a, s) = x^{-\alpha} \exp(px^r)\, \theta^n\{x^{\alpha+\lambda n} \exp(-px^r)\},$$

where $\theta = {}_aT_s \equiv x^a(s + xD_x)$.

In view of the aforementioned observations, it would seem worthwhile to introduce here a general sequence of functions $\{V_n^{(\alpha)}(x; a, k, s)/n = 0, 1, \ldots\}$ defined by means of the following nth differential formula:

$$V_n^{(\alpha)}(x; a, k, s) = \frac{1}{n!} x^{-\alpha} \exp(p_k(x))\, \theta^n [x^\alpha \exp(-p_k(x))], \qquad \ldots(1.1)$$

where $p_k(x)$ is a polynomial in x of degree k and

$$\theta \equiv x^a(s + xD_x), \qquad \ldots(1.2)$$

where a and s are constants.

In what follows, for simplicity of notations, we shall put

$$\theta_0 \equiv x^{a+1}D_x \quad \text{and} \quad \theta_1 = x^a(1 + xD_x).$$

Evidently, the sequence of functions introduced here will incorporate several classical and other polynomials or their various generalizations (including those indicated above).

The object of this note is to investigate particularly the linear generating relations for these functions. Numerous scattered results can be exhibited as special cases of our formulas.

2. Linear Generating Relations

In our investigations we make use of the following properties of the differential operator $\theta \equiv x^a(s + xD)$, $D \equiv d/dx$ (Mittal 1977, Patil and Thakare 1975) :

$$\theta^n(x^\alpha) = a^n \left(\frac{\alpha + s}{a}\right)_n x^{\alpha+na} \qquad \ldots(2.1)$$

$$\theta^n = x^{na}(\delta + s)(\delta + s + a) \ldots (\delta + s + (n-1)a) \qquad \ldots(2.2)$$

$$\theta^n(xuv) = x \sum_{m=0}^{n} \binom{n}{m} \theta^{n-m}(v)\, \theta_1^m(u) \qquad \ldots(2.3)$$

$$\exp(t\theta)(x^\alpha) = x^\alpha(1 - ax^a t)^{-(\alpha+s)/a} \qquad \ldots(2.4)$$

$$\exp(t\theta)(xuv) = x \exp(t\theta)\, v \exp(t\theta_1)\, u \qquad \ldots(2.5)$$

and

$$\exp(t\theta)\,(x^{\alpha} f(x)) = x^{\alpha}(1 - ax^{a}t)^{-(\alpha+s)/a} f[x(1 - ax^{a}a)^{-1/a}]. \qquad \ldots(2.6)$$

Consider

$$\theta^{n}[x^{\alpha}\exp(-p_k(x))] = x \sum_{m=0}^{n} \binom{n}{m} \theta^{n-m} \exp(-\, p_k(x)\, \theta_1^{m}\, (x^{\alpha-1}).$$

The above, in view of (2.2), readily yields

$$V_n^{(\alpha)}(x;\, a, k, s) = x^{an} \sum_{m=0}^{n} \frac{1}{m!\,(n-m)!} \prod_{i=0}^{n-m-1} (s - xp_k'(x) + ia) \times \prod_{j=0}^{m-1} (\alpha + ja)$$

which simplifies to yield the explicit expression

$$V_n^{(\alpha)}(x;\, a, k, s) = (ax^{a})^{n} \sum_{m=0}^{n} \frac{1}{m!\,(n-m)!} \left(\frac{s - xp_k'(x)}{a}\right)_{n-m} \left(\frac{\alpha}{a}\right)_{m}. \qquad \ldots(2.7)$$

From the definition (1.1), we also have

$$V_n^{(\alpha)}(x;\, a, k, s) = \sum_{m=0}^{n} \frac{1}{m!}\, (ax^{a})^{m}\, (\alpha/a)_m\, V_{n-m}^{(0)}(x;\, a, k, s). \qquad \ldots(2.8)$$

The linear generating relations, to be established here, are

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(\alpha)}(x;\, a, k, s)\, t^{n} = (1 - at)^{-(\alpha+s)/a} \exp[p_k(x) - p_k\{x(1 - at)^{-1/a}\}] \qquad \ldots(2.9)$$

$$\sum_{n=0}^{\infty} (ax^{a})^{-n} V_n^{\alpha-an}(x;\, a, k, s)\, t^{n} = (1 + t)^{((\alpha+s)-1)/a} \exp[p_k(x) - p_k\{x(1 + t)^{1/a}\}] \qquad \ldots(2.10)$$

$$\sum_{m=0}^{\infty} \binom{n+m}{n} V_{n+m}^{(\alpha)}(x;\, a, k, s)\, t^{m} = (1 - ax^{a}t)^{-(\alpha+s)/a} \exp[p_k(x) - p_k\{x(1 - ax^{a}t)^{-1/a}\}] \times V_n^{(\alpha)}(x(1 - ax^{a}t)^{-1/a};\, a, k, s). \qquad \ldots(2.11)$$

Proof of (2.9)

We have

$$\sum_{n=0}^{\infty} V_n^{(\alpha)}(x; a, k, s)\, t^n = x^{-\alpha} \exp(p_k(x)) \exp(t\theta) \{x^{\alpha} \exp(-p_k(x))\}. \quad \dots(2.12)$$

Making use of (2.5) in the right-hand side, we thus have

$$\sum_{n=0}^{\infty} V_n^{(\alpha)}(x; a, k, s)\, t^n = x^{-\alpha+1} \exp(p_k(x)) \exp(t\theta) \exp(-p_k(x)) \times \exp(t\theta_1) \{x^{\alpha-1}\}.$$

Now (2.9) would follow at once if we interpret the above right-hand expression by (2.4).

It is not difficult to verify that the operator θ satisfies the operational relation :

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \theta^n \{x^{\alpha-an} f(x)\} = x^{\alpha}(1 + at)^{(\alpha+s-a)/a} f[x(1 + at)^{1/a}]. \quad \dots(2.13)$$

Proof of (2.10)

From the definition (1.1), we readily have

$$\sum_{n=0}^{\infty} V_n^{(\alpha-an)}(x; a, k, s) \left\{\frac{t}{x^a}\right\}^n$$

$$= x^{-\alpha} \exp p_k(x) \sum_{n=0}^{\infty} \frac{t^n}{n!} \theta^n \{x^{\alpha-an} \exp(-p_k(x))\}$$

which, in view of (2.13), yields the required result.

Proof of (2.11)

To prove (2.11) we begin by considering (1.1) and express it as

$$\theta^n [x^{\alpha} \exp(-p_k(x))] = n!\, x^{\alpha} \exp(-p_k(x))\, V_n^{(\alpha)}(x; a, k, s).$$

Now operating on both sides by $\exp(t\theta)$ we obtain

$$\sum_{m=0}^{\infty} \frac{t^m}{m!}\, \theta^{n+m} \{x^{\alpha} \exp(-p_k(x))\}$$

$$= n!\,(1 - ax^a t)^{-(\alpha+s)/a} \exp[-p_k\{x(1 - ax^a)^{-1/a}\}]$$

$$\times V_n^{(\alpha)}(x(1 - ax^a t)^{-1/a}; a, k, s)$$

which in view of (1.1) yields the required result (2.11).

The above result (2.11) is analogous to a property possessed by almost all the classical orthogonal polynomials, and we shall use this property in the next section.

3. Applications

In this section we apply the generating relation (2.9) of the previous section to obtain a finite sum formula for $V_n^{(\alpha)}(x; a, k, s)$ and the generating relation (2.11) to establish a class of bilateral generating relations for these functions.

From the generating relation (2.9) and the simple relation

$$(1 - at)^{-\alpha/a} = (1 - at)^{-\beta/a} \sum_{m=0}^{\infty} \left(\frac{\alpha - \beta}{a}\right)_m \frac{(at)\,m}{m!},$$

it follows that

$$V_n^{(\alpha)}(x; a, k, s) = \sum_{m=0}^{\infty} a^m (m!)^{-1} \left(\frac{\alpha - \beta}{a}\right)_m V_{n-m}^{(\beta)}(x; a, k, s). \qquad \text{...(3.1)}$$

In view of (2.11) $V_n^{(\alpha)}(x; a, k, s)$ belongs to a class of functions $\{\triangle_\mu(x) : \mu$ is an arbitrary complex number$\}$, considered by Srivastava and Lavoie (1975) to obtain bilateral generating relations for this class of functions. Therefore, as a special case of eqns. (105) through (108) of Srivastava and Lavoie (1975), with $\mu = n$, $\gamma_{n,m} = \binom{n+m}{n}$

$$\theta(x, t) = (1 - ax^a t)^{-(\alpha+s)/a} \exp[p_k(x) - p_k\{x(1 - ax^a t)^{-1/a}\}]$$
$$\phi(x, t) = 1 \quad \text{and} \quad \psi(x, t) = x(1 - ax^a t)^{-1/a}$$

we are led to the following bilateral generating relation for $V_n^{(\alpha)}(x; a, k, s)$:

$$\sum_{m=0}^{\infty} V_{\nu+m}^{(\alpha)}(x; a, k, s)\, R_{m,\nu}^{q}(y)\, t^m$$
$$= (1 - ax^a t)^{-(\alpha+s)/a} \exp[p_k(x) - p_k\{x(1 - ax^a t)^{-1/a}\}]$$
$$\times \phi_{q,\nu}[x(1 - ax^a t)^{-1/a}, yt^q],\ \nu = 0, 1, 2, \ldots,$$

where

$$\phi_{q,\nu}[x, t] = \sum_{m=0}^{\infty} \delta_{\nu,m} V^{(\alpha)}_{\nu+qm}(x; a, k, s)\, t^m,$$

$$\delta_{\nu,m} \neq 0$$

and

$$R^q_{m,\nu}(y) = \sum_{k=0}^{[n/q]} \binom{\nu+m}{\nu+qk} \delta_{\nu,k} y^k.$$

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